RECENT MATHEMATICAL TABLES

169[B].—BAASMTC, *Mathematical Tables, Volume IX. Table of Powers giving integral powers of integers. Initiated by* J. W. L. GLAISHER, *extended by* W. G. BICKLEY, C. E. GWYTHER, J. C. P. MILLER, E. J. TERNOUTH. Edited by Miller. Cambridge University Press, 1940, xii, 132 p. 21.5 × 27.9 cm. 15 shillings.

NYMTP, *Table of the First Ten Powers of the Integers from 1 to 1000.* New York, December 1, 1938 (title page), 1939 (outside cover), 80 p. 21 × 35 cm. Reproduced in mimeographed form. Originally listed at $.50; out of print. Not to be reprinted.

These two tables, published within a year of one another, are similar in their purpose, to provide computers with an adequate table of the higher powers of the integers. The first, however, covers a much greater range than the second, although for powers in which the two overlap the second is somewhat easier to use because of the various printing devices which were used to condense the longer work.

The BAASMTC tables originated with a work prepared by J. W. L. GLAISHER more than 70 years ago to which the following reference was made in the 1873 Report of the Committee on Mathematical Tables: "Mr. J. W. L. Glaisher has had formed in duplicate a table giving the first twelve powers of the first thousand numbers, which, after the calculation has been made independently a third time, will be stereotyped and published, probably in the course of 1873; it is hoped that it will help to make the tabulation of mathematical functions less laborious and difficult." But the tables never appeared.

Intrigued by this reference and also by a reference to these tables made in 1905 by A. J. C. CUNNINGHAM, L. J. COMRIE initiated a diligent search for a copy. His efforts were finally rewarded by success and in 1935 H. J. WOODALL presented to the Committee the copy to which Cunningham had made reference. This is a proof copy and is probably the only one in existence. The introduction to the work under review states that "without this copy it is doubtful if the production of the present volume would have been undertaken."

With the original Glaisher computation as a nucleus, the scope of the table was extended to give x^n for the following ranges:

$$n = 2(1)12 \qquad x = 1(1)1099$$
$$n = 13(1)20 \qquad x = 1(1)299$$
$$n = 21(1)50 \qquad x = 1(1)120.$$

Because of the limitations of space, the powers corresponding to $n = 28$ and 29 were curtailed, and for values of n greater than 30, except for $n = 40$ and $n = 50$, the powers were also curtailed. In these abbreviated values the last figure printed is always the true value and is not raised. For convenience in use the number of figures retained is a multiple of five, and the counting of the digits is assisted by small figures printed at the top and bottom of the columns or in the margins. In the incomplete values never fewer than 21 figures are retained; below 100 not more than 25 are given, but in the range to 120 the number may be as great as 28.

"The tables are arranged so that, whenever possible, an opening contains powers of a hundred consecutive integers, and so that consecutive powers of a given integer are normally on the same or adjacent openings. . . . The complete range of powers for each hundred consecutive integers forms a single 'chapter' which is completed before the next is started."

For compactness in printing various devices have been adopted. Thus, we find that "powers have in some cases been combined in pairs so as to give an approximately constant total number of digits. For example, in the first 'chapter,' x^9 and x^{30} appear together on pages 4 and 5, followed by x^{10} and x^{29} on pages 6 and 7, and so on, up to pages 24 and 25, which contain x^{19} and x^{20}. Thus for consecutive powers of, say, 57, we turn forward until we come to x^{20}, then backward (as indicated by a footnote) until x^{30} is reached; another footnote directs us to page 27 for x^{31}, and from there we proceed forward to x^{50}."

Unusual care was taken to insure accuracy in the table. For this purpose the following formula was used,

$$\sum_{x=1}^{X} x^n = \frac{1}{n+1} X^{n+1} + \tfrac{1}{2} X^n + a_1 X^{n-1} - a_3 X^{n-3} + a_5 X^{n-5} - \cdots,$$

where we abbreviate:

$$a_{2r+1} = \frac{n(n-1)(n-2)\cdots(n-r)}{(2r+2)!} B_{r+1},$$

in which B_{r+1} is the $(r+1)$th Bernoulli number. The powers were summed in blocks of 50 and these totals compared with those computed directly from the formula. The figures were also read against those given in other existing tables where the ranges overlapped. This resulted in the detection of a number of errors in other tables, which have been listed in the "Bibliography."[1] No errors were discovered in the calculations of Glaisher, nor in the NYMTP tables, which were loaned by A. N. Lowan for this check.

In addition to the table of powers there is given a table of the binomial coefficients[2] $\binom{n}{r}$ for $n = 2(1)50, r = 2(1)10$ and a table of the numerators and denominators of the coefficients a_{2r+1} for values of $n \leqq 50$.

The NYMTP table was the first publication of that Project. It was computed from the first four powers of the integers given in P. Barlow's *Table of Squares, Cubes. . . .* Because of unusual precautions taken to avoid errors in the final stencils, it was believed that the table was without error. The check by the BAASMTC, which we have mentioned above, confirmed this belief.

