RECENT MATHEMATICAL TABLES

359[A].—A. L. Crelle, *Rechentafeln welche alles Multiplizieren und Dividieren mit Zahlen unter Tausend ganz ersparen, bei grösseren Zahlen aber die Rechnung erleichtern und sicherer machen. Neue Ausgabe besorgt von O. Seeliger* [1907]. *Neudruck, mit Tafeln der Quadrat- und Kubikzahlen von 1–1000.* Berlin, Gruyter, 1944. viii, 501 p. 24.8 × 36.7 cm. See also *MTAC*, v. 2, p. 179.

A. L. Crelle (1780–1855) was the founder (1826) of a notable mathematical research journal still in existence, and author of various volumes, including even one on music (1823). His calculating tables, of which there have been many editions in English, French, and German, was first published in two small volumes over 125 years ago, Berlin, 1820. But in such editions, since 1857 at least, the folio format (each page containing four pages of the first edition) has been in use up to the present. As the title indicates the war edition under review contains two extra pages with the squares and cubes of numbers N, $N = 1(1)999$. A one-volume Japanese edition of Crelle's tables by Tsuneta Yano, Tokyo, 1913, has been referred to, *MTAC*, v. 2, p. 18; our incidental note, v. 1, p. 436, that this was published by an insurance company, is incorrect. This fact was learned after Mr. Edwin G. Beal, Jr., Chief of the Japanese Division in Library of Congress had kindly made a study of the volume for us. He also reported that he could find no record of a 1927 Japanese edition of this work.

R. C. A.

360[A, B, F].—J. Ser, *La Numération et le Calcul des Nombres*, Paris, Gauthier-Villars, 1944, 194 p., 25 × 16 cm.

The author of this work appears to be something of an individualist. He gives but one reference to the work of others and this is to an article on the foundations of mathematics, a subject not covered in this book. The reader will find many new points of view, "méthodes personelles," and unfamiliar nomenclature. Much of this has to do with pencil and paper calculation, a discipline all but unknown in this age of mechanized computation.

The more extensive tables in this work may be described as follows:

(i) An arithmetical table (p. 60–95) giving for the first 1000 integers N the following functions:

$1/N$ to 6D, the first nine multiples of N (these are used to facilitate multiplication and division), N^2, $\sqrt{N}$ to 4D, $\sqrt{10N}$ to 4D, N^3, $N^{1/3}$ to 4D, $(10N)^{1/3}$ to 3D, and $(100N)^{1/3}$ to 3D.

(ii) A small table, p. 96, of powers n^k for $n = 1(1)9$, $k = 1(1)16$, the first nine multiples of $10^{1/2}$, $10^{1/3}$, $100^{1/3}$ (all but five values have last-digit errata) and the fourth roots of $m \cdot 10^k$, $m = 1(1)9$, $k = 1, 2, 3$ to 4D.

(iii) A table of the roots x of the linear congruence

$$Rx \equiv D \pmod{B}$$

where R and B range over the first 50 integers and D is the greatest common divisor of R and B (p. 100–101). The reader will find that values of R are given as column headings.

(iv) A table of the residues with respect to each of the moduli 2, 3, 5, and 7 of the first 210 integers arranged in two ways (p. 122–123).

(v) A factor table for each of the first 1000 integers except multiples of 10. To save space composite numbers are usually broken into only two factors, one of which is often a composite number (p. 144–145).

(vi) A factor table for those numbers between 1000 and 10000 which are prime to 30. Beyond 4020 only the least prime factor is given if the number is composite (p. 146–149).

(vii) A condensed factor table for numbers under 210000, not divisible by 2, 3, 5, or 7. This 12-page table (p. 150–161) is reminiscent of an unfinished project of E. Lebon,[1] and is too complicated to describe here in detail. The reader is warned to study directions before attempting to use the table. For composite numbers whose least factor exceeds 210, the table yields the two factors at one "coup d'oeil," that is, after a little hunting. For other numbers some mental calculation involving the factoring of 10 or sometimes 25 three-digit numbers is necessary.

D. H. L.

[1] E. LEBON, *Table de Caractéristiques de Base 30 030 donnant, en un seul Coup d'Oeil, les Facteurs Premiers des Nombres Premiers avec 30 030 et Inférieurs à 901 800 900*, v. 1, pt. 1, Paris, 1920.

361[B].—ALBERT GLODEN, *Table des Bicarrés X^4 pour $1000 < X \leqslant 3000$*, Luxembourg, author, rue Jean Jaurès 11, 1946. Offset printing on one side of each of 17 leaves, with paper cover. 20.5 × 29.7 cm.

There is no text. The values of X^4 for $X = 1001(1)1099$ check with the values given in BAASMTC, *Mathematical Tables*, v. 9, 1940, p. 122–123. Some of the printing is unclear so that a "3," one case tested, $X = 1003$, might easily be mistaken for a "5." This table was made preparatory to writing the paper reviewed in RMT **348**.

R. C. A.

362[D].—JOSEF KŘOVÁK, *Natürliche Zahlen der Funktion Cotangens für Winkel in Zentesimalteilung von 0^{gr} bis 100^{gr}*. Prague, Landesvermessungsamt Böhmen und Mähren, second ed., 1943. iii, 396 p. 15.5 × 21.5 cm.

The one-page explanation (dated Prague, 1943) of these tables says that they had already been announced in the *Sechsstellige Tafeln der natürlichen Werte der Funktionen Sinus und Cosinus für Winkel in Zentesimalteilung* that the Finance Minister had published in the previous year.

The tables were brought into being for use with trigonometrical survey calculations with twin calculating machines where the accuracy of measurement is of the order of 2'' (0''.6) or slightly better. In other words, it provides six significant figure values for working to $0^g.0001$ or 1'' or one centesimal second or about one third of a sexagesimal second. They are a further outcome of Hitler's decree that German surveyors were to use the centesimal division of the quadrant.

The principal survey problem that is facilitated by cotangents and twin machines is that of intersection, i.e. the determination of the co-ordinates of a point whose bearings from two known points have been measured.

The lay-out is shown by the following table:

Pages	From	To	Interval	Diff. for 1''
1– 20	0^g	1^g	1''	None
21–160	1	15	2	63.6 to 1.5
161–230	15	50	10	2.9 to 0.3
231–316	50	93	10	3.2 to 1.5
317–376	93	99	2	1.5 to 15.9
377–396	99	100	1	None

Where the interval is 2'', mean differences are given for 1''; where the interval is 10'', complete proportional parts for centesimal seconds are given.

The tables have been typed and reproduced from photographic plates. The Bremiker division of the lines has been used throughout. No description of the source of the values is given.

It seems hard to justify the use of six significant figures throughout if the accuracy of measurement is limited to about 1''. For small angles, the last one, two or even three figures are meaningless. The same is true of angles near 100^g, where the number of decimals increases steadily to 11. These extra decimals cannot be of any use in survey work, and are only likely to be a source of confusion.

It is a blemish on the arrangement of the part of the table that is at interval 10'' (156 pages) that the centesimal minutes 0 to 50 are on a ***right-hand*** page and 50 to 100 on the following left-hand page; they should, of course, have been printed at a single opening.

