

the associated decade counters to clear to 0. Besides the preceding, each stepper counter has a direct input which allows the stepper counter to be stepped independently of the decade counters and a clear input which allows the stepper counter to be cleared back to its first stage from any stage. It might be remarked that digit pulses, as well as program pulses, may be brought to the stepper direct input.

We will now return to the problem discussed above to illustrate how the master programmer may be used to link program sequences. Let us suppose that it is desired to repeat the computations described 7 times and then to print. Suppose, furthermore, that computations are to cease after 200 printings. A complete plan for this computation is given in Fig. 4.

In the ENIAC, programs may be linked together either serially, as described above, or the choice of routine may be made to depend on the magnitude of a number. The criterion may be the magnitude of the digit in a specific place of a number or it may be the sign of a number. Either form of magnitude discrimination can be accomplished by sending pulses from some lead of a digit output terminal to a stepper direct input.

The ENIAC has, of course, been designed to handle far more complicated and tedious computations than the one discussed above which required the use of less than 4 percent of the machine's programming capacity. This illustrative problem is, however, useful for indicating how the machine can be instructed in its routine for a particular computation.

H. H. GOLDSTINE & ADELE GOLDSTINE

Institute for Advanced Studies

Scientific Computing in Great Britain

1. Introduction: Commercial Machines in Scientific Computing. The commercial calculating machine has been used in scientific work, after a fashion, for many years. Twenty years ago one or two hand-operated machines were to be found in the mathematics departments of British universities, used by graduate students. But a great stimulus to their better and wider use was given by L. J. COMRIE in lectures given at the London School of Economics in 1926¹ and 1927,² in which he stressed the importance of the intelligent and resourceful use of commercial machines.

About the same time Comrie described the "end-figure" method³ of constructing mathematical tables by subtabulation, and a means of using this method on Hollerith punched-card equipment. Four years later⁴ came his description of the Burroughs Class 11 machine and its use for integration from second differences. About the same time he described his application of the Hollerith tabulator to E. W. BROWN, *Tables of the Moon*, which still stands as a remarkable achievement.

His next great advance⁵ was the use of the six-register National Accounting machine as a "difference-engine" to handle problems in finite differences. As a result there are now fifteen National machines in use in research establishments in Great Britain, several times as many as there are in similar use in the United States, where the machine originated!

The story of the adaptation of commercial machines to uses in the scientific field in Great Britain is almost wholly the story of Comrie's life

and work. The debt owed to him for this, even without considering the great work he has done in table-making and in discovering errors in published tables, is hard to estimate and impossible to repay.

2. Finite Differences and Computing Methods. In this, as in other sections, rather than attempt a comprehensive survey, two or three examples will be chosen, as being of unusual importance, or having some topical interest.

From Comrie's work we choose a device, known as the throw-back, for simplifying ordinary interpolation formulae involving differences.³ Although Comrie invented this device independently, he later discovered that a suggestion for its use had, in fact, been made some years earlier by that great astronomer E. W. Brown. Its advantages, as given by Comrie, are:

"It enables whole columns of differences to be dispensed with in published tables, and saves the user the trouble of looking out (in general) two coefficients and performing two multiplications."

The method can be applied either to Bessel's or to Everett's formula. Bessel's formula can be written in the following form, if, instead of n , the fraction of the tabular interval, we write $y = n - \frac{1}{2}$ and separate the terms involving odd and even differences:

$$u_n = u_0 + n\Delta_1' + \frac{y^2 - \frac{1}{4}}{2.2!} (\Delta_0'' + \Delta_1'') + \frac{(y^2 - \frac{1}{4})(y^2 - \frac{9}{4})}{2.4!} (\Delta_0^{iv} + \Delta_1^{iv}) \\ + \frac{y(y^2 - \frac{1}{4})}{3!} \Delta_1''' + \frac{y(y^2 - \frac{1}{4})(y^2 - \frac{9}{4})}{5!} \Delta_1^v + \dots$$

If we denote the coefficients of the differences of second and higher orders by B'' , B''' , \dots , etc., then

$$\frac{B^{(2n+2)}}{B^{(2n)}} = \kappa_{2n} = -\frac{\frac{1}{4}(2n+1)^2 - y^2}{(2n+1)(2n+2)}, \\ \frac{B^{(2n+3)}}{B^{(2n+1)}} = \kappa_{2n+1} = -\frac{\frac{1}{4}(2n+1)^2 - y^2}{(2n+2)(2n+3)}.$$