H. T. D.

[1] Errors in the following works are listed:
P. Barlow, *New Mathematical Tables . . .*, London, 1814
K. Hayashi, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen . . .*, Berlin, 1926
K. Hayashi, *Fünfstellige Funktionentafeln . . .*, Berlin, 1930
K. Hayashi, *Tafeln für die Differenzenrechnung . . .*, Berlin, 1933
K. Pearson, *Tables for Statisticians and Biometricians*, Part II, London, 1931
H. W. Weigel, $x^n + y^n = z^n$*? Die elementare Lösung des Fermat-Problems . . .*, Leipzig, 1933
L. Zimmermann, *Vollständige Tafeln der Quadrate aller Zahlen bis* 100 009, third ed., Berlin, 1938.

[2] A complete table of the binomial coefficients $\binom{n}{r}$ for $n = 2(1)50$, $r = 2(1)50$, is contained in J. W. L. Glaisher, *Messenger Math.*, s. 2, v. 47, 1917, p. 97–107.

170[B].—*Square Root Divisors*, 1941, 2 p., *Square Root Multipliers*, 1940, 2 p., *Cube Root Divisors*, 1944, 2 p. *Publications* MM 56, 57, 68 of the Marchant Calculating Machine Co., Oakland, California. 21.5 × 28 cm.

These tables are one-page substitutes for more extensive tables of square and cube roots and are supposed to be used in connection with any standard computing machine.

The table of Square Root Divisors, T. 56, gives 8-place values of $2A^{\frac{1}{2}}$ and $2(10A)^{\frac{1}{2}}$ for 166 integral values of A (at irregular intervals) between 100 and 1000. The table of Square Root Multipliers, T. 57, gives for the same values of A the reciprocals of the values in T. 56.

To use T. 56 to find an approximation to the square root of a given number N, we enter the table at the value of A which is nearest to N and take out the value of $2A^{\frac{1}{2}}$ (or perhaps $2(10A)^{\frac{1}{2}}$). Then the approximation

$$N^{\frac{1}{2}} = \frac{N+A}{2A^{\frac{1}{2}}}$$

is obtained by machine division of $N + A$ by the tabulated entry. Thus the method is based on the fact that the arithmetic and geometric means of two nearly equal numbers are very nearly equal.

T. 57 merely replaces the operation of division by multiplication. Probably the average computer will find T. 56 a bit more efficient than T. 57 when his machine has automatic division and the quotient can be copied down while the machine is operating. In some cases when the square root is in the denominator T. 57 might prove more effective.

It may be argued that no table for square root is necessary if one has a computing machine. Most computers are familiar with a square root process based on the identity

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

which is well suited to some machines and which will yield the square root of an occasional number sooner than the operator can locate a table. Tables 56 and 57 will be found more efficient when a great many scattered square roots to 5S are needed quickly.

The Table of Cube Root Divisors, T. 68, gives 6-place values of $3A^{2/3}$, $3(10A)^{2/3}$ and $3(100A)^{2/3}$ for 162 values of A between 100 and 1000. It is used in a similar way to obtain cube roots by the approximation

$$N^{1/3} = \frac{N + 2A}{3A^{2/3}}.$$

The irregular intervals of the argument A of the three tables are so chosen that one application of the process will give the required root correct to at least 5S, no matter what value of N is given. On the reverse sides of T. 56 and 68 are examples of the re-application of the appropriate formulae a second time to obtain approximations correct to 9S and 10S.

The method involved in these tables was apparently suggested by comments of L.J.C. in his "Introduction" to the third edition of *Barlow's Tables* (1930). The general method, which is usually superior to interpolation, may be used with any table of square or cube roots, or indeed any table of squares or cubes. T. 68, however, is decidedly more convenient for cube roots.

The reviewer has two photostatic copies of a 3-page manuscript table of the same sort for 5th roots prepared and supplied by the same company. No doubt this useful table will be published in due course, to replace *Basic Publication* MM 88, of Aug., 1940, 2 p.

Quite another method for cube and 5th roots has been given by Dederick.[1] A one-page table enables the computer to obtain either root to 10S.

D. H. L.

[1] L. S. DEDERICK, "A modified method for cube roots and fifth roots," *Amer. Math. Mo.*, v. 33, 1926, p. 469–472. See also D. H. LEHMER, "On the use of the calculating machine for cube and fifth roots," *Amer Math. Mo.*, v. 32, 1925, p. 377–379. In the "table of cube and fifth roots" for this latter article are the following slips: $\sqrt[5]{10}$, *for* 93193, *read* 93192; $\sqrt[5]{1000}$ *for* 71705, *read* 71706; $\sqrt[3]{1.02}$ *for* 27709, *read* 22710; $\sqrt[5]{5}$, *for* 29662, *read* 29661.

171[D].—A. LINNEBACH, "Mehrkreisige Siebenschaltungen mit ausgeglichener Resonanzkurve," *Elek. Nach. Technik*, v. 20, Oct. 1943, p. 247–248. 21.7 × 28 cm.

There are tables of exact values, and to 6D, of $\cos \frac{(2\nu - 1)}{2n}\pi$, and $\sin \frac{(2\nu - 1)}{2n}\pi$, $n = 1, 2, \nu = 1$; $n = 3, 4, \nu = 1, 2$; $n = 5, 6, \nu = 1(1)3$; $n = 8, \nu = 1(1)4$; $n = 10, \nu = 1(1)5$; $n = 12, \nu = 1(1)6$. Also exact values, and to 5D, of $\cos \frac{2\nu - 1}{24}\pi$, $\sin \frac{2\nu - 1}{24}\pi$, $\nu = 1(1)6$.

