4. $Ch(u) = \frac{1}{2}\int_0^x I_{-\frac{1}{2}}(t)dt = \int_0^u \cosh(\frac{1}{2}\pi t^2)dt,$

$Sh(u) = \frac{1}{2}\int_0^x I_{\frac{1}{2}}(t)dt = \int_0^u \sinh(\frac{1}{2}\pi t^2)dt,$

$x = [0(.001).02(.01)2; 8D]$, $x = \frac{1}{2}\pi u^2$. These are the modified Fresnel integrals.

5. $K(x) = D_x^{\frac{1}{2}} \sin x = (1/\pi^{\frac{1}{2}}) \int_0^x \cos(x - t)dt/t^{\frac{1}{2}},$

$L(x) = D_x^{\frac{1}{2}} \sinh x = (1/\pi^{\frac{1}{2}}) \int_0^x \cosh(x - t)dt/t^{\frac{1}{2}},$

$M(x) = D_x^{\frac{1}{2}}(1 - \cos x) = (1/\pi^{\frac{1}{2}}) \int_0^x \sin(x - t)dt/t^{\frac{1}{2}},$

$N(x) = D_x^{\frac{1}{2}}(\cosh x - 1) = (1/\pi^{\frac{1}{2}}) \int_0^x \sinh(x - t)dt/t^{\frac{1}{2}},$

$x = [0(.02)1(.1)5; 8D]$. These are the one-half derivatives of Riemann and can be expressed in terms of Fresnel integrals and modified Fresnel integrals.

6. $H(x^{\frac{1}{2}}) = (1/\pi^{\frac{1}{2}}) \int_0^x e^{-t}dt/t^{\frac{1}{2}} = (2/\pi^{\frac{1}{2}}) \int_0^{\sqrt{x}} e^{-t^2}dt$

$= (2^{\frac{1}{2}}/\pi) \int_0^x K_{\frac{1}{2}}(t)dt = 2^{\frac{1}{2}}[Ch(u) - Sh(u)],$

$x = \frac{1}{2}\pi u^2 = [0(.001).02(.01)2; 8D]$. This is the error function for the argument $\sqrt{x}$.

7. $D(x^{\frac{1}{2}}) = (1/\pi^{\frac{1}{2}}) \int_0^x e^{t}dt/t^{\frac{1}{2}} = (2/\pi^{\frac{1}{2}}) \int_0^{\sqrt{x}} e^{t^2}dt$

$= 2^{\frac{1}{2}}[Ch(u) + Sh(u)].$

$x = \frac{1}{2}\pi u^2 = [0(.001).02(.01)2; 8D]$. This is closely related to Dawson's integral for the argument $\sqrt{x}$.

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Advanced Development Section

MECHANICAL AIDS TO COMPUTATION

23[Z].—D. R. HARTREE, "The ENIAC, an electronic calculating machine," *Nature*, v. 157, 20 April 1946, p. 527.

Concluding sentence: Its flexibility and speed of operation will make it possible to carry out many numerical calculations, in many fields of investigation, which without its assistance would have been regarded as much too long and laborious to undertake.

24[Z].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. 1: *A Manual of Operation for the Automatic Sequence Controlled Calculator.* Cambridge, Mass., Harvard University Press, 1946, xiv, 561 p. + 9 plates. 20 × 26.7 cm. $10.00.

This volume gives the first really scientific account of the Automatic Sequence Controlled Calculator, the first of the large all-purpose digital calculators developed during the war, and was prepared by a staff of 24, Comdr. H. H. AIKEN in charge, largely through the efforts of Lt. GRACE M. HOPPER. The volume consists of six chapters, a bibliography of numerical analysis, and a sizable appendix in eight parts. There are 16 full-page photographs showing parts of the machine.

Chapter 1, entitled Historical Introduction, gives a short account of early mechanical computers of the "difference engine" type, especially those associated with the names of

PASCAL (1642), LEIBNITZ (1694), C. BABBAGE (1812–33), SCHEUTZ (1834), WIBERG (1863), and GRANT (1871).

Chapter 2 gives a description of the present calculator. In its outward appearance it consists of a 51 foot panel, 8 feet high, with two 6 foot panels extending at right angles from the back. The mechanical parts are driven in synchronism by a 4 H.P. motor. The machine weighs about five tons. The essential parts of the machine consist of mechanical counters and relays for the most part. They may be described very briefly as follows:

1. A register of 60 signed constants, of 23 digits each, manually set on switches. This provides the machine with data, such as coefficients of power series and small empirical tables, which are known in advance of the calculation being performed.