Since the limits of variation of y are $\pm\frac{1}{2}$, these ratios vary only a little; they vary proportionately less as n increases. If they were strictly constant, fourth differences could be accounted for exactly by writing

$$B''(\Delta_0'' + \Delta_1'') + B^{iv}(\Delta_0^{iv} + \Delta_1^{iv})$$

as

$$B''[(\Delta_0'' + \Delta_1'') + \kappa_2(\Delta_0^{iv} + \Delta_1^{iv})] = B''(M_0'' + M_1'')$$

where

$$M'' = \Delta'' + \kappa_2\Delta^{iv}$$

and both fourth- and sixth-order differences could be accounted for by writing

$$M'' = \Delta'' + \kappa_2\Delta^{iv} + \kappa_2\kappa_4\Delta^{vi}.$$

If suitable mean values of κ_2 and κ_4 are chosen, the true modified differences are represented sufficiently well for the residual errors to be neglected. A full table, giving the upper limits of differences with which these modifications may be safely used, is given in the original paper.

The principal drawbacks of the method, as stated by Comrie, are that additional computation is necessary to produce the modified differences and that more responsibility is thrown on the proof-reader, as the modified differences cannot be so readily checked from the other columns of the table. Modified differences have been used extensively in the well-known B.A.A.S., *Mathematical Tables*. In 1932 AITKEN published a method of interpolation⁶ which is most attractive in its simplicity. It is not a formula, but a set of rules to be followed, and, furthermore, the process is exactly the same for both direct and inverse interpolation. It consists in a succession of interpolations by proportional parts. If we have two values u_a and u_b and wish to find the value of u_x , we have

$$u_x = u_a + \frac{x - a}{b - a} (u_b - u_a)$$

$$= \frac{b - x}{b - a} u_a + \frac{x - a}{b - a} u_b$$

i.e. the weighted mean of u_a and u_b . This can be written as

$$u_x = \left| \begin{matrix} u_a & a - x \\ u_b & b - x \end{matrix} \right| \div (b - a);$$

i.e. a cross-product divided by the difference between the right-hand pair of factors. Aitken calls this the linear cross-mean. It is a very simple operation on an ordinary desk calculator.

We will denote this linear cross-mean by $u_x(a, b)$. Now if we have a series of values u_a, u_b, u_c, \dots etc. and take them in pairs we can form several cross-means, each of which will be an estimate of the true value of u_x . The calculation is set out as shown in the table below :

Argument	Function	Cross-Means	Parts
a	u_a		$a - x$
b	u_b	$u_x(a, b)$	$b - x$
c	u_c	$u_x(a, c)$	$c - x$
d	u_d	$u_x(a, d)$	$d - x$

We now treat the cross-means as though they were tabular values, and form cross-means from them, i.e. we calculate

$$u_x(a, b, c) = \left| \begin{matrix} u_x(a, b) & b - x \\ u_x(a, c) & c - x \end{matrix} \right| \div (c - b)$$

and $u_x(a, c, d)$ similarly.

Aitken has shown that this process is the same as using the general divided-difference formula, to the same order of differences as the number of times the process is performed. It can be continued indefinitely, until the linear cross-means have converged to the desired degree of approximation. A numerical example taken from Aitken's paper, is given below.

Adjusted Argument	u	Stage (1)	Stage (2)	Stage (3)	Stage (4)	Parts
-2	5772 1566					-2.68327
-1	5608 8546	53 339732.4				-1.68327
0	5447 8931	.. 371133.4	39 2588.8			-0.68327
1	5289 2109	.. 401987.0	.. 2128.1	227 4.0		0.31673
2	5132 7488	.. 432306.6	.. 1674.9	... 6.6	.. 3.2	1.31673
3	4978 4499	.. 462107.2	.. 1229.9	... 9.3	.. 3.2	2.31673

This method is worthy of serious attention now that the automatically controlled calculating machine is an accomplished fact. For these machines a method is needed which involves the minimum of tabular matter, and a simple set of instructions to the machine. Aitken's method fulfils both these requirements. It converges rapidly, and is self-checking. It is not a formula, but a set of instructions simple enough to be incorporated into a machine routine, requiring neither tables of coefficients nor tabulated differences. Since it does not depend upon the function being tabulated at equal intervals of the argument, the same machine routine can be used both for direct and inverse interpolation. The fact that it requires more multiplications than conventional difference formulae is no obstacle, in view of the speeds of machine operation now contemplated.