L. J. C.

363[D, E, L].—Miklós Imre Hetényi, *Beams on Elastic Foundation. Theory with Applications in the Fields of Civil and Mechanical Engineering.* (*Univ. Michigan Studies. Scientific Series*, v. 16). Ann Arbor, Univ. of Michigan Press, 1946, p. 217–255. 17 × 25.2 cm. $4.50.

The tables include (p. 217–239) graphs and 4D tables $A_x = e^{-x}(\cos x + \sin x)$, $B_x = e^{-x}\sin x$, $C_x = e^{-x}(\cos x - \sin x)$, $D_x = e^{-x}\cos x$, for $x = 0(.001).02(.01)4(.1)8$, $\frac{1}{4}\pi(\frac{1}{4}\pi)\frac{5}{2}\pi$. There are also 5–7S tables with graphs (p. 241–243) of $E_1 = \frac{1}{2}e^x(\sinh x + \sin x)^{-1}$, $F_1 = \frac{1}{2}e^x(\cosh x + \cos x)^{-1}$, $E_{11} = \frac{1}{2}e^x(\sinh x - \sin x)^{-1}$, $F_{11} = \frac{1}{2}e^x(\cosh x - \cos x)^{-1}$, for $x = 0(.05)3(.1)5$.

There are also graphs and 4D tables (p. 245–255) of $Z_1(x) = \operatorname{ber} x$, $Z_2(x) = -\operatorname{bei} x$, $Z_1'(x)$, $Z_2'(x)$, $Z_3(x)$, $Z_4(x)$, $Z_3'(x)$, $Z_4'(x)$ for $x = 0(.01)6$, where

$$Z_3 = \tfrac{1}{2}\operatorname{ber} x - (2/\pi)[R_1 - \operatorname{bei} x(\gamma + \ln \tfrac{1}{2}x)],$$
$$Z_4 = -\tfrac{1}{2}\operatorname{bei} x + (2/\pi)[R_2 + \operatorname{ber} x(\gamma + \ln \tfrac{1}{2}x)], \text{ and}$$
$$R_1 = (\tfrac{1}{2}x)^2 - \frac{\phi(3)}{3!^2}(\tfrac{1}{2}x)^6 + \frac{\phi(5)}{5!^2}(\tfrac{1}{2}x)^{10} - \cdots,$$
$$R_2 = \frac{\phi(2)}{2!^2}(\tfrac{1}{2}x)^4 - \frac{\phi(4)}{4!^2}(\tfrac{1}{2}x)^8 + \frac{\phi(6)}{6!^2}(\tfrac{1}{2}x)^{12} - \cdots, \quad \phi(n) = \sum_1^n 1/k,$$
$$\gamma = .577216\cdots.$$

364[D, P].—Istituto Geografico Militare, Florence, *Tavole per Calcolare le Differenze di Livello nelle Levate Topografiche e per Calcolare le Distanze ridotte all'Orizzonte.* (*Collezione di Testi Tecnici*). Florence, 1943, viii, 195 p. 19.3 × 23.6 cm. Full cloth. The Preface is signed by Prof. Giovanni Boaga, geodetic chief.

L is the distance AB of an object, α the angle of its elevation or depression with reference to the horizontal plane, $D = AC$ the projection of L on this plane, $L' = AE$ the projection of $AC = D$, on L, and $BC = h$.

Table I, p. 3: $D = L\cos\alpha$ to 4D, for $\alpha = 1°(1°)30°$, $L = 1(1)9$.
Table II, p. 14–41: $L' = D\cos\alpha = L\cos^2\alpha$, to 4 or 5S, for $\alpha = 0(5')5°(2')11°(1')19°59'$ and $L = 1(1)9$; also for $L = 1$, $\alpha = 20°(1')45°$.
Table III, p. 43–135: $h = D\tan\alpha$, to 5D, for $D = 1(1)9$, $\alpha = 0(15'')15°(30'')20°(1')45°20'$.
Table IV, p. 191: corrections due to sphericity and refraction, differences of level in meters 1000(100)25900, coefficient of refraction = .06733.
Table V, p. 195: $\tan^2\alpha$, for $\alpha = 0(50')30°50'$, Tables for correction of sphericity and refraction.

The previous edition of this work appeared in 1915 (15.7 × 22.4 cm., 53 p.) and contained four tables. The first, and last two tables are practically equivalent to the first, fourth, and fifth tables of the 1943 edition. T. II (1915) is of the same plan as T. III (1943), but for the range [0(1')45°; 4D]. This 1915 edition was an enlarged and corrected edition of a previously revised and corrected edition, which appeared in 1896.

R. C. A.

365[D, S].—Louis Couffignal, *Tables de Produits de Lignes Trigonométriques.* Paris, Gauthier-Villars, 1943. iii p. + 24 thick paper leaves, printed on only one side. 31.3 × 23.5 cm. Boards, 210 francs.

This volume was prepared under the direction of Dr. Couffignal, the director of the laboratory of mechanical calculation in the Centre National de la Recherche Scientifique.

The author's volume, *Les Machines à Calculer. Leur Principes. Leur Evolutions.* (Paris, 1933, ix, 86 p.) is well known. His doctoral dissertation at Paris was entitled *Sur l'Analyse Mécanique. Application aux Machines à Calculer et aux Calculs de la Mécanique Céleste* (Paris, 1938, 132, 3 p. 4to).

On the back of the title-page of the present volume is a brief preface in French, German and English, and on the opposite page, again in three languages, are "Directions for use of the Tables." We are told that "The establishment of crystal structure from X-ray diagrams demands extensive calculations, where products of two or three cosines occur continually. On the request of several French crystallographers the French National Office of Scientific Research has undertaken to publish tables which might facilitate this kind of work. Besides, such Tables may be useful in a great many cases of harmonic analysis."

The tables give the products $P = f(X) \cdot g(Y) \cdot h(Z)$, where f, g, h are either sine or cosine functions. The arguments X, Y, Z are at interval one hundredth of a circumference, that is, $4^g = 3°.6$. From the table one may read off at once the value of P for any X, Y, Z in 4^g units up to 100^g. For example, to evaluate $P = \sin 41 \cdot \cos 65 \cdot \cos 7$ first turn to $\cos Z = \cos 7$, p. 7 (in the upper right-hand corner of the page). Then on that page columns $\sin X = \sin 41$, $\cos Y = \cos 65$, indicate that $P = -.2850$. The results are all to 4D. The author states that the error in any P is less than $5 \cdot 10^{-5}$.

R. C. A.

366[F].—A. GLODEN, "Compléments aux tables de factorisations de CUNNINGHAM," *Mathesis*, v. 55, 1946, p. 254–256. 16.2 × 25 cm.

