tory, the New York Mathematical Tables Project, the U. S. Naval Observatory, and others. Let us hear from each of those some report of their ". . . other aids to computation."

L. J. C.

[1] My youthful typist once typed "mater" cards; perhaps she thought they were better fitted for reproducing.

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"The use of calculating machines in ray tracing," Phys. So., *Proc.*, v. 52, 1940, p. 246–252.

The Twin Marchant Calculating Machine and its Application to Survey Problems, London, Scientific Computing Service, 1942, 40 p.

"Mechanical computing," Appendix I to DAVID CLARK, *Plane and Geodetic Surveying*, third rev. ed., v. 2, London, 1944, p. 462–473.

"Careers for girls," *Math. Gazette*, v. 28, 1944, p. 90–95.

"Recent progress in scientific computing," *J. Scient. Instruments*, v. 21, 1944, p. 129–135.

RECENT MATHEMATICAL TABLES

316[A–D].—FRANKLIN MARION TURRELL, *Tables of Surfaces and Volumes of Spheres and of Prolate and Oblate Spheroids, and Spheroidal Coefficients.* Berkeley and Los Angeles, University of California Press, 1946. xxxiv, 153 p. Offset print. 13.9 × 21.6 cm. $2.00.

The author of this work is an assistant plant physiologist at the Citrus Experiment Station of the University of California. He tells us that it is necessary to find the areas and volumes of such fruits as lemons, grapefruit, melons, nuts, etc. The lemon is regarded as approximating to a prolate spheroid, and a grapefruit to an oblate spheroid. Assuming that $2a$ and $2b$ are the axes of the ellipse of revolution and $\delta = 2a - 2b$, $\rho = 2b/2a$, e = eccentricity, there are the following tables:

Table 1 (p. 1–3): Surfaces and Volumes of Spheres, $\delta = 0$, d (diameter) $= 1(.1)15$, 3S up to $d = 5.6$, then integral values to the end.

Table 2 (p. 4–133): Surfaces and Volumes of Spheroids, for $2a = N(.1)15$, to 3S or to the nearest integer, last figure uncertain. $N = 1$, $\delta = .1(.1).5$; $N = 1.5$, $\delta = .6, .7$; $N = 2$, $\delta = .8(.1)1$; $N = 2.5$, $\delta = 1.1(.1)1.5$; $N = 3$, $\delta = 1.6(.1)2$; $N = 5$, $\delta = 2.1(.1)2.5$; $N = 7$, $\delta = 2.6$; $N = 7.5$, $\delta = 2.7(.1)3$. The smallest value for $2a$ is 1 and for $2b$ is .5. ρ varies from about .33 to 1. A centimeter is the linear unit used. These values were calculated from the formulae

$$\begin{aligned} &\text{Sphere, } S = \pi d^2,\ V = (1/6)\pi d^3; \\ &\text{Prolate spheroid, } S = 2\pi b^2 + 2\pi(ab/e)\sin^{-1} e, \\ &\qquad V = (4/3)\pi ab^2; \\ &\text{Oblate spheroid, } S = 2\pi a^2 + \pi(b^2/e)\ln[(1+e)/(1-e)], \\ &\qquad V = (4/3)\pi a^2 b. \end{aligned}$$

Table 1 was checked by Tables 21–22, p. 87–91 of *Chemical Engineers' Handbook*, ed. by J. H. PERRY, second ed., New York, 1941. In T. 21 S and V are given for

$$d = [\tfrac{1}{32}(\tfrac{1}{32})\tfrac{1}{2}(\tfrac{1}{16})4(\tfrac{1}{8})12(\tfrac{1}{4})34\tfrac{1}{2}(\tfrac{1}{2})100;\ 5S \text{ or } 6S].$$

In T. 22, V is given for $d = [1(.01)10;\ 4S]$. In F. G. GAUSS, *Fünfstellige vollständige Logarithmische u. trigonometrische Tafeln*, Berlin, 1870, p. 130, there is a table for S and V for r (radius) $= 0(1)100$ to the nearest integer. This same table is given in Hamburg, Sternwarte, *Sammlung von Hilfstafeln, A–F*, Hamburg, 1916, p. A31.

There are also the 20-place tables for the area and volume of a sphere by J. MODER, 1886; see *MTAC*, v. 2, p. 87.

In the case of **Table 2**, the 4-place tables with 3-place arguments of H. B. DWIGHT (*Mathematical Tables* . . . , third impression with additions, 1944, *MTAC*, v. 1, p. 180, p. 114–115) were used for evaluating $\sin^{-1} e$. In other elements the calculations were to 3D or 4D, and the final values of the formulae rounded off to 3S. The values in Tables 1–2 were further checked by considering the series of values in the ratios S_{n-1}/S_n, V_{n-1}/V_n, $n = 1(1)n$. "If the ratios of adjacent values of S or V changed considerably from one ratio to another, the solutions for the pair of arguments concerned were recalculated. If the error was not discovered, recalculation was made of S or V as required for other adjacent pairs of arguments."

Tests of the accuracy of Tables 1–2, when used for estimating surfaces and volumes of citrus fruit, were made by F. M. TURRELL, JANE P. CARLSON & L. J. KLOTZ (Amer. So. Horticultural Sci., *Proc.*, v. 46, 1945, p. 159–165). They found that the mean error between actual surface measurements and tabular values ranged from 2.23% to 5.59% for different kinds of fruit. In the case of volume measurements and tabular values the mean error ranged from 1.95% to 8.35% for different kinds of fruit. Interpolation for Tables 1–2 is discussed and illustrated by examples.

Tables 3–6 (p. 134–137), spheroidal coefficients for extending **Table 2**, were prepared by F. M. TURRELL and A. P. VANSELOW. Their characteristics, methods of calculation, methods of checking, and errors introduced by their use are discussed in Amer. So. Hort. Sci., *Proc.*, v. 47, 1946.

Many different computers are listed as having been connected with the preparation of the tables discussed above.

R. C. A.

317[A, D].—M. J. BUERGER & GILBERT E. KLEIN, "Correction of diffraction amplitudes for Lorentz and polarization factors," *J. Appl. Physics*, v. 17, Apr. 1946, p. 285–306. 20 × 26.7 cm.

The following 12 tables are given:

T. I–II, VI–VII. $y_1 = \frac{1}{2}(1 + \cos^2 2x)$, and $y_1^{\frac{1}{2}}$ for $\sin x = [0(.001).999;\ 4D]$.

I = II = y_1 are really identical tables; also VI = VII = $y_1^{\frac{1}{2}}$. It seems rather extraordinary that in a paper emanating from the Massachusetts Institute of Technology it should not have been recognized that tables for parameters $\sin x = 0(.001).999$ and $\sigma = 2 \sin x = 0(.002)1.998$ are identical.

T. III, VIII. $y_2 = \csc x$, and $y_2^{\frac{1}{2}}$ for $x = [0(0°.1)179°.9$; 4–5S].

T. IV–V, IX–X. $y_3 = (1 + \cos^2 2x)/[2 \sin 2x]$, and $y_3^{\frac{1}{2}}$ for $\sin x = 0(.001).999$; mostly 4D]. IV = V = y_3; IX = X = $y_3^{\frac{1}{2}}$.

T. XI. $(h^2 + k^2)^{\frac{1}{2}}$, $h = 0(1)30$, $k = 0(1)30$, to 4S.

T. XII. $(h^2 + k^2 + hk)^{\frac{1}{2}}$, $h = 0(1)30$, $k = 0(1)30$, to 4S.

Tables for $h^2 + k^2$ and for $h^2 + k^2 + hk$ were published previously by the authors in *J. Appl. Physics*, v. 16, p. 412; in this same article are also the two tables of $1/y_3$ (p. 414, 416), and also two of $1/y_1$ (p. 415, 417). Compare *MTAC*, v. 1, p. 436.

318[C, D].—CARL THEODOR ALBRECHT [1843–1915], *Logarithmic and Trigonometric Tables to Five Decimal Places*. New York, G. E. Stechert & Co., 1946 [the title-page gives "1944" incorrectly; the present edition is a reprint of the new 1944 edition]. vi, 147 p. 15.3 × 22.8 cm. Photolithographed by The Murray Printing Co., Cambridge, Mass. $1.50.

This volume contains the following: a preface by Stechert (p. iii–iv); **Table I,** Logarithms of Numbers (1000–10000, p. 1–37); conversion table for natural and common logarithms (p. 38); **Table II,** Logarithms of sines and tangents, 0(1″) 3° (p. 39–75); conversion table for degrees measure and radian measure (p. 76); **Table III,** Logarithms of [the six] trigonometric functions at interval 1′ (p. 77–122); **Table IV** Natural trigonometric functions of sin, tan, cot, cos, at interval 1′ (p. 123–146, reprinted from E. R. HEDRICK, *Logarithmic and Trigonometric Tables*); Constants (p. 147).

The publishers tell us that the Albrecht Tables in their original form had been used at the Drexel Institute of Technology, Philadelphia, for the past 25 years and that Institute's department of mathematics had suggested the form of the 1944 edition. Albrecht's work appeared first with the title *Logarithmisch-Trigonometrische Tafeln mit fünf Decimalstellen*, Berlin, 1884, xvi, 172 p. The nineteenth ed. was published at Stuttgart in 1930. It seems to have been an offset print of this latter edition which Stechert had made in 1932 by Henri Dupuy, Paris, and distributed in New York; xvi, 176 p. Apparently this edition differs very slightly from the original edition (which we have not seen). Pages 6–126 are the same as p. 1–122 of the volume under review. Then follow a table for addition and subtraction logarithms (p. 127–148), and other miscellaneous tables, formulae and constants (p. 149–176).