In describing the process, Aitken says, "The convergence of the interpolation is indicated to the eye of the computer by the convergence towards equality of consecutive cross-means in the same column at any stage; the process is stopped when such entries agree to an assigned number of digits." It is evident that the eye of the machine could observe this indication equally well, and that such a self-checking method is particularly well suited to the machines of the future. Aitken also says, ". . . in the later stages, when cross-means in the same column converge to agreement in their more important digits, we need form cross-means of the later and differing digits only, since the digits that are common to both will be reproduced under the operation of weighted mean and therefore need be recorded in the first entry only. . . ." This also is within the power of the machines now contemplated.

No better example could be found of the importance of suiting computing methods to the equipment being used. Just as great advantages can be derived from skilful exploitation of commercial equipment by devising methods appropriate to the facilities they provide, so must the most efficient methods be found for the automatically controlled machines now being developed.

3. Differential Equations. Some of the most interesting developments in numerical methods in Great Britain have been in the solution of differential equations. This may seem strange, in view of the fact that the differential analyser is an American development, but is nevertheless true. Between 1912 and 1928 RICHARDSON published several papers on a general method, applicable to both ordinary and partial differential equations, which he christened "The deferred approach to the limit."⁷ The basic principle is both simple and powerful. If, in any ordinary or partial differential equation, the differential coefficients are replaced by finite differences of the same order, using steps of a certain size in the independent variable, then the solution obtained is not the solution of the problem described by the differential equation, but of a corresponding problem in finite differences. Let the solution we require be ϕ_0 , and the solution of the corresponding problem in finite differences be ϕ_h where h is the size of the interval. Then, if the differences are properly centered, Richardson proves for some problems, and suggests that for many others,

$$\phi_h = \phi_0 + \phi_2 h^2 + \phi_4 h^4 + \dots$$

a series in even powers of h , in which the ϕ_2 , ϕ_4 , etc., are independent of h .

It is practicable, therefore, instead of choosing such small intervals in h that ϕ_2 and all higher terms are negligible, to work with a comparatively coarse mesh, repeat the calculations for steps of a different size, and then extrapolate to steps of zero size; for instance, if we have two solutions for interval h , and $\frac{1}{2}h$ respectively, then

$$\begin{aligned}\phi_h &= \phi_0 + h^2\phi_2 + \dots, \\ \phi_{\frac{1}{2}h} &= \phi_0 + \frac{1}{4}h^2\phi_2 + \dots, \\ \phi_0 &= \phi_{\frac{1}{2}h} + \frac{1}{3}(\phi_{\frac{1}{2}h} - \phi_h).\end{aligned}$$

Thus when one solution has steps half the size of the other, we have the simple rule: add one third of the difference to the better approximation.

An interesting and simple application of the method is to calculate π by taking the perimeters of the inscribed square and hexagon in a circle, assuming that the error is inversely proportional to the square of the number of sides. This gives 3.18, which is astonishingly accurate when we consider the crudeness of the two original approximations, namely 2.83 and 3.00. A defect of the method is its proneness to oscillatory errors in open-boundary problems, unless care is taken. In many problems it can be extremely powerful, but one cannot always find limits for the error.

HARTREE and WOMERSLEY⁸ have combined this method with the use of the differential analyser for the solution of certain types of partial differential equations. If we take the ordinary diffusion equation as an example,

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$$

and replace the time derivative by a finite difference, we have

$$\frac{\theta_1 - \theta_0}{h} = \frac{1}{2} \left(\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_0}{\partial x^2} \right)$$

and thus, if θ_0 is known as a function of x for $t = t_0$, say, we have an ordinary differential equation for θ_1 at $t = t_1$. We thus divide up the domain over which the solution is required into a series of strips, and integrate the ordinary differential equations in succession, repeating the whole process for strips of a different size and then extrapolating to strips of zero width. This has been successful in the boundary layer problem, and in many diffusion problems. In general, it is best suited to problems of hyperbolic type, or open-boundary problems.

Another method of solving partial differential equations, applicable to closed boundary conditions, is SOUTHWELL'S relaxation method.⁹ Of it, TEMPLE said,¹⁰ ". . . one of the most powerful methods of computation in mathematical physics and engineering." This technique began as a method of finding the displacements of complicated elastic structures, and has been extended by Southwell and his pupils to cover a remarkably wide field of problems. The basic idea is very simple. Although it may be very difficult to calculate the displacements corresponding to given forces, it is easy to calculate the forces which maintain given displacements. These "constraints" on the elastic system are then "relaxed." The displacement of the system is altered so that the largest of the "unwanted" forces is