The tables referred to are those in which CUNNINGHAM gives[1] (with many incomplete entries) the factors of numbers of the form $x^4 + 1$ for $x < 1000$. The results quoted in this note serve to complete all but 51 entries in this table. Previous addenda by KRAITCHIK[2] and BEEGER[3] are given and have been verified. The new results are by-products of tables of the solutions of the congruence

$$x^4 \equiv -1 \pmod{p}$$

for $p < 500000$ by GLODEN and DELFELD.[4] Those values of x for which $(x^4 + 1)/d$ is a prime between 10^{10} and $25 \cdot 10^{10}$ are listed for $d = 1, 2, 17, 34, 41$, and 82. For some reason the author has failed to list 565 for $d = 1$ and 640, 648 for $d = 2$. Six other factorizations are given for $x = 595, 598, 685, 714, 844, 880$. The author has recently given a similar table[5] to Cunningham's for $1000 < x \leqslant 3000$. The present note closes with a table of the factors of $x^8 + 1$ for $x = 37, 41, 50, 52, 63, 82, 85$, and 87.

D. H. L.

[1] A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1, London, 1923, p. 113–119.
[2] M. KRAITCHIK, *Recherches sur la Théorie des Nombres*, v. 2, Paris, 1929, p. 116–117.
[3] N. G. W. H. BEEGER, *Additions and Corrections to Binomial Factorisations by Cunningham*. Amsterdam, 1933, 1945.
[4] See *MTAC*, v. 1, p. 6; v. 2, p. 71–2, 210–211.
[5] See *MTAC*, v. 2, p. 211.

367[F].—MIKHAIL BORISOVICH OSTROGRADSKIĬ (1801–1861) *Polnoe Sobranie Sochineniĭ Akademika M. B. Ostrogradskogo* [Complete collected works of Academician M. B. Ostrogradskiĭ], v. 2: *Lektsii Algebraicheskogo i Transtsendentnogo Analiza* [Lectures on algebraic and transcendental analysis], Moscow-Leningrad, Academy of Sciences, 1940, 464 p. 17 × 25 cm. Bound, 19 roubles.

This volume of Ostrogradskiĭ's works contains (p. 433–462) his tables of indices and powers of a primitive root modulo p, for all primes under 200. This set of tables first appeared in Akad. Nauk, S.S.S.R., Leningrad, *Mémoires, . . . Sci. Math. Phys. et Nat.* s. 6, v. 3 = *Sci. Math. Phys.*, s. 6, v. 1, "livraison 4," 1836, p. 359–385, and apparently was the first of its kind to be published. These tables were reproduced and extended by JACOBI in 1839 to

$p < 1000$ to form his famous *Canon Arithmeticus* (Compare *MTAC*, v. 1, p. 440). Twelve errata were discovered in Ostrogradskiĭ's tables by Jacobi after the latter had had them set in type:

p	Table	Arg.	*For*	*Read*
71(439)	I	16	15	22
71(439)	I	26	22	15
83(440)	I	25	8	80
127(447)	N	105	107	108
127(447)	N	116	31	71
137(449)	N	108	88	87
167(455)	I	57	128	28
173(456)	I	57	72	92
181(458)	I	16	165	172
181(458)	I	26	172	165
181(458)	N	78	94	64
193(460)	N	155	173	174

More than a century later the tables are now reproduced with the same old errata.

It should be noted that Ostrogradskiĭ's tables give also all the primitive roots of each prime, information not presented in Jacobi's *Canon*.

D. H. L.

Editorial Note: Ostrogradskiĭ's portrait is on a plate opposite p. 64 of *Les Mathématiques dans les Publications de l'Académie des Sciences 1728–1935. Répertoire Bibliographique*, Moscow, Academy of Sciences, 1936. In this v., p. 108, the date of publication of Ostrogradskiĭ's "Tables des racines . . ." is given incorrectly as 1838. The blue cover of "livraison 4" in the Harvard University library copy is dated 1836.

368[F].—Wilhelm Patz, *Tafel der regelmässigen Kettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen von 1–10000*. Leipzig, Akademische Verlagsgesellschaft, 1941. Lithoprinted by Edwards Bros., Ann Arbor, Michigan, 1946, xvi, 282 p. 15 × 22.9 cm. $6.50. Published and distributed in the public interest by authority of the Alien Property Custodian under license number A-412.

The regular continued fraction representing the square root of a positive non-square integer has been the subject of much experimental work and theoretical investigation since the time of Euler. This table will serve as a useful tool in the further work along these lines. For each positive non-square $D \leqslant 10002$ are given the periodic partial quotients $b_1, b_2, \cdots, b_p$ in the expansion

$$\sqrt{D} = b_0 + \frac{1}{b_1 +}\,\frac{1}{b_2 +}\cdots\frac{1}{b_p +}\,\frac{1}{b_1 +}\cdots$$

Here $b_p = 2b_0 = 2[\sqrt{D}]$. Since $b_k = b_{p-k}(k > 0)$, it suffices to give b_k for $k \leqslant \frac{1}{2}p$ and b_p; and this is done except when $p \leqslant 6$. In a majority of cases p is even ($p = 2q$) and b_q is printed with an asterisk. Thus for $D = 178$ and 209 the entries are

$$178|13(2, 1, 12^*, 1, 2, 26), \quad 209|14(2, 5, 3, 2^*, \cdots, 28).$$

In case p is odd a diamond is printed before the expansion. Those values of D which are primes are followed by a small p in the argument column.

The usual method of expanding $\sqrt{D}$ is explained on p. xi–xiii. Writing the nth complete quotient in the form

$$x_n = (\sqrt{D} + P_n)/Q_n = b_n + x_{n+1}^{-1}$$

we have the four formulae of recurrence

$$b_n = (b_0 + P_n)Q_n^{-1},\ P_{n+1} = b_nQ_n - P_n,\ Q_{n+1} = b_n(P_{n+1} - P_n) + Q_{n-1},$$
$$Q_{n+1} = (D - P_{n+1}^2)/Q_n$$

The last of these was used as an "automatic check" on the exactness of the calculation.

It is perhaps worth noting that the first two formulae may be replaced to advantage by the following pair whenever the value of b_n is not at once obvious

$$b_0 + P_n = b_n Q_n + r_n, \quad P_{n+1} = b_0 - r_n,$$

where $0 \leqslant r \leqslant Q$.

The usefulness (and also the number of pages) of this volume would have been more than doubled had the author included the denominators Q_n, as given in the tables of DEGEN,[1] CAYLEY[2] and WHITFORD.[3] These numbers are important in the application of continued fractions, especially to the diophantine equation of Lagrange

$$x^2 - Dy^2 = N$$

Besides this application the importance of this table lies in the wealth of statistical information it gives about the expansion of square roots of integers. The table throws some light on the unsolved questions of whether the period p is even or odd, whether, for a given D, the central partial quotient has the value b_0 or $b_0 - 1$ or not, whether $p < c\sqrt{D}$; and so on. Inspection of the latter part of the table reveals quite a large number of very long expansions. There are 31 values of D for which the period exceeds $2\sqrt{D}$. These range from 1726 to 9949 with periods of 88 and 217 respectively. The ratio $p/\sqrt{D}$ reaches a maximum of 2.2245 at $D = 7606$. Thus there is still room for the conjecture that, for all D, $p < \sqrt{5D}$. These long expansions appear to have more than their share of unit values among their b's. In fact more than 43.62 percent of their partial quotients are equal to unity. The average for all real numbers is only $\log_2 (4/3) = .41503$.