For many years, after 1873, Albrecht was a "Sectionschef" of the Prussian Geodetic Institute. In 1883 he edited the tenth edition of *Logarithmisch-Trigonometrische Tafeln mit sechs Decimalstellen* by CARL BREMIKER (1804–1877), of which there was an eleventh stereotyped ed., published in Berlin in 1890. The English edition (London, 1887), with new material by ALFRED LODGE, was therefore adapted from the Albrecht-Bremiker work.

R. C. A.

319[C, D, E].—GEORG VEGA (1756–1802). **1.** *Thesaurus Logarithmorum Completus, ex Arithmetica Logarithmica*, et ex *Trigonometria Artificiali Adriani Vlacci collectus, plurimis erroribus purgatus, in novum ordinem redactus*, . . . WOLFRAMII *denique tabula logarithmorum naturalium locupletatus a Georgio Vega* . . . , Leipzig, 1794 [also with a briefer German title page]. viii, XXX, 685 p. Size of the Brown University copy 20.5 × 33 cm. P. I–XXX, both Latin Introductio and German Einleitung in parallel columns.

6. Facsimile, *Georg Vega 10 Place Logarithms including Wolfram's Tables of Natural Logarithms. Reprint of the rare edition of* 1794. New York, G. E. Stechert & Co., 31 East Tenth St., New York City, 1946. viii, XXXI, 684 p. Photolithographed by The Murray Printing Co., Cambridge, Mass. 15 × 23 cm. $7.50.

GEORG *Baron* VON VEGA (1756–1802) was an Austrian Army Officer who taught mathematics and was the author of several mathematical works, including tables. His *Thesaurus* of 1794 contains the following tables: T. I (p. 1–310), log N, N = [1(1)100999; 10D], 1(1)1000 without Δ, but the rest with Δ. T. II (p. 311–629), log sines, cosines, tangents, and cotangents, for [0(1″)2°; 10D] and the rest of the quadrant to 88°, at interval 10″, with Δ. This is followed (p. 631–632) by the natural sines for [0(1″)12′; 12D]. The appendix (p. 633–685) includes (p. 634–635) the circular measure of 1°(1°)360°, 1′(1′)60′, 1″(1″)60″, all to 11D; and a small table expressing minutes and seconds as fractions of a degree. P. 641–684 contains a reprint of a remarkable table which took the Dutch artillery officer, J. WOLFRAM, six years to compute, hyperbolic logarithms, that is, T. III: ln N, for N = [1(1)2200 (primes and a few others) 10009; 48D]. These were first published in J. C. SCHULZE, *Recueil de Tables Logarithmiques, Trigonométriques et autres nécessaires dans les Mathématiques Pratiques*, also with German t. p., Berlin, v. 1, 1778, p. 189–259 and in *Astron. Jahrbuch für das Jahr* 1783, Berlin, 1780, part 2, p. 191. In this latter place Wolfram first gave the values for N = 9769, 9781, 9787, 9871, 9883, 9907. Since Vega's value for N = 9883 is in error by a unit in the forty-eighth decimal place,[7] it has been surmised that Vega calculated these six results independently. There are at least 31 common errors in Wolfram and Vega namely: for N = 390, 829, 1087 (2), 1099, 1409, 1900, 1937, 1938, 2022, 2064, 2093, 2173, 2174, 2175, 3481, 3571, 3763, 3967, 4033, 4321, 4757, 5123, 6343, 7247, 7853, 8023, 8837, 8963, 9409, 9623. These are discussed by DUARTE,[1] GRAY,[2] GUDERMANN,[3] KULIK,[4] STEINHAUSER,[5] WACKERBARTH,[6] and in my notes in *Scripta Mathematica.*[7] Another error in Wolfram, N = 4891, was correct in Vega.[7] C. R. COSENS noted the four last-figure unit errors calling for increases in 2173, 2175, and decreases in 1087, 2174.

Tables I–II of Vega are mainly reprints of other works, the first being of A. VLACQ, *Arithmetica Logarithmica . . .*, Gouda, 1628, log N, for N = [1(1)100000; 10D]. Log N for N = [100001(1)100999; 10D], appearing for the first time in Briggs, *Arithmetica Logarithmica*, 1624, to 14D (*MTAC*, v. 1, p. 170; v. 2, p. 94), contrary to what Glaisher[13] 1873 states, p. 139, were calculated by "Lieut. Dorfmund." Table II is not reprinted entirely from Vlacq's *Trigonometria Artificialis . . .*, Gouda, 1633, since the logarithms for the first two degrees were calculated for the work by DORFMUND. Vega took great pains to make his T. I accurate and remarkably succeeded.

Vega's *Thesaurus* was reviewed by GAUSS[8] who found T. I exceedingly accurate, but estimated that in T. II, the trigonometric canon, there were from 31983 to 47746 last-figure (tenth decimal) errors, most of them only amounting to a unit, but some to as much as 3 or 4 units. This table has been discussed at length by GLAISHER[9] and lists of errors were published by GRONAU,[10] M. VON LEBER[11] (who arrived at a result similar to that of Gauss), and HOBERT & IDELER.[12] An abridgment of T. II to 10D, at interval 10′, taking account of all known errors at the time of publication by the Astronomische Gesellschaft, is in C. BÖRGEN, *Logarithmisch-Trigonometrische Tafel auf 11 (bezw. 10) Stellen*, Leipzig, 1908, p. 29–34.

It is also of importance to refer to Vega's list of 120 Errata in the original *Thesaurus*. The first page is at the end of the introduction, p. XXX, in Latin, and the second is on p. 685, in Latin and German; the contents of these two lists are identical. The second sheet was for cutting up so that the correct figures might be pasted over those which were erroneous. In some copies the list is much more complete than in others; "120" appears to be the maximum number of entries.

Glaisher notes[13] "There is a great difference in the appearance of different copies of the work. In some the tables are beautifully printed on thick white paper, with wide margin, so that the book forms one of the handsomest collections of tables we have seen; while in others the paper is thin and discoloured; all are printed from the same type."

Of Vega's *Thesaurus* there have been five reprints, the first two, at least, at Florence, by the Istituto Geografico Militare, in facsimile folio format. These were photozincographic reproductions, and the first,

2. was got out in 1889, in an edition of 250 copies, in order to meet the needs of the geodetic service. It was distributed to all libraries, astronomical observatories, and other scientific institutions in Italy, and was presented at the triennial meeting of the International Geodetic Association in Paris in 1889. This edition was soon exhausted, contains xxx, 684 p. 22 × 33.2 cm. (the copy consulted may have been trimmed while being bound). 121 entries (the last wrirten by hand, for ln 1099) are given on the full Errata sheet, p. xxx, and all of these except 1 were corrected in the text before reproduction. On the half-title is pasted a piece of paper (10.3 × 7.8 cm.) with the legend:

Riproduzione fotozincografica
Dell' Istituto Geograficc Militare
Firenze, 1889.

The work was apparently issued with paper covers of colored paper, on which the half-title, without border, was printed, and a similar piece of paper pasted. There is a copy of this edition at Harvard University. Since the original plates had been preserved the Istituto printed, on thicker paper,

3. Another edition of 200 copies, in 1896. 24 × 34.1 cm. In addition to the corrections made in edition **2** the following were also made in **3** before reproduction:

p. 172. The first three figures, 777, which are to be found corresponding to number 59840 were lowered by one line, and the proper asterisks affixed to the numbers in columns 2, 3, 4.
355. log tan 1° 26′ 12″, *for* 4249, ***read*** 4149 (Westphal),
log cot 1° 26′ 12″, *for* 5751, ***read*** 5851 (Westphal).
414. log cot 9° 5′ 50″, *for* 7008, ***read*** 6908 (Luther).
679. ln 6343, *for* 1623, ***read*** 1633.

The print facsimile is slightly larger than the original, and the generous margins and good paper add to the attractions of the volume. What were thought of as pages i–iii of the original edition are used for "Prefazione alla seconda edizione fotozincografica." There seem to have been changes in the course of printing of this edition since the last paragraph of the "prefazione" is different in two different copies, **3A** and **3B**, at Brown University.

In his *Nouvelles Tables Logarithmiques à 36 décimales*, 1933, Duarte seems to state (p. xxiv), perhaps copying H. Andoyer, *Nouvelles Tables Trigonométriques Fondamentales (Logarithmes)*, Paris, 1911 (p. vi), that the Istituto published

4. A third edition in 1910, presumably at Florence. The *Jahrb. ü. d. Fortschritte d. Math.* for 1910 simply lists an edition of the *Thesaurus* published at Milan. I have not seen a copy of this edition. Andoyer's remark concerning the Florentine editions ("Mais les erreurs de l'édition originale n'ont pas été corrigées"), copied by Duarte, is both misleading and incorrect; misleading, because all but one of the corrections of Vega's Errata sheet were made in the text of editions **2**, **3**, **4**; and incorrect, because other corrections were made in **3** and **4**.

5. The next edition was made in Vienna by G. E. Stechert & Co., New York, in 1923, and is a slightly enlarged print facsimile. 21.5 × 33.7 cm. The half-title of the original edition was replaced by a new title page: *Georg Vega. 10 Place Logarithms including Wolfram's Tables of Natural Logarithms. Reprint of the Rare Edition of 1794.* The Errata sheet is the same as the printed part of the list in **3**, and all of these corrections were made in the text before reproduction of the original edition. Hence Henderson's statements "All the errors of the original appear" (art. "Mathematical Tables," *Encycl. Britannica*, 14th ed., 1929, and *Bibliotheca Tabularum, Mathematicarum*, 1926, p. 162) are misleading and raise a doubt in one's mind as to whether Henderson thought that the presence of the Errata sheet in **5** meant that the errata in the text had not been corrected, or, that he intended his remark to apply only to errors not on Vega's Errata sheet. What was p. 685 is unnumbered and follows p. XXX. This edition was sold for $12.50.