2. A set of 72 electro-mechanical adding-storage registers automatically operated by the machine. These 23-figure counters are for storing and combining those numbers which are produced by the machine in the course of its operation. The high capacity of this element of the machine is one of its best features.

3. A central multiplier and divider. These processes are performed by first building up a small table of the first nine multiples of the multiplicand or divisor and then dealing with this table appropriately.

4. Electro-mechanical tables of $\log x$, 10^x and $\sin x$. Logarithms are computed by the familiar method using four factors of the form $1 + h \cdot 10^{-k}$ ($k = 0(1)3$, $h = 0(1)9$) and the power series for $\log(1 + x)$. A reversal of this process gives 10^x. Power series are used to compute either $\sin x$ or $\cos x$ when $x < \pi/4$, and all other values of the trigonometric functions may be easily derived from these.

5. Three tape-driven interpolators. Besides instructing the machine in the routine of some interpolation process (which may be as high as eleventh order) one of these units may be used to introduce numerical values into the machine.

6. A tape-driven sequence unit for controlling the whole machine. This is a 24-hole tape punched, in advance of the calculation, in three columns of eight holes each. Tapes as long as 5000 steps or rows are sometimes used. This unit is the salient feature of the machine and accounts for its name.

These are the internal parts of the calculator. The machine makes contact with the outside world through the following practically standard IBM equipment: two punch card readers, a card punch and two automatic typewriters. A manually operated tape punch for the preparation of tape for the four-tape units, four large plugboards and a number of control buttons make up the rest of the equipment.

The calculator normally handles 23 significant figures but can be set to deal with 12 or 46. Numbers are transferred from one part of the machine to another in the form of timed electrical pulses of 50 volts over a single buss. The simultaneous transfer of two or more numbers is thus impossible. However some parallel programming is possible during multiplication or division.

The fundamental unit of time in the machine is called the cycle. This is the time required to add two numbers or to move the control tape one step ahead and is three tenths of a second (200 per minute). Multiplication requires 20 cycles while the calculation of $\log x$, 10^x, or $\sin x$ requires about a minute. No data are given on the time required to read or punch a card or to type one line. These operations, whose durations depend somewhat on the amount of data handled, are relatively slow since the cards are fed lengthwise.

The calculator is capable of a certain amount of discrimination. Register, or counter, No. 70, called the choice counter, can be used to change the sign of a quantity if and only if the number in the choice counter is negative. Counter No. 72 may be used to stop the machine when the number in it is less, in absolute value, than a preassigned tolerance. This is used in making automatic checks. Because the machine possesses but a single control tape, discrimination cannot be used to alter the routine of a complex calculation.

Chapter 3 gives an account of the electrical circuits of the calculator as they apply to the separate components. The discussion here is in simplified form. For a complete and accurate account of the circuits and relays as they really exist, the reader is referred to the appendix.

Chapter 4, entitled Coding, is the most difficult one to understand, chiefly because of the apparent lack of order or structure in the system. No doubt there is a code book (perhaps shown in plate XV) in which further explanations are disclosed. As it is, coding appears to be an almost unsurmountable barrier between the machine and the mere mathematician with a problem to solve. To begin with, one is dismayed to find that the 72 storage counters have code numbers as well as serial numbers. Thus counter number 48 has a code number of 65, while counter 29 has a code of 5431. However, this is the same early impatience that one finds in beginning to learn a foreign language. The reader is referred to the book for details of the design and construction of tapes for the many operations of the machine. It would appear that a great deal of care, acumen, and time is necessary to be sure that a tape is efficiently and correctly punched. Once an operational tape has been prepared, however, it may be used and stored for future use when the same type of problem comes up again. This feature makes possible the use of the machine to compute extensive tables of a given function "on the side" while tapes for other shorter programs of higher priority are being prepared. Tables computed in this way are being published as further volumes of the *Annals of the Computation Laboratory of Harvard University* (see RMT **335**).

Chapter 5 gives plugging instructions for the 12 units of the machine requiring plugboards. These latter apparently are of the standard IBM type (although no mention is made as to whether they are demountable or not) and are used to shift the decimal point and to delete digits. Plugboards are used also to control the typewriters and the card readers and punch.