172[E].—NATIONAL DEFENSE RESEARCH COMMITTEE, Division 6, *Tables of the Bipolar Transformation*, compiled by M.I.T. Underwater Sound Laboratory, Report, November 1944, 13 leaves + 3 plates (2 folding). 21.5 × 28 cm. Printed by the photo-offset process from manuscript.

These tables are available only to certain Government agencies and activities.

In all transmission line problems the transformation

$$\tanh[\pi(\alpha + i\beta)] = R + iX = \rho e^{i\phi}$$

is of importance.

The tables give values of R(5D), X(5D), ρ(4D), and ϕ (to the nearest 1′) as functions of $\alpha = 0(.01).2(.02).4(.05).6(.1)1$, and $\beta = 0(.02).5$.

The following formulae are useful:

$$R = \frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha) + \cos(2\pi\beta)}, \quad X = \frac{\sin(2\pi\beta)}{\cosh(2\pi\alpha) + \cos(2\pi\beta)},$$

$$R^2 + [X + \cot(2\pi\beta)]^2 = \csc^2(2\pi\beta),$$

$$[R - \coth(2\pi\alpha)]^2 + X^2 = \operatorname{csch}^2(2\pi\alpha).$$

The first plate (27.8 × 34.3 cm.) is a graph of $\tanh[\pi(\alpha + i\beta)] = \rho e^{i\phi}$, $\phi = 0(0^\circ.5)90^\circ$, $\rho = .1(.01)1(.1)10$, $\alpha = .01(.01).06(.02).2(.05).3(.1).5$, $\beta = .01(.01).06(.02).2(.05).3(.02)$-$.44(.01).5$.

The second plate (27 × 27.8 cm.) shows contours of the transformation $R + iX = \tanh[\pi(\alpha + i\beta)]$, $R = 0(.05)5$, $X = 0(.05)5$, $\alpha = .005(.005).1(.01).25$, $.3(.02).4(.1).6$, $\beta = 0(.02).1$, $.15(.01).4(.05).5$. There are three errors in marking the letters α and β, and 7 places where I is substituted for X.

The third plate (27.8 × 19.1 cm.) displays contours for the equation $\rho e^{i\phi} = \sinh[\pi(\alpha + i\beta)]$, $\rho = .1(.005).2(.01)1(.05)2(.1)8$, $\phi = 0(2^\circ)90^\circ$, $\alpha = .02$, $.05(.05).3(.1).9$, $\beta = .02$, $.05(.05).5$.

173[E].—A. Pucher, "Rechteckplatten mit zwei eingespannten Rändern," *Ingenieur-Archiv*, v. 14, 1943, p. 250–251. 19.3 × 26.8 cm.

There are 8 tables, to 5S, for $x = .25(.05)3.6$, of the following functions:

$$\frac{\sinh\pi x + \pi x\cosh\pi x}{\frac{1}{2}\sinh 2\pi x + \pi x}, \quad \frac{\cosh\pi x + \pi x\sinh\pi x}{\frac{1}{2}\sinh 2\pi x - \pi x}, \quad \frac{\pi x\sinh\pi x}{\frac{1}{2}\sinh 2\pi x + \pi x},$$

$$\frac{\pi x\cosh\pi x}{\frac{1}{2}\sinh 2\pi x - \pi x}, \quad \frac{\sinh\pi x}{\frac{1}{2}\sinh 2\pi x \pm \pi x}, \quad \frac{\cosh\pi x}{\frac{1}{2}\sinh 2\pi x \pm \pi x}.$$

174[E].—P. B. Wright, "Resistive attenuator, pad and network, theory and design" part 2 of a 4-part paper, *Communications*, v. 24, Oct. 1944, p. 64, 66, 68. 19.6 × 27.1 cm.

Tables for $20\log e^\theta(e^{2\theta} = k^2 \geqslant 1) = 0(.01).2(.05).4(.1)4(.5)30(1)60(5)140, 150$ of (a) $\sinh\theta = (k^2 - 1)/2k$; (b) $\cosh\theta = (k^2 + 1)/2k$; (c) $\tanh\theta$; (d) $\tanh\frac{1}{2}\theta = (k - 1)/(k + 1)$; (e) $\cosh^2\theta$; (f) $\sinh^2\frac{1}{2}\theta = (k - 1)^2/4k$; (g) $\operatorname{csch}\theta$; (h) $\operatorname{sech}\theta$; (i) $\coth\theta$; (j) $\coth\frac{1}{2}\theta$; (k) $\operatorname{sech}^2\theta$; (l) $\operatorname{csch}^2\frac{1}{2}\theta$. These tables are to 5–9S.

175[I].—A. N. Lowan & H. E. Salzer. "Coefficients for interpolation within a square grid in the complex plane." *J. Math. Phys.*, M.I.T., v. 23, 1944, p. 156–166. 17.5 × 25.5 cm.