eliminated. Attention is then directed to the largest force remaining, and this is "relaxed" in turn. The process is continued until the remaining "unwanted" constraints lie within the limits of error of the original data. In his book *Relaxation Methods in Engineering Science*, Southwell illustrates by carefully-chosen examples how, by relaxing the constraints in groups, the "relaxer" can make use of his knowledge of the physical nature of the problem to reduce the time required to reach the final solution. It would be quite impossible to name the many types of problem to which these methods have been applied, but they include surveying errors, natural frequencies of oscillating systems, and two-dimensional solutions of Laplace's and Poisson's equations, including plastic torsion. Conformal transformations have been treated by it, biharmonic analysis including stress-distributions in solids of revolution, and the flow of a compressible fluid through a two-dimensional nozzle. Southwell has gone so far as to call this method "a new mathematics for engineering science." As it is a method of successive approximation, it is self-checking, and it would, perhaps, be possible to make it fully automatic but for the fact that to get the best results from it considerable judgment and experience are required.

In the most recent paper¹¹ of the long series describing applications of these methods, it is pointed out that no paper in the series gives an exact account of relaxation methods as now applied to problems in two independent variables, and a summary is given of current methods. The basic principle is still the same. An approximation in finite differences is substituted for the partial differential equation. In effect, a system of equations is obtained which links together values of the wanted function at nodal points of a regular lattice or net. These equations are then solved by the relaxation technique. In this paper, meshes of varying size are used systematically, and a definite technique established for using specially fine meshes in the parts of the region where they are particularly needed.

If this method is to be made fully automatic, it will be necessary to construct a machine which can make a choice of one of several possible calculations according to the results of work done previously. Not only this, but the machine would have to have a vastly greater storage capacity than any present equipment. It would need to have storage mechanism, readily available, for at least a thousand numbers. It seems doubtful whether, with present technique, such a machine would be practicable, in view of the large quantity of equipment that would have to be kept continuously in service.

4. Special Machines. Although the differential analyzer was developed in the United States, there is one development in England that is worth mentioning. This is the small, cheap, differential analyzer, surprisingly accurate in relation to its first cost, to be seen in two of our universities. The first of these was made by HARTREE and PORTER in Manchester, and aroused a great deal of interest. Its accuracy was in the region of 2 to 5 per cent and it was constructed almost entirely of Meccano—a constructional toy. Another similar but more carefully designed machine was built in Cambridge, and turned out some excellent work. Both these machines were intended as models on which to gain experience during the period in which the large machines for these universities were being constructed, but

they have continued to exist in their own right. H. W. S. MASSEY built a third machine at Queen's University, Belfast. This was brought by him to University College, London, just before World War II and was unfortunately destroyed by enemy action.

The calculation of serial correlation coefficients has become an important practical problem in recent years. In 1936 Foster¹² described a simple integrating mechanism for integrating the square of a function which he showed could be adapted for this purpose. It uses the wheel and disc integrator, now familiar on the differential analyzer. Imagine an ordinary planimeter wheel running on a 2-inch (paper covered) disc, and sliding on a rod on which is a pointer for setting on the curve. The curve is moved under the pointer in small finite steps (about $\frac{1}{10}$ -inch). At each step the pointer is set on the curve, and the depression of a key traverses the curve and rotates the disc by a small amount. The planimeter wheel therefore integrates the function and gives its mean value. To obtain the mean square, another disc and wheel are used, resting on the first wheel. These are so arranged that the displacement of the upper wheel from the centre of the upper disc is the same as the displacement of the lower wheel from the lower disc. The upper disc is driven by the lower wheel, and therefore is rotated an amount proportional to the displacement of the lower wheel. Thus the amount by which the upper planimeter wheel is rotated at any step is proportional both to its displacement and the amount of rotation of the upper disc, i.e. to the square of the displacement of the lower wheel. To calculate serial correlation coefficients, three of these integrators are used, linked together by an averaging motion. If the displacements of the two outer integrators are x and y respectively, then the displacement of the third is $\frac{1}{2}(x + y)$, and thus one run of the curve through the device gives simultaneously Σx^2 , Σy^2 and $\frac{1}{2}\Sigma(x + y)^2$ from which all the quantities for calculating the correlation coefficient are obtained.