There are three errata listed on p. xiv: $D = 2872$, *for* 1, 2, 2, 4, *read* 1, 2, 4; $D = 4170$, *for* 2, 1, 3, 3, *read* 2, 1, 4, 3; $D = 4966$, *for* 1, 4, 1, 2, *read* 1, 4, 2, 2. The first still occurs in the 1946 edition while the last two have been corrected. Nevertheless the above list is given. This confusing bit of editing led the reviewer to recalculate the expansions for $D = 4170$ and 4966. BEEGER has pointed out (*MTAC*, v. 2, p. 88) that in the 1941 edition the diamond sign is printed one line too low at $D = 6938$, 6949, 6953, and 9698. The first three of these misprints occur also in the present edition but the diamond is two lines too low at 9697 and is missing at 9698. The author has compared his table with those of DEGEN ($D \leqslant 1000$), CAYLEY ($1001 \leqslant D \leqslant 1500$), WHITFORD ($1501 \leqslant D \leqslant 2012$) and THIELMANN[4] (about 140 isolated D's under 10^4). No errata in these tables are quoted although the first and third are known to contain 2 and 4 erroneous continued fraction expansions respectively.[5] The table of ROBERTS[6] for all primes $D = 4n + 1 < 10^4$ was not available to the author. A comparison of these two tables would give a very good idea of the reliability of the one under review.

D. H. L.

[1] C. F. DEGEN, *Canon Pellianus* . . ., Copenhagen, 1817.

[2] A. CAYLEY, "Report of a committee appointed for the purpose of carrying on the tables connected with the Pellian equation from the point where the work was left by Degen in 1817," BAAS, *Report, 1893*, p. 73–120; also *Collected Mathematical Papers*, v. 13, 1897, p. 430–467. [These tables were computed by C. E. BICKMORE.]

[3] E. E. WHITFORD, *The Pell Equation*, New York, 1912, p. 164–190.

[4] M. VON THIELMANN, "Zur Pellschen Gleichung," *Math. Annalen*, v. 95, 1926, p. 635–640.

[5] D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, 1941, p. 138, 171.

[6] C. A. ROBERTS, "Table of the square roots of the prime numbers of the form $4m + 1$ less than 10000 expanded as periodic continued fractions," *Math. Magazine*, v. 2, p. 105–120, 1892.

369[F].—HEINRICH TIETZE, "Einige Tabellen zur Verteilung der Primzahlen auf Untergruppen der Gruppe der teilerfremden Restklassen nach gegebenem Modul," Akad. d. Wiss., Munich, *Abh.*, *Math. Nat. Abt.*, n.s., no. 55, 1944. 31 p. 22.3 × 28.5 cm.

This paper contains 26 short tables giving information about the distribution of primes in certain sets of arithmetical progressions the last term of which is denoted by L. The actual forms considered are $km + r_j (j = 0, 1, \cdots)$ for the 26 following values of m and r_j:

Table	m	$\phi(m)$	r_j	L
1	10	4	1, 9	571
2	10	4	1	571
3	8	4	1	449
4	8	4	1, 3	457
5	8	4	1, 5	449
6	8	4	1, 7	449
7	9	6	1, 4, 7	487
8	9	6	1, 8	487
9	30	8	1, 29	3511
10	30	8	1, 11	3511
11	30	8	1, 17, 19, 23	1913
12	30	8	1, 7, 13, 19	1879
13	30	8	1, 11, 19, 29	1901
14	26	12	1, 3, 9, 17, 23, 25	1091
15	26	12	1, 3, 9	1069
16	26	12	1, 25	1091
17	26	12	1	1093
18	262	130	Quadratic residues	3931
19	262	130	5th power residues	3929
20	262	130	10th power residues	3911
21	262	130	13th power residues	3929
22	262	130	26th power residues	3467
23	262	130	65th power residues	3929
24	262	130	1	298943
25	262	130	259	298153
26	262	130	17	297911

Under multiplication modulo m, the set of r's in each case forms a group Γ, in fact a subgroup of the group H of the $\varphi(m)$ numbers $\leqslant m$ and prime to m.

Let $\Pi_H(x)$ and $\Pi_\Gamma(x)$ denote the number of primes $\leqslant x$ belonging respectively to H and Γ modulo m. If there are h elements of Γ and if $\phi(m) = hi$, then, according to the prime number theorem (generalized), $\Pi_H(x)$ and $i\Pi_\Gamma(x)$ are asymptotically equal and approach $\phi(m)x/m \ln x$. The tables give values of these step functions together with the difference

$$\Delta(x) = \Pi_H(x) - i\Pi_\Gamma(x)$$

for $x \leqslant L$.

Since $\Delta(x)$ is a step function it suffices to tabulate it only at the values of x where it changes value, that is at primes belonging to Γ (mod m). These primes are denoted by N and form the arguments of the tables. This makes $i\Pi_\Gamma(N)$ merely a list of consecutive multiples of i, and this column might well have been omitted. As x varies from one value of N to the next, $\Delta(x)$ increases because $\Pi_H(x)$ increases. The value of $\Delta(x)$ just before the next value of N is denoted by $\Delta^*(N)$ and is tabulated also.

The modulus 262 is chosen because 131 is the least prime having 3, 5, 7, 11, 13 as quadratic residues, 17 and 259 are the least and greatest primitive roots of 131. The last three large tables, especially table 24 might some day prove useful as a list of primes of these forms. Cunningham's observation that the form $km + 1$ contains fewer primes $\leqslant x$ than $km + l$, $l \neq 1$, (l, m coprime) does not seem to hold for $m = 262$. In fact $\Delta(N)$ in table 24 changes sign very often.

D. H. L.

370[F].—I. M. VINOGRADOV, *Osnovy Teorii Chisel* [Fundamentals of the Theory of Numbers], Moscow-Leningrad, (**a**) third ed., 10 000 copies, 1940, 111 p. + an errata sheet. 12.7 × 18.7 cm. Bound, 3 roubles. (**b**) Fourth ed., 3 000 copies, 1944, 142 p. + an errata sheet. 13.8 × 20 cm. Paper bound, 4 roubles.

(a) This interesting little volume contains two kinds of tables:

(1) Tables of indices and powers of a primitive root modulo p for $p < 100$ (p. 104–109). These are based on least primitive roots and so are identical with tables of WERTHEIM,[1] and USPENSKY & HEASLET.[2] A comparison with the latter table reveals no discrepancy.

(2) Table of least primitive roots of primes $p < 3000$ (p. 110–111).

Three errata may be noted: $p = 1013$, *for* 2, *read* 3; $p = 2593$, *for* 10, *read* 7; $p = 2999$, *for* 7, read 17.

(b) In this edition the first group of tables (p. 135–140) is the same as in the third edition but the table of least primitive roots of primes < 3000, in the third edition, has been corrected and enlarged to primes < 4000 (p. 141–142).

D. H. L.

[1] G. WERTHEIM, *Aufgangsgründe der Zahlenlehre*. Brunswick, 1902, p. 412–417.
[2] J. V. USPENSKY & M. A. HEASLET, *Elementary Number Theory*. New York and London, 1939, p. 477–480.