The usual theory of polynomial interpolation holds in the complex domain as well as on the real axis. Thus, given an analytic function $f(z)$ of the complex variable $z = x + iy$ and $n + 1$ arbitrary points $z_0, z_1, \cdots, z_n$ within the domain of regularity of $f(z)$, there will be one, and only one, polynomial $\phi_n(z)$ of nth degree which will assume the prescribed values $f(z_k)$ for $k = 0, \cdots, n$. The degree of approximation of $\phi_n(z)$ to $f(z)$ can be judged from Taylor's formula. The polynomial $\phi_n(z)$ can be represented in many forms, but we are here concerned only with the standard Lagrange form

$$\phi_n(z) = \sum_{k=0}^{n} \frac{P_\nu(z)}{P_\nu(z_k)} f(z_k),$$

where $P_\nu(z) = (z - z_0) \cdots (z - z_{\nu-1})(z - z_{\nu+1}) \cdots (z - z_n)$.

Originally, this formula has been regarded as a purely theoretical tool. However, in the real domain tables of the coefficients $P_\nu(z)/P_\nu(z_k)$ are now available for the typical case of equally spaced z_k. With such tables[1] one has only to perform the $n + 1$ multiplications by $f(z_k)$, and the products are accumulated in any modern computing machine without clearing. Thus, using tables and a modern computing machine, the Lagrange interpolation formula will lead to the required value $\phi_n(z)$ without any writing and auxiliary computations. In the complex plane the situation is different only in that a multiplication of two complex numbers requires four ordinary multiplications.

The present paper provides tables of the quadratic and cubic Lagrange interpolation polynomials if the three or four given points are vertices of a typical square: z_0, $z_1 = z_0 + h$, $z_{-1} = z_0 + ih$, $z_2 = z_0 + h + ih$ (h real), and it is desired to interpolate for a point $z = z_0 + \Delta z$ within the square. In the usual way one puts $\Delta z = h\,(p + iq)$ so that $0 \leqq p \leqq 1$, $0 \leqq q \leqq 1$. We have then to interpolate at the point $P = p + iq$ of the unit square. The quadratic and cubic formulae are written in the form

$$\phi_2(z) = L_{-1}^{(2)}(P)f(z_{-1}) + L_0^{(2)}(P)f(z_0) + L_1^{(2)}(P)f(z_1)$$

and

$$\phi_3(z) = L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2).$$

The seven coefficients $L_{-1}^{(2)}(P)$ etc. are tabulated (*p.* 160–166), each in the form of a double-entry table with the arguments p and q varying from 0 to 1 in steps of .1. The values are exact, which means that the coefficients $L_k^{(2)}$: $L_{-1}^{(2)}(P)$, $L_1^{(2)}(P)$ are given to 3D and $L_0^{(2)}(P)$ to 2D, while the coefficients $L_k^{(3)}(P)$, $k = -1(1)2$, are given to 4D.

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[1] NYMTP, *Tables of Lagrangian Interpolation Coefficients*, New York, 1944.

176[I].—H. E. Salzer, "Table of coefficients for inverse interpolation with central differences," *J. Math. Phys.*, M.I.T., v. 22, 1943, p. 210–224. 17.5 × 25.5 cm.

The author has donė a genuine service to practical interpolation by tabulating not only the coefficients for inverse interpolation by central differences, as given in the article under review, but also for a similar tabulation of the coefficients of the formula with advancing differences; see the review by D. H. L., *MTAC*, p. 315 f. The difficult problem of finding inverse values from tables which are computed to arguments that are not sufficiently close to assure linear interpolation, is thus made much simpler by means of these tables.

The tables computed by Salzer are based upon the inverse of what is commonly called the Laplace-Everett formula for interpolation. If $f(x)$ is the function tabulated for $x = x_0$, $x_0 + h$, $x_0 + 2h$, etc., and if p lies between 0 and 1, then the Laplace-Everett formula may be written:

$$(1)\quad f(x + ph) = f(x) + p\Delta f(x) - [E_2(q)\delta^2 f(x) + E_2(p)\delta^2 f(x + h)] + [E_4(q)\delta^4 f(x) + E_4(p)\delta^4 f(x + h)] - \cdots$$

where $q = 1 - p$, $\delta^2 f(x)$, $\delta^4 f(x)$ etc. are central differences, and

$$E_2(p) = p(1^2 - p^2)/3!, \qquad E_4(p) = p(1^2 - p^2)(2^2 - p^2)/5!\text{ etc.}$$

If we make the further abbreviations:

$$m = \frac{f(x + ph) - f(x)}{\Delta f(x)}, \qquad d_0{}^r = \frac{\delta^r f(x)}{\Delta f(x)}, \qquad d_1{}^r = \frac{\delta^r f(x + h)}{\Delta f(x)},$$

then equation (1) can be written in the form:

$$p = m + E_2(q)d_0^2 + E_2(p)d_1^2 - E_4(q)d_0^4 - E_4(p)d_1^4 + \cdots.$$

This formula can now be inverted by means of Lagrange's formula for the inversion of functions of the form $p = m + \lambda\phi(p)$, subject to the condition $|\lambda\phi(p)| < r$, where $r = |p - m|$. This inversion leads to the following expansion:

$$p = m + {}_mC_3 d_0^2 + \tfrac{1}{6}m(1 - m^2)d_1^2 + A(m)d_0^4 + B(m)d_1^4 + C(m)(d_0^2)^2 + \cdots,$$

where ${}_mC_3$ is the third binomial coefficient and the functions $A(m)$, $B(m)$, $C(m)$, etc. are polynomials in m. If one defines a product of the form $(d^p)^r(d^q)^s$ as being a term of order $pr + qs$, then the inversion is carried out through terms of sixth order. The original calculation of the coefficients by H. T. D.[1] were checked and one printing error detected, the first number in the coefficient of the term $d_0^2(d_1^2)^2$ being 1 instead of 2.