6. Since the unsold stock of **5** was stored in Leipzig and destroyed during the recent war, G. E. Stechert & Co. decided to make another reprint at a time when paper and binding material shortages were acute. Hence the miniature photolithographed copy under review. The size of the new print page as compared with the original, **1**, is about as 34:61. The new margins are very narrow, but the print is clear, the type is adequately large for use, and the volume is much easier to handle than the large folio.

During the past 150 years various errata lists for Vega's *Thesaurus*, besides those already mentioned, have been published.[14] So far as T. I is concerned Peters believed that he listed all of Vega's 303 errors (301 unit errors, and 2 of 2 units in the tenth decimal place) on p. xvi of his work, 1922, but he omitted the unit error in 100330, noted by Lefort. Peters' list, mainly a correction and expansion of Lefort's lists,[15] almost wholly duplicated by Leber,[11] is reprinted, with extra information about the entries, in JAMES HENDERSON, *Bibliotheca Tabularum Mathematicarum*, Cambridge, 1926, p. 94–95.

The reader naturally inquires, Has Vega's work ever been superseded? The answer is, partially. T. I [N = 1(1)100 000] and T. III N = [1(1)146 (primes) 9973] by J. T. PETERS, *Zehnstellige Logarithmentafel*, v. 1: *Zehnstellige Logarithmen der Zahlen von* 1 *bis* 100 000, Berlin, 1922, and an auxiliary table, [v. 3, 1919], showing corrections for second differences; T. III with 10 errors (N = 829, 1087, 1409, 3967, 6343, 7247, 8837, 8963, 9623, 9883), is in the Appendix. Two of Vega's errors (in 3571, 7853) are here corrected. There is a table log N, N = [10 000(1)100 000; 20D], δ^{2n} by A. J. THOMPSON, *Logarithmetica Britannica* . . . , 9 parts, Cambridge, 1924–1937; the final published part, no. 2, N = 20 000–30 000, is in the press. But for T. II no 10D table covering Vega's range has been published since 1794. Andoyer's *Nouvelles Tables Trigonométriques Fondamentales* (*Logarithmes*), Paris, 1911, contains a 14D table, at interval 10″, which is as complete, in interval, for 3°–88°. But these volumes are expensive, and the first one is excessively difficult, if not impossible, to secure, except in a library. Since the Vega volume is reasonably accurate it still fills a decided need.

No comprehensive review of editions of Vega should fail to refer to a ten-place logarithm table of numbers 10000(1)100009, Δ, issued with author's name, W. W. DUFFIELD, then superintendent of the U. S. Coast and Geodetic Survey, in its *Report for 1896*, Washington, 1897, appendix, no. 12, "Logarithms, their nature, computation, and uses, with logarithmic tables of numbers and circular functions to ten places of decimals—Part I," p. 395–722 (the table occupying p. 422–721. 22.8 × 38.5 cm.). On p. 397 we find the following:

"In the accompanying logarithmic tables all the mantissae have been computed to twelve places of decimals, and whenever the eleventh and twelfth places exceeded 50 the tenth place has been increased by unity, or 1; but whenever the eleventh and twelfth places were 50 or less than that number the tenth place has not been increased."

"When these computations were begun I was not aware that Baron George von Vega had preceded me in his Thesaurus Logarithmorum Completus. But my own results have been carefully compared with those of Von Vega, and whenever any difference was detected the computation was made anew. In this way many serious errors (undoubtedly typographical) in Baron von Vega's tables have been discovered and corrected."

The gross falsity of the superintendent's statement concerning computations of his staff (to which no reference is made) was the basis of much comment on their part. With devastating completeness Peters showed that the Duffield staff carried out computations as stated to only about 26004, but thereafter simply copied Vega's table. The explanation of the three later numbers which Duffield had correct (31653, 38051 and 60704), while Vega was incorrect, is that Duffield had his computers carry through the work for a few later random values. There is a one-to-one correspondence between the 267 other errors in Vega and Duffield. Up to 26 004, the 1897 work had corrected 33 Vega errors. See also Henderson, *l.c.* WILLIAM WARD DUFFIELD (1823–1907) served on the staff of General Pillow in the Mexican War, 1847–48, and during the Civil War, 1861–65, commanded the 4th Michigan infantry. He was brevetted major-general in 1863, elected state senator for Michigan in 1878 and appointed chief engineer for railways in Michigan, New York, Illinois and Texas; and U. S. engineer of improvements on Wabash and White rivers in 1892. He was superintendent of the U. S. C. G. S. 1894–98 (*Encycl. Americana*). From a scientific genius, BENJAMIN PEIRCE, superintendent 1867–74, to a scientific pigmy!

R. C. A.

[1] F. J. DUARTE, (a) *Nouvelles Tables de Log n! à 33 décimales depuis n = 1 jusqu'à n = 3000*, Paris, 1927, p. III; (b) *Nouvelles Tables Logarithmiques à 36 Décimales*, Paris, 1933, p. xxii.

[2] P. GRAY, *Tables for the Formation of Logarithms & Antilogarithms*, London, 1865, p. 39.

[3] C. GUDERMANN, *J. f. d. reine u. angew. Math.*, v. 9, 1832, p. 362.

[4] J. P. KULIK, *Astron. Nach.*, v. 3, 1825, cols. 191–192.

[5] A. STEINHAUSER, *Hilfstafeln zur präcisen Berechnung zwanzigstelliger Logarithmen . . .*, Vienna, 1880, p. 1.

[6] A. F. D. WACKERBARTH, R.A.S., *Mo. No.*, v. 27, 1867, p. 254.

[7] R. C. A., *Scripta Math.*, v. 4, 1936, p. 99, 293. See also *MTAC*, v. 1, p. 57.

[8] C. F. GAUSS, "Einige Bemerkungen zu Vega's *Thesaurus Logarithmorum*," *Astron. Nachr.*, v. 32, 1851, cols. 181–187; also in his *Werke*, v. 3, 1866 and 1876, p. 257–264.

[9] J. W. L. GLAISHER, "On logarithmic tables," R.A.S., *Mo. No.*, v. 33, 1873, p. 440–451; see also an appended letter of J. N. LEWIS. See also v. 32, 1872, p. 288–290, and v. 34, 1874, p. 471–475.

[10] J. F. W. GRONAU, "Tafeln für die hyperbolischen Sectoren und für die Logarithmen ihrer Sinus und Cosinus," Natur. Ges., Danzig, *Neueste Schriften*, v. 6, no. 4, 1862, p. vi. Lists 99 errors in T. II.

[11] M. VON LEBER, *Tabularum ad Faciliorem et Breviorem, in Georgii Vegae "Thesauri Logarithmorum" magnis Canonibus, Interpolationis Computationem utilium, Trias*, Vienna, 1897. He lists 272 errors in the tenth place of T. I, of which all but five are unit errors; the three serious ones had been corrected by Vega himself. Peters showed that 8 other entries by Leber as errors in Vega, were in fact correct. Also 2148 errors in T. II.

[12] J. P. HOBERT & L. IDELER, *Neue Trigonometrische Tafeln für die Decimaleintheilung des Quadranten*, Berlin, 1799, p. 350–351. There are here 168 corrections of T. II, 157 final-unit errors, 10 2-unit errors, 1 3-unit error.

[13] J. W. L. GLAISHER, B.A.A.S., *Report, 1873*, p. 138.

[14] Other lists are as follows:

K. KNORRE, *Astr. Nach.*, v. 7, 1829, col. 62. Error in T. I.
R. LUTHER, *Astr. Nach.*, v. 44, 1856, cols. 239–240. Error in T. II.
E. SANG, R. So. Edinburgh, *Proc.*, v. 8, 1875, p. 376. 40 unit errors, and one error listed by Vega, in T. I, $N = 20071$–29703.
D. J. M. M'KENZIE, *Bull. Sci. Math.*, s. 2, v. 4, 1880, p. 31f. Error in T. III.
A. WESTPHAL, *Astr. Nach.*, v. 114, 1886, cols. 333–334. Error in T. II.
J. FRISCHAUF, *Astr. Nach.*, v. 174, 1907, col. 173. 2 errors in T. I.
G. WITT, *Astr. Nach.*, v. 178, 1908, cols. 263–266. 23 errors in T. II.
P. ADRIAN, *Astr. Nach.*, v. 198, 1914, cols. 167f, 327f. Errors in T. I.

[15] F. LEFORT, Paris, Observatoire, *Annales, Mémoires*, v. 4, 1858, p. [148]–[150]. 300 errors listed in T. I. Peters' table does not adopt 7 of the final-digit unit changes demanded by Lefort for 26188, 29163, 30499, 31735, 34162, 34358, 60096. There are 25 very serious errors in T. I of 1, listed by Lefort, but all of these are in Vega's Errata list.

F. LEFORT, R. So. Edinburgh, *Proc.*, v. 8, 1875, p. 571–574, 587; also by E. SANG, p. 586–587. Lefort lists 287 last-figure errors (all except 5, one unit in the tenth decimal place) in T. I, 2 in Table II, and 2 in T. III (1099, 7853). There is a duplication of statement of 7 correct logarithms as erroneous. But furthermore, Lefort notes 6 errors, all in Vega's list of Errata, which implies inclusion of 100330 since the unit error for 10033 is listed, but it is not actually stated, as in Lefort.

320[D].—BENGT STRÖMGREN (1908–), *Optical Sine-Tables giving seven-figure values of* $x - \sin x$ *with arguments* x *and* $\sin x$. Geodaetisk Institut, Copenhagen, *Skrifter*, s. 3, v. 5, 1945, 63 p. 22.8 × 28.6 cm.