Two other instruments make use of optical methods for constructing periodograms. The first of these, described by Foster¹³ in 1930, is an instrument for periodogram analysis. The observations to be analyzed are made into a long narrow "lantern slide," rather like a "variable area" sound track, somewhat extended and magnified. This is illuminated by diffused light, which then falls upon a harmonic grating, i.e. a grating in which the light transmitted is proportional to $(1 + a \cos 2\pi ux)$. (Such gratings are prepared by photographing a sine curve with a short-focus cylindrical lens in front of the camera lens.) The light then falls upon a ground-glass screen. The grating can be moved along an optical bench so that its position relative to the screen and the source can be varied. It can be shown that the illumination on the ground-glass screen will be in form of a series of light and dark fringes if there is a period in the original data which bears the same ratio to the period of the grating as the ratio of the distances of the source and the screen from the grating. The stronger the period, the more intense the bands on the screen. By this method, outstanding periodicities in textile yarns, due to machine faults, can be traced rapidly. If, instead of a harmonic grating, a grating could be used which gives a transmission proportional to the height of the original curve, suitably contracted (say in the ratio 2 : 3) along its length, then the grating, if placed at the corresponding distance, will display a series of light and dark bands, the illumi-

nation at any point being proportional to the corresponding serial correlation coefficient.

MARTINDALE¹⁴ has described a correlation periodograph which makes use of this principle. The grating is replaced by a replica of the curve on a reduced scale and a cylindrical lens. The amount of light is measured by replacing the screen by a narrow vertical slit, long enough to collect all the light passing through the highest ordinates of the curves, and collecting the light on to a photo-cell. The photoelectric current is amplified and measured on a galvanometer. The measurements give accurate relative values of the correlation coefficients; but the present arrangement of the instrument does not permit of calibration to give absolute values.

Descriptions of the three machines have been collected in a paper read before the Research Section of the Royal Statistical Society.¹⁵ At the same meeting CUNNINGHAM and HYND¹⁶ gave a description of a relay machine, devised by BARNES and WEIR, for calculating serial correlation coefficients. In this the observations (which must lie within the limits ± 63) are coded on five-hole teletype tape in binary notation, and multiplication is carried out in the binary scale. The results are displayed on indicating lamps. The speed is one multiplication per second, but even this is much slower than the speed that can be achieved on standard punched-card equipment with less limitation on the observations.

5. Conclusion. There is a great need, with the recent growth of the use of numerical methods, for a critical and exhaustive survey of computing methods, with a study of their field of usefulness in connection with present-day equipment. The examples given are only "small samples" from a vast mass of published material. Before undertaking the design of special machines performing single calculations, the methods and equipment already available should be carefully studied. It is also worth stressing that the great new machines now under development should be provided with "instructions" based on methods suited to their capabilities. Aitken's method of interpolation is a particularly good example of the kind of method that should be looked for, and in the future it will be worth while to think in terms of processes of this kind, rather than formulae in the conventional sense.

J. R. WOMERSLEY

¹ L. J. COMRIE, *Accountants' J.*, v. 45, 1927, p. 42.

² L. J. COMRIE, *Office Machinery Users' Assoc., Trans.*, 1927, p. 30.

³ L. J. COMRIE, *R.A.S., Mo. Notices*, v. 88, 1928, p. 506.

⁴ L. J. COMRIE, *R.A.S., Mo. Notices*, v. 92, 1932, p. 523.

⁵ L. J. COMRIE, *R. Statistical So., J.*, v. 3, 1936, suppl., p. 87.

⁶ A. C. AITKEN, *Edinburgh Math. So., Proc.*, s. 2, v. 3, 1932, p. 56.

⁷ L. F. RICHARDSON & J. A. GAUNT, *R. So. London, Trans.*, v. 226A, 1927, p. 299f.

⁸ D. R. HARTREE & J. R. WOMERSLEY, *R. So. London, Proc.*, v. 161A, 1937, p. 353.

⁹ R. V. SOUTHWELL, *Relaxation Methods in Engineering Science*, Oxford, University Press, 1940. This gives references to the earlier papers.

¹⁰ G. F. J. TEMPLE, *R. So. London, Proc.*, v. 169A, 1939, p. 476f.

¹¹ D. N. DE G. ALLEN, R. V. SOUTHWELL & GILLIAN VAISEY, *R. So. London, Proc.*, v. 183A, 1945, p. 258. This paper gives references to earlier papers.

¹² G. A. R. FOSTER, *Textile Institute, Manchester, J.*, v. 27, 1936, p. T37.

¹³ G. A. R. FOSTER, *Textile Institute, J.*, v. 22, 1930, p. T18.

¹⁴ G. A. R. FOSTER, *Textile Institute, J.*, v. 58, 1946.

¹⁵ G. A. R. FOSTER, *R. Statistical So., J.*, v. 13, 1946, suppl. (in the press).

¹⁶ L. B. C. CUNNINGHAM & W. R. B. HYND, *idem*, 1946.