Exclusive of the coefficients of the terms of second order there are 15 coefficients to be tabulated, which the author has designated by $A(m)$, $B(m)$, $\cdots$, $O(m)$. The first two coefficients ${}_mC_3$ and $m(1 - m^2)/6$ are to be found in the recently published *Tables of Lagrangian Interpolation Coefficients* of NYMTP (*MTAC*, p. 314f), and are accordingly omitted by the author. The remaining 15 coefficients are computed to 10D. Those of fourth order, namely $A(m)$ through $E(m)$ are given over the range $m = 0(.001)1$ and the author makes use of the relationships:

$$A(m) = B(1 - m), \qquad -C(m) = D(1 - m), \qquad E(m) = -E(1 - m),$$

to reduce the space occupied by the tables. The coefficients of sixth order are given over the range 0(.1)1.

H. T. D.

[1] H. T. Davis, *Tables of the Higher Mathematical Functions*, v. 1, Bloomington, Ind., 1933, p. 82–83.

177[J].—H. Stenzel, "Das Schallfeld eines Strahlers in einer Mediumschicht mit schallweicher und schallharter Begrenzung," *Ann. d. Phys.*, s. 5, v. 43, June, 1943, p. 30, 18. 13.8 × 21.5 cm.

$$\text{Table of } \psi(x) = \frac{1}{2x} - \frac{1}{1^2 - x^2} + \frac{2}{2^2 - x^2} - \frac{3}{3^2 - x^2} + \cdots$$

for $x = [.005(.005).5; 4D]$.

On p. 18 $iH_0^{(1)}(x)$ is given, to $4D$, for $x = .6084$, .4156, .4712i, .9024i, 1.266i, 1.610i, 1.943i, 2.272i, 2.595i, 2.918i, 3.238i, 3.558i, 3.876i, 4.194i, 4.512i, 4.829i, 5.145i, 5.462i, 5.794i, 6.094i.

178[L].—P. K. Bose, "On confluent hypergeometric series," *Sankhyā. The Indian J. Statistics*, v. 6, Feb., 1944, p. 407–412. 22.5 × 29.2 cm.

For integral and half integral values of a and c the confluent hypergeometric series ${}_1F_1(a; c; x)$ occurs in F-Statistic under non-null hypothesis, in Studentized D^2-Statistic and in the expression for the multiple correlation coefficient for a particular type of parent population. Tables with 5S are given here for $a = 2(1)15$ and with 4S for $a = 16(1)25$. The values chosen for c are 1, 2, 3, 4 and the values of x are 2, 3, 3.8, 5.7, 7, 8.4, 9, 10.5, 13.5. A short table is given also to show the agreement between the value calculated from the series and the value calculated by recurrence formulae starting from the known values for $a = 2, 3$; $c = 1, 2$.

These new tables include some values of x not considered in previous tables. In those of Webb and Airey[1] for $c = 1(1)7$, $a = -3(.5)4$ the range with 4S was $x = 1(2)6(1)10$. In Airey's first table[2] for $2c = -3(2)3$, $2a = -8(1)8$ the range with 6S was $x = 0(.1)1(.2)3(.5)8$. In the tables of Gran Olsson[3] c has the values 1, 2, 3 but a is not an integer, 4D being given. In the tables of Chappell[4] $c = 1$, $a = \tfrac{1}{2} - k$, $k = 1(1)10$, $kx = .1(.1)1.5, 2(1)10$ with 4D.

The functions for which c is an integer occur in many physical problems some of which are mentioned by Webb and Airey.[1] The Bessel function $I_n(z)$ in which n is an integer or half an odd integer is related to these functions, so also are the polynomials of Laguerre and Sonin. The case $c = 3$ occurs in the work of SEN[5] on stresses in some rotating circular disks of varying thickness while the function ${}_1F_1(a;\, n + 2;\, -2ky)$ occurs in the work of LAMB[6] on atmospheric oscillations, and some calculations were impracticable on account of the lack of tables. The polynomials of Hermite correspond to the case in which $2c$ is 1 or 3. Some tables of these functions are already available. The author indicates that further numerical results are forthcoming.

Some numerical results relating to the roots of Laguerre polynomials $L_n(x) = F(-n;\, 1;\, x)$ suitable for numerical integration over the range $(0, \infty)$ have been given by KOSHLIAKOV.[7] Tables of the function

$$H(m, a, x) = \exp(-ix)F(m + 1 - ia;\, 2m + 2;\, 2ix),$$

with 6D, have been published by LOWAN & HORENSTEIN[8] for $m = 0(1)3$, $a = 0(1)10$, and $x = 0(1)10$. The 2 in $2ix$ has been omitted on the cover of the reprints.

H. B.

[1] H. A. WEBB and J. R. AIREY, "The practical importance of the confluent hypergeometric function," *Phil. Mag.*, s. 6, v. 36, 1918, p. 129–141.