371[G, L].—A. COLOMBANI, "La théorie des filtres électriques et les polynomes de Tchebichef," *Jn. de Physique et de Radium*, s. 8, v. 7, Aug. 1946, p. 231–243. 21.3 × 26.6 cm. Compare *MTAC*, v. 1, p. 125, 149f, 385, RMT **381**, **383**.

There are two tables for so-called Chebyshev polynomials, p. 236–237. **T. I** gives $S_n(x)$ for $x = -2(.1)0$, $n = [2(1)10; n\text{D}]$. Also zeros of S_n to 5D. **T. II** gives values of $X_n(x) = S_n(x) - S_{n-1}(x)$, for $x_1/x_2 = x - 2 = -4(.1)0$, $x = -2(.1)2$, $n = [1(1)10; n\text{D}]$. Also zeros of $X_{10}(x)$ to 5D. Figs. 2–4, p. 234–235, are graphs of $X_n(x)$ for $n = 1(1)10$.

$$S_n(x) = \sin(n+1)\theta/\sin\theta, \quad x = 2\cos\theta.$$

372[I].—H. E. SALZER, "Coefficients for facilitating the use of the Gaussian quadrature formula," *Jn. Math. Physics*, v. 25, 1946, p. 244–246. 17.5 × 25.5 cm.

In the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} a_i f(x_i) + R_n$$

the sum extends over the roots x_i of the Legendre polynomial $P_n(x)$. As these roots are not equally spaced, it is not possible to test the smoothness of a set of computed ordinates $f(x_i)$ by straight-forward differencing, as one would do in the case of a Cotes type formula.

To examine the $(n-1)$st difference of this set of ordinates one must resort to divided differences. Much of the cumbersome calculation attending the general divided difference process can, in this case, be avoided. In fact this difference can be written

$$\sum_{i=1}^{n} C_i^{(n)} f(x_i)$$

where the coefficients are simply

$$C_i^{(n)} = 2^{-n}\binom{2n}{n}/P_n'(x_i).$$

This paper contains a small table of these coefficients for $n = 3(1)10$. The accuracy is 8D for $n = 3, 4, 5$; 7D for $n = 6, 7, 8$; and 6D for $n = 9, 10$. The author fails to indicate that in his notation the roots x_i are so ordered that

$$x_1 < x_2 < \cdots < x_n,$$

a fact which the user of the table will need to know.

D. H. L.

373[I, L].—H. A. RADEMACHER & I. J. SCHOENBERG, "An iteration method for calculation with Laurent series," *Quart. Appl. Math.*, v. 4, July 1946, p. 142–159. 17.5 × 25.4 cm.

In the authors' words, "the purpose of this paper is to describe a method whereby rational or algebraic operations with Laurent series may be performed with high accuracy at the expense of a reasonable amount of labor."

The main problem considered is that of solving numerically

$$f(w, z) \equiv a_0(z)w^m + a_1(z)w^{m-1} + \cdots + a_m(z) = 0 \tag{1}$$

for the coefficients of the Laurent series of a particular branch of $w(z)$, where the $a_i(z)$ are regular and uniform functions of z in the ring

$$R: \quad r_1 < |z| < r_2$$

and where neither $a_0(z)$ nor the discriminant $D(z)$ of (1) is zero in R.

The procedure suggested by the authors is first to find an initial approximation and then to use an iteration scheme based on a modification of Newton's algorithm. In this modification only one division is needed and that is a preliminary one. A method is given of obtaining by trigonometric interpolation a Laurent polynomial,

$$F_n(z) = \sum_{j=-n}^{n} c_{n,j}z^j,$$

as a first approximation. It is proven that $F_n(z)$ approaches the solution as $n \to \infty$. One of the points of this paper is that it is preferable to start with a small value of n and then to iterate rather than to use $F_n(z)$ for a large value of n.

The iteration scheme is as follows: Since the discriminant $D(z)$ is not zero there are polynomials $\phi(w)$ and $\psi(w)$, with coefficients which are polynomials in the $a_i(z)$ divided by $D(z)$, such that,

$$\phi(w)f(w, z) + \psi(w)\frac{df(w, z)}{dw} = 1.$$

The modified Newton's algorithm can now be expressed by the recurrence formula

$$w_{r+1} = w_r - f(w_r, z)\psi(w_r).$$

This has the usual quadratic convergence of the Newton algorithm. The authors state that this method has been used previously by Schwertfeger for the numerical solution of ordinary algebraic and transcendental equations.

The authors show that in the special case of solving

$$a(z)w(z) - 1 = 0$$

the recurrence formula reduces to

$$w_{r+1} = w_r(2 - aw_r),$$

which is the formula described by Hotelling for inverting matrices. Therefore, in reciprocation of a Laurent series one can use an inequality of Hotelling and Lonseth to obtain a limit for the error due to stopping after any number of steps.

As an illustration the authors compute the coefficients w_n of the Laurent expansion of the reciprocal of $-J_0(\sqrt{13z})$ between the first two positive roots of that function. The entire computation to 9D is exhibited in tabular form for w_n with $-29 \leqslant n \leqslant 32$. The remaining coefficients are numerically smaller than 10^{-9}. The paper also contains a description of how the methods of calculating with Laurent series apply to calculations with absolutely convergent Fourier series.

In conclusion one can say that this article presents in a very convenient form a solution to the problem considered, especially for those who will have to do actual computations of this sort.

ABRAHAM HILLMAN

NBSMTP

374[K].—LA MONT C. COLE, "A simple test of the hypothesis that alternative events are equally probable," *Ecology*, v. 26, 1945, p. 204. 16.5 × 25.4 cm.

Table III, values of $P = 2^{1-n}\sum_0^k \frac{n!}{E!(n-E)!}$, for $k = 0(1)12$, $n = [2(1)35; 5D]$.
"Table gives proportion in *both* tails of $(\frac{1}{2} + \frac{1}{2})^n$. For larger values of n use $t = (n - 2E)n^{-\frac{1}{2}}$. In general, a value is statistically significant ($P < 0.05$) if $E \leqslant \frac{1}{2}n - n^{\frac{1}{2}}$."

Extracts from text

375[K].—FREDERICK E. CROXTON & DUDLEY J. COWDEN, "Tables to facilitate computation of sampling limits of s, and fiducial limits of sigma," *Industrial Quality Control*, v. 3, July 1946, p. 18–21. 21.6 × 27.9 cm.

For samples of size N drawn from a normal distribution with known variance σ^2, upper and lower percentage points of the distribution of s/σ are given in **Table 1** entitled: "Values of s/σ at selected probability points for various sample sizes." The sample standard deviation is $s = [\sum (x - \bar{x})^2/N]^{\frac{1}{2}}$. The probability points of the distribution of s/σ are given in pairs for probabilities α and $1 - \alpha$, with $\alpha = .001, .005, .01, .025, .05, .10$, for $N = [2(1)30; 3D]$. An approximation is given for $N > 30$. A table similar to the present one for $\alpha = .001, .005$ and for $N = 2(1)15$ is given in Amer. Standards Assoc., *Control Chart Method of Controlling Quality During Production*, no. ASA Z 1.3—1942, p. 40.