In the fields of optics and astronomy, it has long been realized that tables giving the quantity $x - \sin x$ are useful in numerical calculations of the properties of optical systems. Some previously published tables are as follows:

1. A. M. LEGENDRE, *Exercices de Calcul Intégral*, v. 3, 1816, p. 178. Table of $\frac{1}{4}(2x - \sin 2x)$ for $x = [0(1°)90°; 10D]$, Δ^2.
2. J. F. ENCKE, "Ueber die Berechnung der Bahnen der Doppelsterne," *Berliner Astron. Jahrb. f. 1832*, 1830, p. 297–304; also in J. F. ENCKE, *Ges. mathem. u. astron. Abhandlungen*, v. 3, Berlin, 1889, p. 71–78, $2x - \sin 2x$, for $x = [0(10')90°; 5D]$, Δ.
3. A. STEINHEIL & E. VOIT, *Handbuch der angewandten Optik*, v. 1, Leipzig, 1891, p. 271–314, $x^0 - (\sin x)^r$, $x = [0(10'')2°\ 46'40''; 0''.001]$, $[2°\ 46'\ 50''(10'')30°; 0''.01]$. This table is not included in the English translation, 2 v., 1918–19.
4. R. HEGER, *Fünfstellige logarithmische und goniometrische Tafeln . . .*, Leipzig, 1900, p. 83; second edition, 1913, p. 83, $x = [0(1°)179°; 4D]$.

5. J. BAUSCHINGER, *Tafeln zur theoretischen Astronomie*, Leipzig, 1901, p. 142–146. This table does not appear in the 1934 edition. x = [0(1′)3°; 0″ .0001], [3°(1′)40°; 0″ .01], Δ, as in no. 3.
6. J. R. AIREY, B.A.A.S., *Report 1916*, p. 88–89; x = [0(.00001).001; 11D].
7. HENRI CHRÉTIEN, *Nouvelles Tables des Sinus Naturels spécialement adaptées au calcul des combinaisons optiques* . . . , Paris, 1932, p. 10–27, to x = .181842, i.e. to *c.* 10° 25′, 6D critical; then to x = .52661, i.e. to *c.* 30° 10′, 5D critical. See *MTAC*, v. 1, p. 16.

Strömgren's two new tables (p. 11–62) are of $x - \sin x$ with arguments x and $\sin x$ each = [0(.0001).5; 7D], Δ. {On p. 63 is a single-page table of $\tan x$ for x = [0(.001).2; 7D], Δ.} The author remarks that Chrétien's tables give sufficient accuracy for most purposes. "However, in investigations of objectives of astronomical instruments of medium, or long, focal length, it is desirable to carry out trigonometrical ray-tracing of higher accuracy. The tables of the present publication have been prepared in order to meet this need." For early arguments, 0(.0001).04, values are given to 8D.

For the argument x in radians the function $x - \sin x$ was tabulated at interval .001, from the 10D table of BAASMTC, *Mathematical Tables*, v. 1, 1931. From this, a corresponding 10D table at interval .0001 was calculated by interpolation. The table thus obtained was checked by differencing. To complete the check a duplicate table was computed by cumulative addition of the differences, and then compared with the original.

Next the table was abridged from 10D to 7D. All tabular values in the 10D table ending in 498, 499, 500, 501, 502 were recomputed to 12D with the aid of the series expression for $\sin x$, and abridged accordingly. Since this revision did not lead to any changes, the 7D values should be correct to the last decimal place.

For values of x up to .04 the table was abridged from 10D to 8D. All tabular values ending in 48, 49, 50, 51, 52 were recomputed to 12D and abridged accordingly.

With $\sin x$ as argument 11D values of $x - \sin x$ were first computed for $\sin x$ = .01(.01).53 from the BAASMTC volume referred to above, by a process of backward interpolation utilizing the addition formulae valid for the sine function. These values were checked with the aid of Briggs table of 1633. An interpolation to tenths was carried out, and the values obtained rounded off to 10D, and checked by differencing. Checking and abridgment were then carried out as before.

Pages 4–10 are mostly occupied with an explanation of the use of the tables.

Strömgren has been an ordinary professor of astronomy at the University of Copenhagen and director of the University's astronomical Observatory since 1940.

R. C. A.

321[F].—D. P. BANERJEE, "On a theorem in the theory of partition," Calcutta Math. So., *Bull.*, v. 37, 1945, p. 113–4.

This note contains a short table (for $n \leqq 70$) of the number $Q(n)$ of partitions of n into parts which are both odd and distinct. The author is unaware of a table of WATSON[1] giving this function for $n \leqq 400$. A comparison of the two tables shows no discrepancy. The theorem referred to in the title is to the effect that $p(n)$, the number of unrestricted partitions of n is odd or even according as $Q(n)$ is odd or even. This theorem is noted by Watson[1] and MACMAHON[2] and is actually much older, since it follows at once from the theorem of SYLVESTER[3] that $Q(n)$ is also the number of self-conjugate partitions of n. The recurrence formula used to compute the table is the same as that used by Watson.

D. H. L.

[1] G. N. WATSON, "Two tables of partitions," London Math. So., *Proc.* s. 2, v. 42, 1937, p. 550–556.

[2] P. A. MACMAHON, "The parity of $p(n)$, the number of partitions of n, when $n \leqslant 1000$," London Math. So., *J.*, v. 1, 1926, p. 226.

[3] J. SYLVESTER, "On a new theorem in partitions" and "Note on the graphical method in partitions," Johns Hopkins Univ., *Circulars*, v. 2, 1883, p. 70–71; *Collected Math. Papers*, v. 3, 1909, p. 680–684.

322[G].—MAURICE B. KRAITCHIK, "On certain rational cuboids," *Scripta Mathematica*, v. 11, July–Dec. 1945 [publ. Aug. 1946], p. 317–326. 16.5 × 24.7 cm.

Denoting the edges of a rectangular parallelopiped or cuboid by x, y, z and the diagonals of the faces by X, Y, Z, we shall have

$$x^2 + y^2 = Z^2, \quad y^2 + z^2 = X^2, \quad z^2 + x^2 = Y^2. \tag{1}$$

If m, n, p are integers such that $m^2 + n^2 = p^2$, the following are general solutions:

$$\begin{cases} x = m(p^2 - 4n^2), & y = 4mnp, & z = n(p^2 - 4m^2) \\ X = n(p^2 + 4m^2), & Y = p^3, & Z = m(p^2 + 4n^2). \end{cases} \tag{2}$$

On p. 326 is a table of 50 cuboids which cannot be derived from formulae (2) or from each other. The first and last cuboids of this table are $z = 85 = 5\cdot17$, $x = 4\cdot3\cdot11$, $y = 16\cdot9\cdot5$, $z = 2636361 = 27\cdot7\cdot13\cdot29\cdot37$, $x = 8\cdot5\cdot17\cdot41\cdot107$, $y = 32\cdot3\cdot5\cdot11\cdot17\cdot37$.

Extracts from the article

323[I, O].—I. J. SCHOENBERG, "Contributions to the problem of approximation of equidistant data by analytic functions. Part A.—On the problem of smoothing or graduation. A first class of analytic approximation formulae", *Quart. Appl. Math.*, v. 4, Apr. 1946, p. 45–99. The tables, p. 91–99, are by Mrs. MILDRED YOUNG. 17.7 × 25.4 cm.

The problem of interpolation may be conceived in a very general form as follows: Let $f(x)$ be a function concerning which information is tabulated for a discrete set of values of x, say a sequence $\{x_n\}$. (The function may not even be defined elsewhere.) It is required to define a function $F(x)$ subject to specified restrictions and such that it bears a specified relation to $f(x)$. If the specified relation is that, for every n,

$$F(x_n) = f(x_n), \tag{1}$$

the formula $F(x)$ is called an *ordinary interpolation formula;* on the other hand if we only require (1) to hold approximately, and emphasize the restrictions which make $F(x)$ smooth, $F(x)$ is called a *smoothing* or "modified" *interpolation formula*. There are advantages in understanding the term interpolation broadly enough to take in other forms of relationship, so as to include such matters as mechanical quadratures, but this is irrelevant for our present purposes.

The mth degree Lagrange interpolation formula gives a polynomial $F(x)$ of degree $\leqq m$ which is an ordinary interpolation formula for a set consisting of $m + 1$ distinct points. Under certain assumptions this $F(x)$ is an accurate approximation near the middle of the interval spanned by these $m + 1$ points; this suggests piecing such polynomials together so as to get a formula valid over a wider range. But when this is done, the resulting $F(x)$ has discontinuities, either in the function or in its first derivative, at the points where the pieces are joined together. For some purposes this is a grave disadvantage. In dealing with empirical data it is frequently desirable to form a very smooth function $F(x)$ for which (1) holds only to within the accuracy of the data; while $F(x)$ may be computed and smoothed to a far greater accuracy than this, in order to avoid excessive rounding errors in the subsequent calculation. This has led to a more general study of interpolation, to which Schoenberg's paper is a notable contribution.

A general class of interpolation formulae consists of those of the form

$$F(x) = \sum_n f(n)\cdot L_n(x), \tag{2}$$

where the L_n are determined by the x_n independently of $f(x)$. In case the x_n are equidistant and extend to infinity both positively and negatively, it is natural to take $x_n = n$ and

$$L_n(x) = L(x - n), \tag{3}$$

so that (2) becomes

$$F(x) = \sum_{n=-\infty}^{\infty} f(x_n)L(x - n). \tag{4}$$

This is the type considered by Schoenberg. He requires in addition that $L(x)$ be an even function—a restriction which is something of a nuisance since it is not always convenient to maintain it in programming computations for the automatic machines. In Chapters I and II of the present paper he develops a general theory of such formulae using the method of Fourier analysis. In this he makes, for technical reasons, the restriction that $L(x)$ vanish exponentially at infinity. The function $L(x)$ he calls the *base function*. The study of interpolation formulae of type (4) thus reduces to that of their base functions; and it is convenient to speak of such a base function as giving rise to an interpolation formula of such and such type.