Chapter 6, entitled Solution of Examples, gives 13 short examples seven of which are completely worked out to the point where they could be put on the machine. Example 7, for instance, consists in evaluating the two linear forms (quadrature formulae)

$$\Delta I_n = K(-f_{n-1} + 8f_n + 5f_{n+1}), \quad \overline{\Delta I}_n = K(5f_n + 8f_{n+1} - f_{n+2})$$

where $K = .000833333333$, given the values $f_0, f_1, \cdots, f_{600}$ on punch cards. The quantities ΔI and $\overline{\Delta I}$ are to be compared and $I = \sum \Delta I_n$ accumulated and typed. To get the problem started requires 93 cycles of coding on a beginning tape. Then a main control tape of 103 cycles is set on the machine and run for 500 revolutions. The whole calculation requires at most 205 minutes.

Obviously this is an important chapter for those readers who might want to adapt their problem to this calculator. One's zeal in attempting to master these techniques is apt to be dampened a little by the final paragraph of the chapter to the effect that the procedures outlined in the examples are already a year old and have been improved by permanent changes in the wiring of the calculator. There is no doubt that actual contact with the machine itself is the very best way to learn of its details.

Perhaps the most useful part of the volume, for the general reader, is its Bibliography of Numerical Analysis (p. 338–404). It "is composed principally of those references that have been found useful during the one and one-half years of operation of the Automatic Sequence Controlled Calculator." The material is arranged in 23 topics (four with subtopics) ranging from Historical Background to Integral Equations. This bibliography of about 1200 titles should prove immensely useful to any worker in the field of computation.

D. H. L.

25[Z].—E. Laurila, "Ein Produktintegraph," Suomen Tiedeakatemia, Helsingfors, *Toimituksia*, *Annales*, s. A, *I. Mathematica-Physica*, no. 29, 1945, 12 p. 17.6 × 24.6 cm.

A linkage mechanism is described for evaluating the integral $\int_0^x f(x)\, g(x)\, dx$. While there is no particular novelty in the development, the simple design may be found attractive where a low-cost but relatively low-precision device is required.

S. H. C.

EDITORIAL NOTE: See V. BUSH, F. D. GAGE & H. R. STEWART, "A continuous integraph," Franklin Institute, *J.*, v. 203, 1927, p. 63–84. Among other things this instrument integrates the product of two functions, making use of the principle of electrical integrating watt hour-meter, combined with a moving table.

NOTES

59. ADMIRALTY COMPUTING SERVICE.—In *MTAC* we have recently had occasion more than once to refer to publications of this Service, for example, v. 2, p. 31, 35, 36, 39, 40, 80. "At the end of the War, the Admiralty Computing Service was providing a fairly comprehensive mathematical and computational service which was not only meeting all demands from Admiralty sources but was also able to offer informal assistance to the other Services, Government departments and contractors who had no comparable facilities at their disposal." This is a quotation from an article in *Nature*, v. 157, 4 May 1946, p. 571–573, entitled, "Mathematics in Government Service and industry. Some deductions from the war-time experience of the Admiralty Computing Service." Actual computations were carried out at the Nautical Almanac Office by a group under the direction of its Superintendent, D. H. SADLER, and "arrangements were made whereby scientific workers in the universities and elsewhere could be employed as consultants." Most of the members of the computing groups were recently transferred to the permanent National Mathematical Laboratory in the Mathematics Division of the National Physical Laboratory at Teddington. This Division is under the direction of Mr. J. R. WOMERSLEY. In view of our introductory article in the present issue of *MTAC* it is interesting to note in the *Nature* survey that "our experience fully confirms the statements made on many occasions by Dr. L. J. Comrie, . . . that full exploitation of the capabilities of the commercial calculating machines (including the National accounting machine and the Hollerith) is usually the most efficient way of dealing with problems, and specially designed calculating machines and instruments are necessary only for large-scale investigations of infrequent occurrence."

60. COEFFICIENTS IN AN ASYMPTOTIC EXPANSION FOR $\int_a^b e^{P(u)}du$.—The expansion for $\int_a^b e^{P(u)}du$ which is given below is well known in principle, since it is the result of merely continuing an integration by parts where at each successive step $e^{P(u)}du$ is replaced by $\frac{de^{P(u)}}{P'(u)}$. It has been employed extensively by several mathematicians of the NYMTP, especially for the purpose of extending the range of the usual asymptotic series by expressing the remainder $\int_a^b F(u)du$ as $\int_a^b e^{P(u)}du$. This method is often applicable to an integral of a real oscillatory function by considering it as either the real or imaginary part of a complex integral over the range a to b, provided that $P' \equiv P'(u)$ has no zeros in that range while $P^{(m)} \equiv P^{(m)}(u)$ are sufficiently small in comparison with P'.

The purpose of the expansion below is to enable one to obtain many terms of an asymptotic expansion, while avoiding the excessive labor of differentiating more and more complicated expressions that arise in repeat-