Table 2, entitled "Values of σ/s for use in computation of selected fiducial limits of σ for various sample sizes," may be used to obtain confidence or fiducial limits for σ. Confidence levels available are .998, .99, .98, .95, .90, .80 for $N = 2(1)30$. The entries in this table are principally reciprocals of the entries of **Table 1**. A similar table for confidence levels .90, and .98, $N = 5(1)30$, appears in E. S. PEARSON, *The Application of Statistical Methods to Industrial Standardisation and Quality Control*, London, British Standards Institution, 1935, p. 69.

The tabulated values of the tables under review were derived principally from CATHERINE M. THOMPSON, "Table of percentage points of the χ^2 distribution," *Biometrika*, v. 32, p. 187f, 1941. (See *MTAC*, v. 1, p. 78). But the .999 points of **Table 1** were derived from R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural, and Medical Research*. London, 1938, Table IV, p. 27, while the .001 points of **Table 1** were derived from tables of F shown in F. E. CROXTON & D. J. COWDEN, *Applied General Statistics*. New York, 1939, p. 878–879.

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EDITORIAL NOTE: It may be remarked that it was exactly on the pages quoted as sources, FISHER & YATES, 1938, p. 27, and CROXTON & COWDEN, 1939, p. 878, that we have listed errors in the tables in question, namely: *MTAC*, v. 1, p. 324, and 86.

376[K].—V. L. GONCHAROV, *Teoriīa Veroīatnosteĭ* [Theory of Probabilities]. Moscow and Leningrad, 1939, 427 p. + errata slip. 14.4 × 21.7 cm. Bound, 11 roubles. An edition of 5000 copies.

This government ordnance industry publication has three small tables on its last seven pages. In the notation of the FMR, *Index*, these are

(a) $H(x) = 2\pi^{-\frac{1}{2}}\int_0^x e^{-t^2}dt$, for $x = [0(.05)2.2; 4D]$, $[2.2(.05)2.75, 3; 6D]$, $[3.5, 4; 9D]$.
(b) $H(\rho x)$, where $\rho = .4769362762\cdots$ is the root of $H(x) = \frac{1}{2}$, for $x = [0(.01)3.4(.1)5.4; 5D]$.
(c) $xH(\rho x) + \rho^{-1}\pi^{-\frac{1}{2}}e^{-\rho^2x^2}$ for $x = [0(.05)5.2; 4D]$.

All three tables give first differences.

Strange to say, the very well known function $H(x)$ is not tabulated correctly in (a). There are two errata: $t = 2.4$, *for* .999312, *read* .999311; $t = 2.7$, *for* .999868, *read* .999866. A last-figure error occurs in (b); in the final entry $t = 5.40$, *for* 99972, *read* 99973.

The function tabulated in (c) is essentially the second iterated integral:

$$\frac{1}{\rho\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\int_0^x \int_0^{\rho t} e^{-\theta^2} d\theta dt.$$

This table appears to have several last-figure errors. For example, $t = 1.05$, *for* 1466, *read* 1468; $t = 1.10$, *for* 1493, *read* 1494.

For a discussion of the tables the reader may consult p. 150 and 217.

D. H. L.

377[K].—*Mrs.* CATHERINE M. (THOMPSON) GRYLLS & *Mrs.* MAXINE MERRINGTON, "Tables for testing the homogeneity of a set of estimated variances," *Biometrika*, v. 33, June 1946, p. 302–304. A preface by H. O. HARTLEY & E. S. PEARSON occupies p. 296–301. 19.3 × 27.3 cm.

These tables are designed to provide 1 per cent. and 5 per cent. points for testing the hypothesis that the variances of several normal populations, as estimated from two or more independent observations on each, are equal; or hypotheses equivalent thereto. The basis for computation was an approximation of HARTLEY[1] to the distribution of a statistic suggested by BARTLETT[2] which differs in weighting factors from the likelihood ratio statistic proposed by NEYMAN & PEARSON,[3] and which has been found more powerful in certain cases by BISHOP & NAIR.[4]

Hartley's approximation depends on three parameters: the number of populations k, and two functions of the degrees of freedom ν_t of the estimates of the population variances,

$$c_1 = \sum_t \left(\frac{1}{\nu_t}\right) - \frac{1}{N}, \quad c_3 = \sum_t \frac{1}{\nu_t^3} - \frac{1}{N^3}, \quad \text{where} \quad N = \sum_t \nu_t$$

The percentage points vary but little with c_3. The tables are double entry, giving for each pair k and c_1, two values which are approximately the extremes with respect to variation in c_3. An auxiliary table aids interpolation with c_3 in the occasional case when these extremes cover the computed statistic.

The percentage points are given to 2D(3 or 4S) for $k = 3(1)15$ and (the entire range of) $c_1 = 0(.5)5(1)10(2)14$. A historical note, several illustrative examples and a discussion of the accuracy of the approximation, are provided in the preface. It is found that the approximation is "very good" if the degrees of freedom all exceed 2, and is "adequate" if some are as small as 2.

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[1] H. O. HARTLEY, "Testing the homogeneity of a set of variances," *Biometrika*, v. 31, 1940, p. 249–255.

[2] M. S. BARTLETT, "Properties of sufficiency and statistical tests," R. Soc. London, *Proc.*, v. 160A, 1937, p. 268–282.

[3] J. NEYMAN & E. S. PEARSON, "On the problem of k samples," Akad. umiejetności, *Bull. Intern.*, 1931A, p. 460–481.

[4] D. T. BISHOP & U. S. NAIR, "A note on certain methods of testing for the homogeneity of a set of estimated variance," R. Statist. Soc., *Jn.*, v. 102, *Suppl.* v. 6, 1939, p. 89–99.

378[K, V].—CECIL HASTINGS & MARGARET PIEDEM, *Miscellaneous Probability Tables*, calculated and checked under the direction of Dr. H. H. GERMOND, 1942–1944. Applied Mathematics Panel, National Defense Research Committee, Note no. 14, New York, July 1944. ii, 65 p. Offset print. 21.3 × 27.8 cm. These tables are not available for public distribution.

The main table (p. 6–37 and introduction p. 1–5) is devoted to the two-dimensional normal distribution function (or error function)

$$V(h, q) = (2\pi)^{-1} \int_0^h \int_0^{qx/h} e^{-\frac{1}{2}W} dydx, \; W = x^2 + y^2.$$

The double entry table to 5D has h and q/h as independent variables, the former appearing in rows, the latter in columns; $h = 0(.01)4$, $q/h = .1(.1)1$. Differences are tabulated in the main rows and columns between the corresponding entries. The differences within columns have usually only one digit and never exceed 85 units; the differences in the rows keep to three digits. The need for the tables arose in probability problems associated with bombing and fragmentation damage. NBSMTP supplied the key values of $V(h, q)$ from which the Table was subtabulated. The computation was carried on for some time before the appearance of C. NICHOLSON, "The probability integral for two variables," *Biometrika*, v. 33, part 1, April, 1943, p. 59–72, where a table of $V(h, q)$, to 6D, is given for $h = .1(.1)3$, $q = .1(.1)3, \infty$.