[2] J. R. AIREY, "The confluent hypergeometric function," B.A.A.S., *Report*, 1926, p. 276–294; 1927, p. 220–244.

[3] R. GRAN OLSSON, (a) "Biegung kreisförmiger Platten von radial veränderlicher Dicke," (b) "Tabellen der konfluenten hypergeometrischen Funktion erster und zweiter Art," *Ingenieur-Archiv*, v. 8, 1937, p. 81–98, 99–103.

[4] G. E. CHAPPELL, "The properties of a new orthogonal function associated with the confluent hypergeometric function," Edinburgh Math. So., *Proc.*, v. 43, 1925, p. 117–130.

[5] B. SEN, "Note on the stresses in some rotating circular disks of varying thickness," *Phil. Mag.*, s. 7, v. 19, 1935, p. 1121–1125.

[6] H. LAMB, "On atmospheric oscillations," R. So. London, *Proc.*, v. 84A, 1910, p. 551–572.

[7] N. S. KOSHLIAKOV, "O vychislenii po formule mekhanicheskikh kvadratur opredelennykh integralov s beskonechnymi predelami" [On the calculation of integrals to infinite limits by means of formulae of mechanical quadratures], Akad. Nauk, Leningrad, *Izvestiîa*, s. 7, *Fiziko-matematicheskoe otdelenie*, v. 7, 1933, p. 801–808.

[8] A. N. LOWAN & W. HORENSTEIN, "On the function $H(m, a, x) = \exp(-ix)F(m+1-ia;\, 2m+2;\, 2ix)$," *J. Math. Phys.*, M.I.T., v. 21, 1942, p. 273–283.

179[L].—BAASMTC, *Mathematical Tables, Volume VI. Bessel Functions, Part I, Functions of Orders Zero and Unity.* Cambridge, University Press, 1937, xx, 288 p. 21.5 × 28 cm. Compare *MTAC*, p. 282f. £2.

The circular cylinder is the most symmetrical body after the sphere. This is perhaps the reason why Bessel functions which arise in the solution of boundary problems for domains with circular symmetry are the most widely used transcendental functions after the circular functions which enter in the solution of boundary problems for domains with spherical symmetry. From the standpoint of the frequency of occurrence, the Bessel functions of orders zero and unity are undoubtedly the most important. Moreover, the recurrence formula between Bessel functions of three consecutive integral orders make it a relatively simple matter to generate the values of Bessel functions of integral orders from those of orders zero and unity. These considerations serve to stress the great contribution to science of the tabular volume before us, devoted to Bessel functions of orders zero and unity. This volume may truly be said to have disposed of the tabulation of the Bessel functions of orders 0 and 1 for all time.

A description of the volume will serve to prove the truth of this statement. The Bessel functions $J_0(x)$ and $J_1(x)$ are solutions of the differential equation $x^2y''+xy'+(x^2-n^2)y=0$ for $n = 0$ and $n = 1$ respectively, which are finite at $x = 0$. The bulk of the volume under review (170 out of 288 pages) is devoted to the tabulation of these functions. Ten-place values of $J_0(x)$ and $J_1(x)$ and second central differences are tabulated at intervals of .001 in the range from 0 to 16 and at intervals of .01 in the range from 16 to 25. Beyond $x = 25$, the auxiliary functions $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ appearing in the asymptotic expansions $J_0(x) = A_0(x) \sin x + B_0(x) \cos x$; $J_1(x) = B_1(x) \sin x - A_1(x) \cos x$ are tabulated in the range from 25 to 1150 at intervals (ranging from .1 to 10) so chosen that second central differences are adequate for the maximum attainable accuracy. In the range from 1000 to 6000 the auxiliary functions are tabulated at intervals of 100 together with "modified" second central differences. Beyond $x = 6000$, one term each in the expansions of the auxiliary functions suffices to generate values to eight-place accuracy. Thus to all intents and purposes the volume under review disposes of the tabulation of $J_0(x)$ and $J_1(x)$ over the entire range of the argument from 0 to infinity. The situation is quite similar with the other Bessel functions tabulated in the volume with some exceptions to be discussed.

The functions of the second kind $Y_0(x)$ and $Y_1(x)$ have a logarithmic singularity at the origin and a table of these functions would not be interpolable near the origin unless the interval is very small. For this reason the authors have tabulated 8-place values of the functions $Y_0(x) - \frac{2}{\pi} J_0(x) \ln x = C_0(x)$ and $x\left\{ Y_1(x) - \frac{2}{\pi} J_1(x) \ln x \right\} = C_1(x)$ in the range from 0 to .5 at intervals of .01. The functions $D_0(x) = \frac{2}{\pi M} J_0(x)$ and $D_1(x) = \frac{2}{\pi M} J_1(x)$, where $M = \log e$, are also tabulated alongside the values of $C_0(x)$ and $C_1(x)$. With the aid of the four auxiliary tables just described, one may compute $Y_0(x) = C_0(x) + D_0(x) \log x$ and $Y_1(x) = C_1(x)/x + D_1(x) \log x$. As is the practice throughout the entire volume, second central differences are tabulated with the values of the auxiliary functions in question.