Two types of base functions are considered by Schoenberg more in detail. The first type are what the reviewer would call *broken polynomials*, i.e. their graphs are formed by piecing together a finite number of polynomial arcs. This type includes ordinary Lagrangean interpolation when centered and pieced together, as above described, in such a way as to maintain symmetry. Such functions $L(x)$ are characterized by (1) the maximal degree m of the constituent polynomials; (2) the class C^{μ}, i.e. the number μ of continuous derivatives; (3) the highest degree k of polynomials for which $F(x) = f(x)$, and (4) the span s. The second type are analytic functions derived from broken polynomials by considerations in the theory of heat flow. In fact if we set

$$L(x, t) = (\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(u-x)^2/t} L(u, 0)du, \tag{5}$$

then $L(x, t)$ is a solution of the partial differential equation of heat flow such that

$$\lim_{t\to 0} L(x, t) = L(x, 0). \tag{6}$$

For each fixed t, $L(x, t)$ is then an analytic smoothed form of $L(x, 0)$, the smoothness increasing with t.

In the present Part A (after the general introductory chapters) the broken polynomials are of degree $k - 1$ and class C^{k-2}. Such a curve (polynomial) Schoenberg calls a *spline* curve (polynomial) of order k because the splines used by draftsmen make such curves for $k = 4$. This is the maximum number of continuous derivatives a broken polynomial can have without reducing to an ordinary polynomial. It turns out that a spline of order k must have span at least k. If the span is k the spline is uniquely determined except for a constant factor. With a certain normalization this spline, called a *basic spline of order k*, is $M_k(x)$, viz.

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2 \sin \frac{1}{2}u}{u}\right)^k e^{iux}du = \frac{1}{(k-1)!}\delta^k x_+^{k-1}, \tag{7}$$

where δ is the central difference operator and

$$x_+^{k-1} = \begin{matrix} x^{k-1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{matrix} \tag{8}$$

Any spline polynomial can be expressed as a linear combination of basic splines $M_k(x - n)$. The interpolation formula based on $M_k(x)$ is exact for degree 1 and converts every polynomial $f(x)$ of degree $k - 1$ into an $F(x)$ of the same degree.

The analytic base function derived from $M_k(x)$ by (5) is

$$M_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_k(u, t)e^{iux}du \tag{9}$$

where

$$\psi_k(u, t) = e^{-\frac{1}{4}tu^2}\left(\frac{2 \sin \frac{1}{2}u}{u}\right)^k. \tag{10}$$

Further functions are then defined thus

$$\phi_k(u, t) = \sum_{n=-\infty}^{\infty} M_k(n, t) \cos nu = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu, t). \tag{11}$$

$$L_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} e^{iux}du. \tag{12}$$

$$L_k(x, t, \epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon + \phi_k(u, t)}{\epsilon + (\phi_k(u, t))^2} \psi_k(u, t)e^{iux}du. \tag{13}$$

Of these $M_k(x, t)$ gives rise to a smoothing interpolation formula which is exact for the degree 1 and preserves the degree $k - 1$; $L_k(x, t)$ gives rise to an ordinary interpolation formula which is exact for the degree $k - 1$; while $L_k(x, t, \epsilon)$, which reduces to $L_k(x, t)$ for $\epsilon = 0$ and to $M_k(x, t)$ as $\epsilon \rightarrow \infty$, is a compromise between these two. It increases in smoothing power (in a sense which is explained in the paper) as ϵ increases.

The formula

$$F(x) = \sum_{n=-\infty}^{\infty} f(x_n)L_k(x - n, t, \epsilon) \tag{14}$$

turns out to be equivalent to the following pair (Schoenberg calls the first one f_ν, but we have here used $f(x)$ in another sense)

$$h_\nu = \sum_{n=-\infty}^{\infty} f(n)\omega_{\nu-n}^{(k)}(t, \epsilon) \tag{15}$$

$$F(x) = \sum_{\nu=-\infty}^{\infty} h_\nu M_k(x - \nu, t) \tag{16}$$

where the $\omega_\nu^{(k)}(t, \epsilon)$ are the coefficients in the expansion

$$\frac{\epsilon + \phi_k(u, t)}{\epsilon + (\phi_k(u, t))^2} = \omega_0 + 2\sum_{\nu=0}^{\infty} \omega_\nu^{(k)}(t, \epsilon) \cos \nu u$$

$$= \sum_{\nu=-\infty}^{\infty} \omega_\nu^k(t, \epsilon)e^{i\nu u}.$$

The advantage of this is that $M_k(x, t)$ damps out as $\exp(-x^2)$ whereas $L_k(x, t)$ as $\exp(-x)$ only.

The tables in the Appendix give certain values for $k = 4$, $t = .5$. These tables, computed by Mrs. Young, are as follows:

Table I: $M_4(x, \frac{1}{2})$, $M_4'(x, \frac{1}{2})$, $M_4''(x, \frac{1}{2})$, for $x = -5(.1) + 4$. These are conveniently arranged for subtabulating the given table $f(n)$ to tenths; values where the arguments differ by an integer are in the same column.

Table II: $\omega_n^{(4)}(\frac{1}{2}, \epsilon)$ for $\epsilon = 0(.1)1$, $n = [0(1)26; 8D]$. (Note that $\omega_n^{(k)}$ is symmetric in n.)

Table III: $L_4(x, \frac{1}{2}, \epsilon)$ and $L_4''(x, \frac{1}{2}, \epsilon)$ for $\epsilon = 0(.1)1$, $x = [0(.5)26; 8D]$. This table is for use with (14) for subtabulation to halves. On account of the slow damping of $L_k(x, t, \epsilon)$ use of (15) and (16) with Tables I and II is recommended for subtabulation to tenths.

HASKELL B. CURRY

Pennsylvania State Coll.

324[K].—SIEGFRIED KOLLER, *Graphische Tafeln zur Beurteilung statistischer Zahlen.* Dresden and Leipzig, Steinkopff, second ed. 1943. Lithoprinted by Edwards Bros., Ann Arbor, Michigan, 1945, x, 73 p. 19.5 × 27.4 cm. $3.00. Published and distributed in the public interest by authority of the Alien Property Custodian under license no. A-634.

This book presents in graphical form the numerical information required for some of the most usual routine tests of statistical significance. Except for insignificant variations the book covers only material generally used by statisticians but usually presented by means of numerical tables. In the preface to the second edition the author complains that the graphical method usually meets a certain resistance; it must be said, nevertheless, that in the present case its advantages are far from obvious. According to the author the graphs require "considerably less" space than numerical tables. The justification of this claim could be disputed on the basis of a comparison of some graphs in the book under review with the corresponding tables in R. A. FISHER, *Statistical Methods for Research Workers* (eighth ed., Edinburgh and London, 1941), which is generally used by statisticians. The principal advantage of the graphical method, it is said, is that it does not require numerical interpolation. This claim is very general. In the particular case of the statistical tables with which we are concerned the accuracy required is so small and the spacing of usual tables is so dense that, in the reviewer's opinion, numerical interpolation does not present a problem to speak of. Instead of interpolation the present graphs require one to find and follow a curve in a very densely drawn family of curves and to read off one of its ordinates by means of a dense and rather unsharp coordinate net on a non-uniform scale; alternatively to join two points by a straight line and to read off the position of its intersection with a third line provided with a scale. For the last purpose the author suggests the use of a glass ruler on which a line is marked. The use of the usual transparent rulers or triangles is said to lead to inaccuracies (where they are used, a second reading with the upper side turned down is said to be necessary). A few trials made the reviewer doubtful as to the claim that this procedure is "more handy, more convenient, and faster than the use of numerical tables."

The book contains an introduction (p. 1–13) explaining the fundamental notions. There are in all seventeen graphs, all of which appear on odd numbered pages, mostly with the reverse blank. Facing each graph are directions for use and empirical examples. Wherever required, the odd page following the graph contains a mathematical definition of the function presented. In the statistical graphs the arbitrary "confidence level" has been chosen as $\epsilon = .0027$. This value corresponds to the old fashioned 3σ-rule and is used instead of the one and five per cent levels customary in the American and British literature. Special attention is paid to small samples; large number approximations are used only when they prove to be within the error limits of the graphs. The question of accuracy is not discussed in detail.

Only **Table 7** presents a simple graph of a function: the ϵ-point of Student's t-distribution as a function of the degrees of freedom n for $n > 10$. For $n \leqslant 10$ the values are given numerically to 2D; almost the same accuracy can be obtained from the graph. The remaining graphs are of three different types.

Group 1. Here functions of one variable are represented by means of two scales (divisions) along the same straight line. The scales are non-uniform since the spacing of the lines of division has had to be adjusted to the requirements of readability. **Table 2** gives x^2 for $1 < x < 10$ along five segments of some 19 cm. each. **Table 9** gives the ϵ-point of the χ^2-distribution with a number of degrees of freedom $n > 40$ (20 cm.). For $n \leqslant 40$ the values are given numerically in a short table which makes the accompanying graph look almost ridiculous. The same arrangement is found in **Table 10** giving $r = t(t^2 + n)^{-\frac{1}{2}}$ where t is the function of n mentioned in the description of **Table 7**. Values for $n \leqslant 30$ are given numerically, for $n > 30$ along a 19 cm. long scale. **Table 11a** represents (along some 90 cm.) the function

$$z = \tfrac{1}{2} \ln \frac{1 + r}{1 - r}$$

which is of use in correlation theory. Similarly, **Table 14** gives the normal density function for $0 < t < 4$, and **Table 15** its integral.

Group 2 contains nomograms of functions $z = f(x, y)$ of two variables. Here each of the variables is represented by the subdivision of one of three parallel lines. The value of z is found by bringing the z-line to intersection with the line joining the given x- and y-points. Thus **Tables 1a** and **1b** serve to find the product or quotient of two numbers.