Pages 39–45 cover tables for $H(x) - xH'(x)$, where $H(x)$ is defined by

$$H(x) = 2(\pi)^{-\frac{1}{2}} \int_0^x e^{-t^2} dt.$$

The range is [0(.001)3(.01)4.3; 7D]; for $x > 4.3$ the entries would equal unity throughout. (The entries in the first column on p. 45 are misprinted: the last zero should be deleted everywhere, and the last line on p. 44 should be deleted. This table was computed by using NBSMTP, *Tables of Probability Functions*, v. 1, 1941.) It is stated that the rounding errors will occasionally amount to one unit in the last digit. There follow, p. 46–49, tables of the inverse of the function $H(x) - xH'(x)$ covering the entire significant range, namely [0(.001)1; 5D].

The next table, p. 51–62, gives values of the function $y = 1 - (1 + x)e^{-x}$. These tables will be useful in particular in connection with the Poisson distribution. The range is [0(.001)5(.01)10(.1)15; 5D]. Again, y is practically constant for $x > 15$.

Two small tables conclude the collection. On p. 63 we find values of the product xy where y is defined by the equation

$$xy = 1 - e^{-y}.$$

The range is x = [0(.01)1; 5D], Δ^2. Finally, on p. 65 is a table of

$$\phi(x) = xe^{-x^2} \int_0^x e^{t^2} dt$$

for the range [2(.1)7; 5D], Δ^2.

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EDITORIAL NOTE: In our notes on Dawson's or Poisson's integral we have listed an earlier table of $\phi(x)$, *MTAC*, v. 1, p. 323, N. KAPZOV & S. GWOSDOWER, *Z. f. Physik*, v. 45, 1927, p. 133. This table is for the range x = [.1, .5, .8(.2)1.2(.05)2.2; 5D]; Δ, 1.5–2.2. The use for such a table there, arose in discussion of oscillations in electron tubes.

379[L].—D. CHALONGE & V. KOURGANOFF, "Recherches sur le spectre continu du soleil," *Annales d'Astrophysique*, v. 9, 1946, p. 69–96. 21.5 × 27.4 cm.

Appendix I contains two tables of "la fonction Γ incomplète d'argument négatif":

$$\tilde{\Gamma}_x(\alpha) = \int_0^x t^{\alpha-1} e^{+t} dt$$

T. A, p. 94. x = [0(.01).1; 4D]; α = [.1(.1)1],
T. B, p. 95–96. α = [.01(.01)1; 4D]; x = [0(.1)1.1].

The authors state that the tables constitute "un extrait, pour le domaine qui nous intéresse ici, d'une table plus étendue qui paraîtra prochainement." No details of the calculation are given.

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380[L].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. 3: *Tables of the Bessel Functions of the First Kind of Orders Zero and One;* v. 4: *Tables of the Bessel Functions of the First Kind of Orders Two and Three*, by the Staff of the Laboratory, Professor H. H. AIKEN, Technical Director, Cambridge, Mass., Harvard Univ. Press, 1947. ii, xxvii, 652 p. and viii, 652 p. 19.5 × 26.7 cm. $10.00 + $10.00. Compare *MTAC*, v. 2, p. 176f, 185f. The offset printing of these volumes is of outstanding excellence.

P. iii, "Staff of the Computation Laboratory" [11 members and 13 assistants listed].

P. vi, "Preface" by Professor Aiken. Quotations: In Nov. 1944 a conference at the Naval Research Laboratory was called to discuss the tabulation of Bessel functions of the first kind and of high order, which at that time were needed in connection with various research problems of interest to the Navy. As a result the computation project of the Bureau of Ships was directed to tabulate the required functions. During discussion it became clear that if $J_{100}(x)$ was to be accurate to ten decimal places, an adding, multiplying, and storage capacity of more than forty digits would be required of the Automatic Sequence Controlled Calculator. Since the multiplying unit of the calculator already supplied forty-six digits and the algebraic sign, it was only necessary to link two normal storage registers, each comprising twenty-three digits and the algebraic sign, to form a single adding storage register covering forty-six digits and the algebraic sign.

On the first of January 1946, the Computation Project was transferred to the Bureau of Ordnance.

P. ix–xxii, "Introduction" by RICHARD M. BLOCH

Part I. The Bessel Functions, p. ix–xi,
Part II. The Computation of the Tables, p. xii–xviii,
Part III. Interpolation in the Tables, p. xix–xxii.

Part I. Because of the important applications of Bessel Functions to many physical phenomena, they have been the subject of intensive investigation for many years. Recent advances in the theory of frequency modulations, resonance in cavities, waves in various media, vibration theory of structures, and other problems of physics and engineering have greatly increased the need for extensive tables of $J_n(x)$ and $Y_n(x)$ covering a large range both of the order and of the argument.

The Staff of the Computation Laboratory is at present engaged in the tabulation of $J_n(x)$, $0 \leqslant x < 100$, $n = 0(1)100$. The present volumes contain tables for $n = 0(1)3$, $x = [0(.001)25(.01)99.99; 18D]$.

The Bessel functions satisfy the two relations

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x). \tag{1}$$

For machine computation, the successive application of the recurrence formula (1) provides the most feasible method of obtaining the high order functions. Since ten decimals place accuracy is to be maintained in the tables of $J_n(x)$ for $4 \leqslant n \leqslant 100$, it was necessary to investigate the cumulative loss of accuracy which arises in the repetitive use of (1) as the computation proceeds. If $J_{m-1}(x)$ and $J_m(x)$ ($m = 3$ for $x < 2$, $m = 2$ for $x \geqslant 2$) are the two basic functions upon which the recurrence is constructed, the maximum number of decimal places lost is eleven. Consequently the low order functions $J_0(x)$, $J_1(x)$ and $J_2(x)$ were computed correct to twenty-three places of decimals. Since the figures were available, the tables of $J_n(x)$ ($n = 0, 1, 2, 3$) have been printed to 18D, despite the fact that interpolation within these tables to full accuracy would be extremely difficult with the present manual aids to computation.

The Automatic Sequence Controlled Calculator is so arranged that after all final results are automatically checked, they are printed by the typewriters controlled by the machine itself. Certain tables of Bessel functions to eighteen or more D were read against the values computed at the Computation Laboratory. These comparisons were made with the values of the functions listed in the following three tables: MEISSEL, as in GRAY, MATHEWS, & MACROBERT, *A Treatise on Bessel Functions*, 1931, p. 286–299, 18D, $1 \leqslant x \leqslant 24$, $\Delta x = 1$, $n = 0(1)3$; HAYASHI, *Tafeln der Besselschen, Theta-, Kugel-, und anderer Funktionen*, 1930, p. 52–59, $n = 0(1)3$, $.01 \leqslant x \leqslant 100$, selected Δx, 22–103D; ALDIS, R. Soc. London, *Proc.*, v. 66, 1900, p. 40–43, $n = 0, 1$; $0 \leqslant x \leqslant 6$, $\Delta x = .1$, 21D. No discrepancies were observed. ***Part II.*** Values of $J_n(x)$ given in other tables were not used. All numerical constants including the coefficients of the ascending power series, the asymptotic series and those required for interpolation, were evaluated at the Computation Laboratory, regardless of the availability of such material from external sources.