In addition to these functions the values of $Y_0(x)$ and $Y_1(x)$ are also given in the range from 0 to .5 with or without modified second central differences. Beyond $x = .5$ and through $x = 25$, eight-place values are given (with ordinary or modified second central differences) for arguments at intervals of .01. Beyond $x = 25$, the auxiliary functions $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ above mentioned may be used for the computation of $Y_0(x)$ and $Y_1(x)$ by means of the relations $Y_0(x) = B_0(x) \sin x - A_0(x) \cos x$ and $Y_1(x) = - A_1(x) \sin x - B_1(x) \cos x$.

The Bessel functions $I_0(x)$ and $I_1(x)$ are solutions of the differential equation $x^2y'' + xy' - (x^2 + n^2)y = 0$, for $n = 0$ and $n = 1$ respectively, which remain finite at $x = 0$. Eight-place values of these functions and second central differences are given in the range from 0 to 5 at intervals of .001. Beyond $x = 5$, the related functions $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ are tabulated at intervals of .01 from $x = 5$ to $x = 10$ and at intervals of .1 from $x = 10$ to $x = 20$.

The functions $K_0(x)$ and $K_1(x)$, the second fundamental solutions associated with $I_0(x)$ and $I_1(x)$, have a logarithmic singularity at the origin and therefore for the sake of interpolability the functions $E_0(x) = K_0(x) + I_0(x) \ln x$ and $E_1(x) = x\{K_1(x) - I_1(x) \ln x\}$ are tabulated together with the products $F_0(x) = -\frac{1}{M} I_0(x)$ and $F_1(x) = \frac{1}{M} I_1(x)$ for x ranging from 0 to .5 at intervals of .01. In addition $K_0(x)$ and $K_1(x)$ are given for x ranging from 0 to 5 at intervals of .01. Beyond $x = 5$, the values of $e^xK_0(x)$ and $e^xK_1(x)$ are given alongside the values of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ in the same range and for the same interval as the last-named function.

To round out the description of the volume under review, let it be mentioned that it contains the first 150 zeros of $J_0(x)$ and $J_1(x)$ and the corresponding values of each of the functions at the roots of the other; the first 50 zeros of $Y_0(x)$ and $Y_1(x)$ together with the corresponding values of $Y_1(x)$ and $Y_0(x)$ at these zeros, a one-page table of e^x and e^{-x}, a one-page table of the Everett coefficients of the second difference, and a $2\frac{1}{2}$ page table of the Besselian coefficients of the double second difference.

In addition to a general introduction devoted to a description of the tables and of their preparation, each of the first eight major tables is preceded by a page giving the definition of the functions, recurrence relations, and other pertinent information. The typography of the volume is excellent and as far as accuracy is concerned the reviewer has the feeling after reading the Introduction that the authors were fully justified in their remark that "There is every reason to believe that the tables are completely free from error."

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180[L].—FEDERAL TELEPHONE AND RADIO CORP., *Reference Data for Radio Engineers*, New York, 1943 (fourth printing Aug. 1944), p. 197–200. 13.8 × 21.6 cm.

Here are 6 tables of Bessel functions: T. 1–2, $J_0(x)$ and $J_1(x)$, $x = [0(.1)15.9; 4D]$; T. 3–5, $J_n(x)$, $n = 2, 3, 4$, $x = [0(.1)4.9; 4D]$; T. 6, $J_n(p)$, $p = 1(1)14$, $n = [0(.5)10; 4S]$.

181[L].—NYMTP, "Table of $f_n(x) = \dfrac{n!}{(x/2)^n} J_n(x)$", *J. Math. Phys.* M.I.T., v. 23, 1944, p. 45–60. 17.5 × 25.3 cm.

Under BAASMTC 2 (*MTAC*, p. 283) reference was made to elaborate ms. tables of $J_n(x)$ with the following ranges:

$$n = 0(1)20, \quad x = [0(.1)25; 10D]; \quad n = 0(1)12, \quad x = [0(.01)10; 8D].$$

Though these tables were made available to the authors of this table of $f_n(x)$, they have preferred to recompute and publish a new table, since although $J_n(x)$ may be given to 8 or 10 decimals, over a considerable range there are but one or two significant figures, whereas the function here adopted has all nine decimals significant over almost the whole range, particularly for n large, where the defect of the J_n table is most noticeable.

The first tables of these functions were calculated by A. WALTHER, S. GRADSTEIN & K. HESSENBERG and published in JAHNKE & EMDE, second ed., 1933, p. 250–258, with the notation

$$\Lambda_n(x)\ [\text{J \& E}] \equiv f_n(x)[\text{NYMTP}], \quad \text{for} \quad n = 0(1)8, \quad x = [0(.02)9.98; 4\text{–}5D].$$

There appears no advantage in changing from the symbol Λ introduced by Emde, to the symbol f, which may even cause confusion as it is commonly needed for a general functional symbol. The symbol Λ will therefore be used in this review.

This table is not only of great value for the actual numerical values thereby made available, but also is of theoretical interest. The functions selected for tabulation and the method of starting with the largest values of n and working down to small n, are clearly the means by which nature intended that Bessel Functions should be computed.