Table 6 lacks mathematical significance; it is an auxiliary to **Table 5** and permits to reduce samples of unequal size to equivalent samples of equal size. **Table 8** serves to compute $z = (x^2 + y^2)^{\frac{1}{2}}$. **Table 11b** represents

$$3\left(\frac{1}{x-3} + \frac{1}{y-3}\right)^{\frac{1}{2}}.$$

The square root is the standard deviation of the difference of the z-values belonging to the correlation coefficients for two samples of sizes x and y. The z-values are obtained from **Table 11a**. These tables serve to test the significance of the difference between two correlation coefficients. To the second group belongs also **Table 12**, serving to compute partial correlations, and giving the function of three variables $r = (\alpha - \beta\gamma)[(1-\beta^2)(1-\gamma^2)]^{-\frac{1}{2}}$. Here the α-point and the r-point are found as described above for x and z while instead of the simple y-point one has now to find the intersection of a β-curve and a γ-curve.

Group 3. This group too is concerned with functions $y = f(x, n)$ of two variables, but this time they are represented by the one-parametric families of curves obtained by keeping n fixed. **Table 3** serves to test the hypothesis that a given sample comes from a Bernoulli population with probability p. For various n the curves represent $p_u - p$ as function of p, where np_u is the least number such that the probability of np_u or more successes is not greater than $\epsilon/2$. The hypothesis is to be rejected if the observed number of successes lies outside the interval (np_u, np_l), where p_l is the point on the p_u-curve with abscissa $1 - p$. **Table 4** gives the ϵ-confidence limits for estimating the unknown parameter p from the number of successes in a sample of n (cf. C. J. CLOPPER & E. S. PEARSON, "The use of confidence or fiducial limits illustrated in the case of the binomial," *Biometrika*, v. 26, 1934, p. 404f). **Table 5** serves to test the significance of the difference of the number of successes in two samples of n trials each. The curves represent the maximum permissible difference under the arbitrary assumption that the true value of p is exactly the arithmetic mean of the two frequencies of successes. Such an assumption is not justified by theory. However, even if it is accepted the test will, in general, lead to two different results depending on whether one starts with the smaller observed frequency and estimates an upper bound for the larger one, or starts with the larger frequency and estimates the smaller one. Finally, **Table 13** illustrates Fisher's z-distribution on which analysis of variance is based.

W. FELLER

Cornell University

325[L].—GERTRUDE BLANCH, "On the computation of Mathieu functions," *J. Math. Phys.*, v. 25, 1946, p. 1–20. 17.2 × 25.3 cm.

Techniques for computing the characteristic numbers and Fourier coefficients of the periodic Mathieu functions are now well known. The present paper gives a method by which the accuracy of the characteristic number may be rapidly and *systematically* improved.

To fix ideas, consider an even solution of

$$d^2y/dt^2 + (\alpha - 2\theta\cos 2t)y = 0 \tag{1}$$

of period π, which may be expressed in the form

$$y = \sum_{r=0}^{\infty} B_{2r}\cos 2rt. \tag{2}$$

In order that (1) may be satisfied, the B_m must satisfy a three-term recurrence relation, and writing

$$G_m = B_m/B_{m-2}, \quad H_m = 1/G_m, \quad k_m = (\alpha - m^2)/\theta \tag{3}$$

this may be written in the alternative forms,

$$G_m = k_{m-2} - H_{m-2} \text{ for } m \geq 5 \tag{4a}$$

$$G_m = 1/(k_m - G_{m+2}) \text{ for } m \geq 3 \tag{4b}$$

together with

(4c) $$G_2 = k_0, \quad G_4 = k_2 - 2H_2 = 1/(k_4 - G_6).$$

To satisfy these, α must have one of a set of characteristic values. If α' is an approximation to one of these, the relations (4c) and (4a) apparently determine in succession $G_2, G_4, \cdots$. But this is not, in fact, the case, for if (2) is to converge, then ultimately $|G_m| < 1$, and since k_m increases rapidly, so also does H_m, with the result that significant figures are rapidly lost. Hence we must also start from some large value of m (so large that B_m is insignificant) and come backwards by use of (4b), meeting at H_p, where p is some convenient value of m (corresponding generally to the coefficient B_p greatest in absolute value). Agreement between the values of H_p determined forwards and backwards is a test of the accuracy of α', but the most significant contribution in this paper is the masterly way in which the discrepancy between these values is made to yield a better value of α, by an application of what is essentially Newton's method of successive approximation to the root of an equation.

If $\alpha = \alpha' + \lambda$ is the true value, and H_{p1} and H_{p2} are the values of H_p determined respectively by using (4a) forwards and (4b) backwards, then it is shown that, if squares and higher powers of λ are negligible,

(5) $$\lambda = \theta(H_{p1} - H_{p2})/(R_{p1} + R_{p2})$$

where

(6a) $$R_{p1} = (B_{p-2}^2 + B_{p-4}^2 + \cdots + 2B_0^2)/B_p^2$$

(6b) $$R_{p2} = (B_p^2 + B_{p+2}^2 + \cdots)/B_p^2$$

(with slight modifications if $p = 0$ or 2), and the consequent changes in the G and the H are listed.

Formulae for the other three classes of function differ only in detail.

Complete numerical details are given for two examples. The first exhibits the method, and the enormous increase in accuracy which can accrue if α' is really close to α, 9D accuracy being converted into at least 18D accuracy by only one application—by a complete recalculation of the G and H, using the new value of α. The second shows how to use the formulae for the errors in the H if only a slight (9D converted to 12D) increase in accuracy is desired. The second example also shows some points which arise when (as for higher orders) the G_m do not decrease uniformly in numerical value with increasing m.

W. G. BICKLEY

326[L].—N. J. DURANT, "Struts of variable flexural rigidity," *Phil. Mag.*, s. 7, v. 36, Aug. 1945, p. 572–577 [publ. Apr. 1946]. 17 × 25.3 cm.

The differential equation

$$EId^2y/dx^2 + Py = 0$$

for a strut of variable cross-section (E is Young's modulus, I the second moment of the cross-section, y the transverse deflection at a distance x measured along the strut, and P the longitudinal thrust) can be integrated in terms of Bessel functions if $I = I_0e^{-kx/l}$ (I_0, k, constants, l the length of the strut); this is a reasonable approximation in the case of some engineering structures. If the strut is fixed at one end and free at the other, the critical thrust is $\gamma EI_0/l^2$, where $\gamma = (k\theta/2)^2$, θ is the smallest positive root of the equation

$$J_0(\beta\theta)Y_1(\theta) - Y_0(\beta\theta)J_1(\theta) = 0$$

in which $\beta = e^{\frac{1}{2}k}$. Table I, p. 576, gives k to 5S; and β, θ, $\beta\theta$, γ and 4γ to 4D, for $c = e^{-k} = .025, .05, .1(.1)1$.

W. G. BICKLEY

Imperial College
London, S. W. 7

327[L].—H. B. DWIGHT, "Table of the Bessel functions and derivatives J_2, J_1', J_2', N_2, N_1', N_2'," *J. Math. Phys.*, v. 25, May, 1946, p. 93–95. 17.2 × 25.3 cm.

Here are tables of $J_2(x)$, $J_1'(x)$, $J_2'(x)$, $Y_2(x)$, $Y_1'(x)$, $Y_2'(x)$ [we see no reason for using German notation], for $x = .01(.01).2(.1)10$. J_2' for $x = 2(.1)10$, and J_2 and J_1' throughout, are to 8D; J_2', for $x = .01(.01).2(.1)1.9$, are to 6 or 7D. Y_1' and Y_2', for x from 1.4 to 10, are to 6D; and for other values of x, to 6 or 7S. Y_2 are for $x < .2$ to 6S; for $.2 \leq x < 5$, to 7S; for $x \geq 5$ to 6D.

328[L].—V. FOK, "The distribution of currents induced by a plane wave on the surface of a conductor," Akad. Nauk USSR, Moscow, *J. of Physics*, v. 10, no. 2, 1946, p. 135–136.

The tables are identical with those already described in RMT **309**.

329[L].—F. I. FRANKL, "K teorii sopel Lavalia͡" [On the theory of the Laval nozzle], Akad. Nauk USSR, Moscow, *Izvestiia͡, seriia͡ matem.*, v. 9, p. 421, Nov. 1945.

There are here three tables to 4D.

I. $Z_1(t) = F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; t)$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})} \int_0^1 u^{-\frac{2}{3}}(1-u)^{-\frac{5}{6}}(1-ut)^{\frac{1}{3}} du$$

for $t = -.5(.1)+1$.

II. $Z_2(t) = -F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; 1-t)$

for $t = 0(.1)1.5$.

III. $Z_3(t) = F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; 1-t) - F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; t)$

for $t = 0(.1)1$.

330[L].—E. NISKANEN, "On the deformation of the earth's crust under the weight of a glacial ice-load and related phenomena," Suomalainen Tiedeakatemia, Helsingfors, *Toimituksia, Annales*, s. A., III. *Geologica, Geographica*, no. 7, 1943.

Table II, p. 34–36, gives the values of $P_n(x)$, and $T_n(x) = P_{n-1}(x) - P_{n+1}(x)$, for $n = [0(1)61; 6D]$, $x = \cos\theta$, $\theta = 6°, 9°, 30°$. The values of P_n, $n = 1(1)32$, were taken from H. TALLQVIST, *Sechsstellige Tafeln der 32 ersten Kugelfunktionen* $P_n(\cos\theta)$, Soc. Scient. Fennicae, *Acta*, n.s. A, v. 2, no. 11, 1938, p. 3–43, and the values of $\cos\theta$ were taken from J. T. PETERS, *Einundzwanzigstellige Werte der Funktionen Sinus und Cosinus zur genauen Berechnung* . . . , Akad. d. Wissen., Berlin, *Abh.*, *Phys.-Math. Kl.*, 1911, Anhang, no. 1. The terms 33(1)61 were computed by means of the formula

$$P_n = [(2n-1)/n]P_{n-1} - [(n-1)/n]P_{n-2}.$$

For $n = 52(1)61$, some final figures in parentheses "can be erroneous," so that, e.g., for $n = 61$ the last three or four places of values of the functions are of doubtful accuracy.

331[L].—JØRGEN RYBNER, "Fourieranalyse af frekvensmodulerede Svingninger med Savtakvariation af Øjebliksfrekvensen (Kipmodulation)," *Matematisk Tidsskrift B*, 1946, *Festskrift til N. E. Nørlund*, second part, p. 97–112. 15.2 × 24.1 cm.

Let Ω, $\Delta\Omega$, f, be positive numbers, and let $\Phi(t)$ be a function of period $1/f$ which in the interval $-\frac{1}{2}f \leqslant t \leqslant \frac{1}{2}f$ equals $f\Delta\Omega t^2$. By using the developments

$$\cos f\Delta\Omega t^2 = \tfrac{1}{2}a_0 + a_1\cos\omega t + a_2\cos 2\omega t + \cdots,$$
$$\sin f\Delta\Omega t^2 = \tfrac{1}{2}a_0' + a_1'\cos\omega t + a_2'\cos 2\omega t + \cdots,$$

where ω is defined by the relation $2\pi/\omega = 1/f$, the author represents the function $u = \sin [\Omega t + \Phi(t)]$, "Kipmodulation," in the form

$$u = \sin \Omega t(\tfrac{1}{2}a_0 + a_1 \cos \omega t + \cdots) + \cos \Omega t(\tfrac{1}{2}a_0' + a_1' \cos \omega t + \cdots).$$

The coefficients a_n, a_n' here depend on the parameter m (modulation index) defined by the formula $m = \Delta\Omega/\omega$. Using Lommel's formulae for Bessel functions the author tabulates (T. 1) the numbers $\frac{1}{2}a_n(m)$, $\frac{1}{2}a_n'(m)$, $M = \frac{1}{2}(a_n^2 + a_n'^2)^{\frac{1}{2}}$, M^2, and $y = m\pi$ = [3(3)30; 6D], n = 0(1)4. Table 2 gives the values for $m = 5$, n = [0(1)10; 6D].

A. Zygmund

Univ. of Pennsylvania

332[L].—Olof E. H. Rydbeck, "On the propagation of radio waves," Chalmers Tekniska Högskola, *Handlingar*, no. 34, 1944. 170 p. 17.5 × 24.8 cm.

On p. 160–166 are "Tables of cylinder functions of order $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$." There are tables of the following 12 functions for x = [0(.02)1(.2)8; 4–5S]: $J_{\pm 2/3}(x)$, $Y_{2/3}(x)$, $I_{\pm 2/3}(x)$, $I_{\pm 1/3}(x)$, $|H_{2/3}^{(1)}(x)|$, $Y_{2/3}(xe^{-\frac{1}{2}3i\pi})$, $|H_{2/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})|$, $iY_{1/3}(xe^{-\frac{1}{2}3i\pi})$, $|H_{1/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})|$. For the same range of x there are also tables of Phase $H_{2/3}^{(1)}(x)$, Phase $H_{2/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})$, and Phase $H_{1/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})$, each to the nearest 1′.

Rydbeck refers to G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 1922, where there are tables of $J_{\frac{1}{3}}(x)$, $Y_{\frac{1}{3}}(x)$, $|H_{\frac{1}{3}}^{(1)}(x)|$, which are tabulated for x = [0(.02)16; 7D]. He also mentions tables of $J_n(x)$, $I_n(x)$, $n = \pm\frac{1}{3}, \pm\frac{2}{3}$, for x = [0(.2)8; 4D], by A. Dinnik, in *Archiv Math. Phys.*, s. 3, v. 18, 1911, p. 337–338. No such tables are given at this reference, but only tables of $n!J_n(x)$, $n = \pm\frac{1}{3}, \pm\frac{2}{3}$. For information concerning Dinnik tables of $J_n(x)$, $I_n(x)$, $n = \pm\frac{1}{3}, \pm\frac{2}{3}$, in 1914 and 1933, see *MTAC*, v. 1, p. 286, 287; for other tables see also p. 237–238.

333[L, M].—Great Britain Nautical Almanac Office, *Tables of the Incomplete Airy Integral*, for the Department of Scientific Research and Experiment, Admiralty Computing Service, April 1946. ii, 15 p. + 1 folding plate. 20.3 × 30.5 cm. Reproduced by photo offset from typescript. This publication is available only to certain Government agencies and activities. Erratum slip for p. 7, dated May 1946.

This report contains values (p. 11–15) of the incomplete Airy integral

$$F(x, y) = \pi^{-1}\int_0^{\pi} \cos (xt - yt^3)dt \tag{1}$$

for x = [− 2.5(.1) + 4.5; 4D], y = 0(.02)1. The last figure tabulated should not be in error by more than one unit; except for small y and negative x the error is unlikely to exceed .6.

The tables can be interpolated in the x-direction using second differences, but for accurate interpolation in the y-direction fourth differences are required; if second differences only are used, the maximum error to be expected is two units.

If new variables $Y = \sqrt[3]{3y}$ and $X = x/Y$ are introduced then, with $T = Yt$, the integral for $F(x, y)$ may be rewritten in the form

$$F(x, y) = (\pi Y)^{-1}\int_0^{\pi Y} \cos (XT - \tfrac{1}{3}T^3)dT \tag{2}$$

which compares immediately with

$$Y^{-1}Ai(- X) = (\pi Y)^{-1}\int_0^{\infty} \cos (XT - \tfrac{1}{3}T^3)dT. \tag{3}$$

Here the complete Airy integral Ai is one of the solutions of the differential equation

$$\frac{d^2v}{dx^2} = xv. \tag{4}$$

$Ai(x)$, with an appropriately defined second solution $Bi(x)$, has been tabulated by J. C. P. Miller, see *MTAC*, v. 1, p. 283.

The corresponding differential equation satisfied by the incomplete Airy integral is easily shown to be

$$\frac{\partial^2 F}{\partial x^2} + \frac{x}{3y} F = \frac{1}{3\pi y} \sin (x\pi - y\pi^3). \tag{5}$$

3. *Method of computation*, p. 2–4; 4. *Numerical integration of the differential equation*, p. 4–9. The opportunity is taken here to describe in some detail a method, used in H. M. Nautical Almanac Office for some time, for the numerical integration of second order linear differential equations. The principle seems to be largely due to B. V. NUMEROV ("Méthode nouvelle de la détermination des éphémérides en tenant compte des perturbations," Observatoire Astrophysique Central de Russie, *Publications*, v. 2, Moscow, 1923), who published many papers from 1923 onwards developing the method and applying it to the simultaneous integration of the three (non-linear) second-order equations defining the motion of a particle under the Newtonian attraction of the bodies in the solar system.

Extracts from introductory text

334[L, M].—GREAT BRITAIN, Nautical Almanac Office, Department of Scientific Research and Development, Admiralty Computing Service, *Tabulation of the Function* $f(x, y) = \int_0^\infty \frac{e^{-k}\{J_0(kx) \cosh (ky) - 1\}}{\sinh k} dk$. Offprint reproduction of handwriting and typescript tables; 7 p. + 1 folding plate. No. SRE/ACS 47, February, 1945. 20 × 33 cm. These tables are available only to certain Government agencies and activities.

The function $f(x, y)$ is the solution to a two-dimensional potential problem, being that solution in the strip $0 \leqslant y \leqslant 1$ of the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{1}$$

which satisfies the conditions

$$\frac{\partial f}{\partial y} = 0 \text{ on } y = 0 \text{ for } x \neq 0, \quad \frac{\partial f}{\partial y} = 1/(x^2 + 1)^{3/2} \text{ on } y = 1,$$

and $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sim O[1/(x^2 + y^2)]$ near the origin and also for large values of x.

$f(x, y)$ is tabulated to 4D in the ranges

$$x = 0(.1)5, \quad y = 0(.1)1, \quad \Delta_x'', \quad \Delta_y''.$$

The methods of computation employed, and the checks which were applied, suggest that the last figure given is seldom in error by more than one unit, and never by more than two units.

The integral was evaluated in two stages. First, values were calculated from the series

$$f(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{1}{[x^2 + (2n - y)^2]^{\frac{1}{2}}} + \frac{1}{[x^2 + (2n + y)^2]^{\frac{1}{2}}} - \frac{1}{n} \right\} \tag{2}$$

at interval .2, on the bounding lines $x = 0$, $x = 5$, $y = 0$, $y = 1$. These values were then used as boundary conditions for the differential equation (1), and the function obtained in the interior of the rectangle by relaxation methods.

The values at interval .2 were interpolated to an interval of .1, an interval more convenient for tabulation and subsequent interpolation. For interpolation purposes, the second differences in both directions are tabulated on the same line as the function.

On the folding plate are "Values and contours of $f(x, y)$," unit 10^{-4}.

Extracts from introductory text

335[L, M].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. 2: *Tables of the Modified Hankel Functions of Order one-third and of their Derivatives.* Cambridge, Mass., 1945. xxxvi, 235 p. 20.2 × 26.5 cm. $10.00. See MAC 24.

Bessel functions of orders $\frac{1}{3}$ and $-\frac{1}{3}$ for real and pure imaginary arguments have been very extensively tabulated by the N.Y.M.T.P. For the case of complex arguments very little has been done. (See BATEMAN & ARCHIBALD—*Guide to Tables of Bessel Functions*, *MTAC*, v. 1, p. 205–308, 1944). The volume under review goes a long way towards filling the existing lacuna.