The $\Lambda_n(x)$ functions have also a valuable property perhaps not envisaged by the authors, opening up possibilities of double-entry tables interpolable in n as well as in x. If curves be plotted of $\Lambda_n(x)$ against n for a fixed parameter $x[1(1)10]$, it is seen that for $n > x$ or thereabouts the curves (monotonic, asymptotic to 1 for $n = \infty$) suggest this possibility. To test this, values of $\Lambda_n(5)$, $n = 5(1)20$, extracted from the tables and differenced, were interpolated for $n = 12.5, 13.5, 14.5$ using 8th difference Everett (the 8th differences only affect a unit in the 9th (last) decimal). Compare my inefficient efforts to interpolate $J_n(1)$ between $n = 1$ and 2 to 7D, requiring 6 differences, although fundamental values had been laboriously computed at interval .1 [*MTAC*, p. 99, where the interval erroneously given as .01 was later corrected, p. 132]. The three values at $n = 12.5, 13.5, 14.5$ satisfied the recurrence relation $\Lambda_{n-1}(x) = \Lambda_n(x) - [x^2\Lambda_{n+1}(x)]/4n(n+1)$; and repeated applications thereof yielded $\Lambda_{2.5}(5) = .08083\ 8726$.

Now $\dfrac{25}{3}\Lambda_{2.5}(5) = \left(\dfrac{5\pi}{2}\right)^{\frac{1}{2}} J_{2.5}(5) = \left(\dfrac{3}{5^2} - 1\right)\sin 5 - \dfrac{3}{5}\cos 5$, on making the obvious substitution of $\Gamma(n+1)$ for $n!$ when n is not integral. Taking sin 5 and cos 5 from BAASMTC, *Mathem. Tables*, v. 1, and computing the true value of $\Lambda_{2.5}$, the result is .08083 87261, with which the value obtained by interpolation agrees so far as it goes. Hence it appears that the table can be interpolated for fractional n to nearly the full accuracy of the original table, provided we start from $n > 2x$ or thereabouts.

So it seems that it would be possible, without excessive labour, to derive a double-entry table for $n = 0(.1)20$, $x = [0(.1)10; 7D]$, that would be interpolable in both n and x without going further than second differences at most. Since $\Lambda_n(x)$ is an even function of x, it would be unnecessary to tabulate for x negative (compare $J_n(x)$, for fractional real n and negative real x, complex). But difficulties arise for Λ_n when n is negative, since $\Lambda_n = \infty$ for n a negative integer. Even conversion to J_n is of little practical use here [$J_n(.1)$ fluctuates from zero to -38 and back, between $n = -1$ and -2]. The only practical solution would appear to be interpolation (if required) in the region n positive, followed by the use of recurrence formulae.

In equation (3), p. 46, the n in the denominator of the last fraction should be deleted so as to read $x^2/4(n+1)(n+2)$.

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182[L].—G. N. Watson, *A Treatise on the Theory of Bessel Functions.* Second edition, Cambridge: at the University Press, New York: The Macmillan Co., 1944, viii, 804 p. 18.7 × 23.2 cm. $15.00.

This offset print of the first edition of 1922 seems to have been in the press for more than three years since Watson's preface to the second edition is dated "March 31, 1941." It is as follows (apart from acknowledgments of assistance):

"To incorporate in this work the discoveries of the last twenty years would necessitate the rewriting of at least chapters XII–XIX; my interest in Bessel functions, however, has waned since 1922, and I am consequently not prepared to undertake such a task to the detriment of my other activities. In the preparation of this new edition I have therefore limited myself to the correction of minor errors and misprints and to the emendation of a few assertions (such as those about the unproven character of Bourget's hypothesis) which, though they may have been true in 1922, would have been definitely false had they been made in 1941."

Hence the numbers of pages in the first and second editions are the same. In the Bibliography, p. 753–788, the only change seems to be an addition of a title on p. 788. We note that equations (3), (4), p. 81 have been corrected. In *MTAC*, p. 307, we have already listed the tables in this work, p. 666–752. The errors which we there noted in T. I–II have now vanished but many others still remain; see *MTAC*, p. 296, and *MTE* **58**, **60**, where a beginning has been made in listing such errors. This reprint has filled a great need.

R. C. A.

183[M].—H. W. Lindemann, "Innenbackenbremsen," *Automobiltechnische Z.*, v. 46, Aug. 1943, p. 367–372; also English transl., "Internal shoe brakes," *Engineers' Digest* (Amer. ed.), v. 1, Sept. 1944, p. 558–563. 19.7 × 27.2 cm. English ed., v. 9, Aug., 1944, p. 230–235.

On p. 562 is a three-place table of eight integrals, from 0 to θ, of (a) $\sin t \cos t$; (b) $\sin^2 t$; (c) $\cos^2 t$; (d) $\cos t$; (e) $\sin t$; (f) $\sin^2 t \cos t$; (g) $\sin t \cos^2 t$; (h) $\sin^3 t$, for $\theta = 0(2^\circ)180^\circ$.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **169** (Barlow, Hayashi, Weigel, L. Zimmermann), **170** (Lehmer), **172** (N.D.R. Comm.), **176** (Davis), **181** (NYMTP); N**26** (Ageton).

55. J. M. Bates, "Zeros of a class of polynomials associated with Bateman's k-function," Iowa State College *J. Sci.*, v. 12, 1938, p. 474.

The first 5 zeros of $J_{1/3}[(2/3)x^{3/2}] + J_{-1/3}[(2/3)x^{3/2}]$ are here given as follows:

2.338107, 4.137258, 5.520555, 6.786701, 7.944136.