The differential equation

$$\frac{d^2u}{dz^2} + zu = 0$$

possesses a general solution of the form

$$u = Az^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}z^{3/2}) + Bz^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}z^{3/2}).$$

As the authors remark, the direct tabulation of the Bessel Functions of orders $\pm\frac{1}{3}$ is inadvisable since these functions are not single-valued. For this reason and having in mind the solution of a certain problem in wave propagation, the authors chose for tabulation the functions

$$h_1(z) = \frac{k}{\pi i}\int_{-\infty}^{\infty e^{\frac{1}{3}i\pi}} e^{zt+\frac{1}{3}t^3}dt = g + \tfrac{1}{3}i\sqrt{3}(g - 2f)$$

$$h_2(z) = \frac{k^*}{-\pi i}\int_{-\infty}^{\infty e^{-\frac{1}{3}i\pi}} e^{zt+\frac{1}{3}t^3}dt = g - \tfrac{1}{3}i\sqrt{3}(g - 2f)$$

where k and k^* are constants and

$$f = (\tfrac{2}{3})^{\frac{1}{3}}z^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}})$$
$$g = (\tfrac{2}{3})^{\frac{1}{3}}z^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}}).$$

It may be remarked that the functions $h_1(z)$ and $h_2(z)$ above defined are linear combinations of

$$\Lambda_{-\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}}) \quad \text{and} \quad z\Lambda_{\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}})$$

where

$$\Lambda_\nu(z) = \Gamma(\nu + 1)J_\nu(z)/(\tfrac{1}{2}z)^\nu.$$

Moreover it may be shown that $\Lambda_\nu(\frac{2}{3}z^{\frac{3}{2}})$ where $\nu = \pm\frac{1}{3}$ is a well behaved function of z in the neighborhood of the origin, and would have been just as suitable for tabulation as the functions adopted by the authors.

The volume contains six tables, four short ones and two long ones. The contents follow:

Table I: First 23 coefficients (11 or 12D) of the Maclaurin series for f, g, f', g'.
Table II: First 14 coefficients in the asymptotic series for $h_1(z)$ and $h_2(z)$.
Table III: Various constants.
Table IV: Zeros of $h_2(z)$ and $h_2'(z)$; $|z_0| < 6$.
Table V: $h_1(z)$ and $h_1'(z)$ to 8D; $\Delta x = \Delta y = 0.1$; $|x + iy| \leqq 6$.
Table VI: $h_2(z)$ and $h_2'(z)$ to 8D; $\Delta x = \Delta y = 0.1$; $|x + iy| \leqq 6$.

In tables V and VI the values of the functions are given only for the upper half of the z-plane; values in the lower half may be obtained from the given entries by means of simple "reflexion" formulae given with the tables. Beyond $|x + iy| = 6$ the asymptotic expansions included in the introduction furnish convenient means of computing $h_1(z)$ and $h_2(z)$, though not always, with an accuracy comparable to the tabular values. The volume contains an informative introduction describing various properties of the functions and their relations to other functions such as Airy integrals and Bessel Functions, of various fractional orders. The introduction also contains contour lines for $R(h_1)$ = const. and $I(h_1)$ = const. as well as a number of other useful graphs.

Even though the interval of tabulation of $h_1(z)$ and $h_2(z)$ is fairly coarse, the problem of interpolating to the full accuracy of the table is possible though somewhat difficult because of the complicated interpolation formulae needed. By utilizing the properties of harmonic functions the interpolation formulae may be considerably simplified for purposes of subtabulation.

Each value in the table was computed ab initio from the power series expansions and checked by duplicate calculation utilizing different equipment wherever possible. A partial check was afforded by the "Wronskian" relation.

A useful bibliography on Bessel Functions of order $\frac{1}{3}$ and Allied Functions is included. This volume is the first in a series of volumes on Bessel Functions computed on the "Automatic Sequence Controlled Calculator." The result is certainly impressive when one considers that according to the authors it took the machine merely the equivalent of 45 days to complete the tables. It may well be said that this basic table is the first in the new era of high speed computing techniques.

WILLIAM HORENSTEIN

NYMTP

EDITORIAL NOTE: The next six volumes of the Harvard *Annals of the Computation Laboratory*, for which computations have been completed, are to be as follows: **3.** $J_0(x)$, $J_1(x)$, and **4.** $J_2(x)$, $J_3(x)$, for x = [0(.001)25(.01)100; 18D], rounded off from computations to 23D. **5.** $J_4(x)$, $J_5(x)$, $J_6(x)$; **6.** $J_7(x)$, $J_8(x)$, $J_9(x)$; **7.** $J_{10}(x)$, $J_{11}(x)$, $J_{12}(x)$; **8.** $J_{13}(x)$, $J_{14}(x)$, $J_{15}(x)$, all for x = [0(.001)25(.01)100; 10D]. These six volumes are each to contain about 600 p. For a similar range it is planned to issue further volumes tabulating $J_n(x)$, n = 16(1)100. The director of the Computation Laboratory and the editor of the *Annals* is HOWARD H. AIKEN, associate professor of applied mathematics in Harvard University. The staff is housed in a new two and one-half story building, specially built for its activities. There will be ample room for other machines to be added to their remarkable Automatic Sequence Controlled Calculator (see *MTAC*, v. 2, p. 91, and *MAC* **24**) set up in a square room about 60 feet on a side.

336[V].—NYMTP, *Table of* $F(v) = \frac{1}{v}\frac{d}{dv}\ln J_6(v)$. No. OP265-2-97-11. New York, 1942. 11 leaves on one side of each leaf. Mimeographed. 20.7 × 33 cm. This publication is available only to certain Government agencies and activities.

The values of $F(v)$ are tabulated as a function of u where $u = v^2/100$ (v in meters per second), and where J_6 is the Army retardation function, determined experimentally, for the 3-inch Common Steel Shell Model 1915 as tabulated by the Aberdeen Proving Ground, Maryland, in 1932.

The NYMTP table of $F(v)$ differs from the 1932 Army table of $F(v)$, no. N-1-35, tabulated at the Army Proving Ground, Aberdeen, Maryland, in intervals as follows:

Army table: u = 0(10)800(1)1500(10)4000(100)8600.
NYMTP table: u = 0(1)100(2)500(10)1000(2)2000(10)8600;
6D to 89; 7D to 1410; 8D to 3830; 9D to 8600.

The function $F(v)$ occurs in various formulae for the computations of differential variations in velocity, air density, wind weighting factors, ballistic coefficient, etc., which are applicable in modifying computed trajectories of projectiles. If x and y represent the coordinates of a projectile on the original computed trajectory, x and y satisfy the differential equations $x'' = -Ex'$ and $y'' = -Ey' - g$, where $E = J_6(v)H(y)/C$; C is the ballistic coefficient. The coordinates on the modified trajectory at the instant t are $x + \xi$ and $y + \eta$.

A formula for δE, the part of the change in the retardation due to the variations in x', y' and y, is[1]

$$\delta E = E[F(v)\cdot v\Delta v + \eta \cdot d \ln H/dy]$$

where $v\Delta v = (x'\xi' + y'\eta')$, $H(y) = e^{-.0001036y}$. Since numerical integration must be used in determining the differential corrections to be applied at each point of the trajectory it is necessary to have tabulated values of $F(v)$.

PAUL D. THOMAS

Bureau of Ordnance, Navy Department

[1] D. JACKSON, *The Method of Numerical Integration in Exterior Ballistics*, Washington, 1921, p. 24; this was a text-book prepared in the office of the Chief of Ordnance, 1919.

EDITORIAL NOTE: These tables were later superseded by smoother functions based on more recent firings. In Jackson's publication is an extended table of $H(y)$.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **319** (Duffield, Lefort, Vega), **332** (Rydbeck); **N62** (Corey, Hardy & Rogosinski, Harvard, Zygmund, etc.).

88. H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 2, Bloomington, Indiana, 1935, p. 29.

There are three serious errors on this page, in $\psi'(x)$ which Davis defines as $d^2 \ln \Gamma(x)/dx^2 = d^2 \ln (x-1)!/dx^2$, and in $\log \psi'(x)$.

For $\psi'(.05) = 401.552357\ 342115$, *read* $401.532357\ 342115$;
For $\psi'(.11) = 84.077927\ 249967$, *read* $84.059535\ 747392$;
For $\log \psi'(.11) = 1.92468\ 19966$, *read* $1.92458\ 69871$.

J. W. WRENCH, JR.

4211 Second St. N.W.
Washington 11, D. C.

89. FMR, *Index*, 1946. See *MTAC*, v. 2, p. 13–18, 136.

A. Apart from a few definite errata noted below we have also indicated inconsistencies, indefinite statements, and a couple of notation changes, which the authors may desire to consider when preparing a new edition. The authors clearly recognized that slight blemishes of this kind existed, because of various elements entering into the preparation and publication of their work. Changes are in italics. See also MTE 90.

P. 23, 2.3 Higher *Positive* Integral Powers [2.5 Higher Negative . . .].
P. 25, **2.3** Higher *Positive* Integral.
P. 33, l. 7, $n = 440 \times 2^p$.
P. 35, **3.14** 27 dec. Thoman.
P. 48, **4.18**, l. 4, 4.02*1*.
P. 51, **4.41**, *for* 4.412, *read* 4.4121; *for* 4.413, *read* 4.4132.
P. 76, **4.9333**, *for* 2^m, *read* 2^q.
P. 100, last l., Cauchy 1882, why not 1827? Also p. 101, **5.7115**.
P. 111, last l., *de* Decker; p. 124 and 125, 5 d., *de* Lella.
P. 151, **9.24**, l. 2, δ for d (longitude)? [see **9.23**].
P. 192, **13.4** 4 dec. *for* 10^4, 10^5, 10^6, *read* $10^{4(1)6}$?
P. 200, 7 dec. Brownlee 1923 (Russell, which one?).
P. 208, **14.92**, *for the heading* "Tables of x," *read* "Inverse tables relating to $B_x(p, q)$"?
P. 251, **17.33** heading to make uniform with **17.35**, *read* $G_0(x)$ and $G_1(x)$, then on next line $G_n(x) = -\frac{1}{2}\pi Y_n(x)$.