The ascending power series

$$J_n(x) = \sum_{r=0}^{\infty} k^{2r+n} a_{r,n} \left(\frac{x}{2k}\right)^{2r+n},$$

where $a_{r,n} = (-1)^r/r!(n+r)!$, and k is a normalizing factor, was used to evaluate $J_n(x)$ for $n = 0(1)3$ over the range $0 \leqslant x < 2$ with increment $\Delta x = .001$, and for $n = 0(1)2$ over the range $2 \leqslant x \leqslant 25$ with increment .01. The values of $10^m \cdot a_{r,n}$ computed to 50D, are given for $r = 1(1)60$, $0 \leqslant m \leqslant 168$, $n = 0(1)2$.

For the range $0 \leqslant x < 2$, $k = 1$; $2 \leqslant x < 10$, $k = 5$; $10 \leqslant x < 20$, $k = 10$; $20 \leqslant x \leqslant 25$, $k = 20$. For $n = 3$ we have $|10^m \cdot a_{r,3}|$, $r = 0(1)15$, $1 \leqslant m \leqslant 28$, p. xxiii–xxxi.

There are similar tables (p. xxxii–xxxvii) for the various asymptotic expansions.

Extracts from introductory text

381[L].—C. W. JONES, J. C. P. MILLER, J. F. C. CONN, & R. C. PANKHURST, "Tables of Chebyshev polynomials," R. Soc. Edinb., *Proc.*, v. 62A, no. 21, 1946, p. 187–203. 17.5 × 25.5 cm.

The main object of the article under review is to present a table of the Chebyshev polynomials $C_n(x) = 2\cos(n \arccos \frac{1}{2}x)$ for $n = 1(1)12$ and $x = 0(.02)2$. The tabulated values are either exact or given to 10D. In addition to the table of $C_n(x)$, the article contains also short tables of the functions $(4 - x^2)^{\frac{1}{2}}$, $(2 + x)^{\frac{1}{2}}$, $(2 - x)^{\frac{1}{2}}$ and $\arccos \frac{1}{2}x$ required in the applications of Chebyshev polynomials discussed in Dr. Miller's article, "Two numerical applications of Chebyshev polynomials" (RMT **383**).

The tabular material is preceded by an excellent introduction giving the definition of the Chebyshev polynomials $C_n(x)$ and $S_n(x)$ and of other related functions, the differential equations and recurrence relations satisfied by these functions, the explicit power series expressions of these functions, the expressions of the twelve powers of x in terms of Chebyshev polynomials $C_n(x)$, the orthogonality relations and the generating functions for each of the functions under consideration.

The reviewer agrees with the authors' remarks that the tables will be of particular importance to computers. One application particularly worth mentioning is the process of interpolation by means of Chebyshev polynomials (RMT **383**). The efficacy of this process of interpolation is illustrated by the following observation: In a certain region of the Mathieu functions ms. in preparation by the NBSMTP, interpolation to the full accuracy of the table would require the use of a formula involving differences up to the ninth order; the corresponding interpolation formula in terms of $C_n(x)$ requires only the first five Chebyshev polynomials.

Forty percent of the entries of the table under review were proofread against the corresponding entries in the more extensive table of Chebyshev polynomials prepared by the NBSMTP (see *MTAC*, v. 1, p. 125); no discrepancies were discovered.

ARNOLD N. LOWAN

382[L].—N. W. McLACHLAN (a) "Computation of the solution of Mathieu's equation," *Phil. Mag.*, s. 7, v. 36, June 1945 (publ. Jan. 1946), p. 403–414. 17 × 25.5 cm. (b) "Mathieu functions and their classification," *Jn. Math. Phys.*, v. 25, Oct., 1946, p. 209–240. 17.3 × 25.4 cm.

(a) This paper deals with the computation of the solutions of

$$\frac{d^2y}{dz^2} + (a - 2q\cos 2z)y = 0 \tag{1}$$

where a, q are real parameters. For certain characteristic values, the solutions of (1) are periodic, of period π or 2π. Those characteristic values which give rise to even solutions of period π and 2π are denoted by a_{2m} and a_{2m+1}, respectively. The characteristic values giving rise to odd solutions of period 2π and π are denoted by b_{2m+1} and b_{2m+2}, respectively. The curves $a = a_m(q)$ and $a = b_m(q)$ separate the $(a - q)$ plane[1] into regions in which the solutions are "stable" or "unstable." When the parametric point (a, q) lies between a_m and b_{m+1}, the solution is "stable"; that is, two independent solutions of (1) may be written in the form

$$y = \sum_{r=-\infty}^{\infty} c_r \frac{\cos}{\sin} (2r + p + \beta)z. \tag{2}$$

In (2), $p = 0$ if the subscript m in a_m is even, and $p = 1$ if m is odd. According to currently accepted theory, there exists a unique positive value of β less than unity corresponding to every point (a, q) in this region, such that the solution (2) remains finite as z approaches infinity through real values.

The value of a which, for a fixed q, determines β, will be denoted by $a_{m+\beta}$. Between $a_m(q)$ and $b_{m+1}(q)$ there lies a family of iso-β curves, i.e., β = constant. When β turns out to be a proper fraction p/s (in its lowest terms), the solution will be periodic, of period $2\pi s$. When β is irrational, the solutions of (2) will be non-periodic.

If the parametric point (a, q) lies between b_m and a_m, the solutions are "unstable"; that is, no solution of the form (2) exists. By Floquet's theorem, there does exist a solution of the form

$$y = e^{\mu z} \sum_{r=-\infty}^{\infty} c_r e^{(2r+p)zi}, \tag{3}$$

where $p = 0$ if m is even in b_m, a_m, and $p = 1$ if m is odd. When (a, q) lies in an unstable region, μ is real. It may be readily seen that when μ is a purely imaginary number, (3) yields the solutions (2).

The most important contribution of the paper is to show how β may be determined, in the stable region, from a knowledge of the characteristic values a_m and b_m. Let a point (a, q) of a stable region be given and let it be desired to determine β. The author improves upon Ince's method by obtaining some good *first approximation* to β. By inverting the known series for "a" in terms of $(m + \beta)$ and q, the author obtains the following approximation[1] to β:

$$\beta = \left[a - \frac{(a-1)q^2}{2(a-1)^2 - q^2} - \frac{(5a+7)q^4}{32(a-1)^3(a-4)} \cdots \right]^{\frac{1}{2}} - m. \tag{4}$$

In (b) the above expansion is extended to include, in the radical, the term $-$ $(9a^2 + 58a$

$+ 29)q^6/64(a-1)^5(a-4)(a-9)$. This reviewer believes that, if the first three terms are left in their present form, the term involving q^6 should be[2] $(-37a^3 + 319a^2 - 587a + 17)q^6/64(a-1)^5(a-4)^2(a-9)$. However, the series (4) is useful only when the expression under the radical converges rapidly enough with the given terms; and whenever the contribution from the corrected term involving q^6 is small, the uncorrected term will not be much larger numerically.

When (4) cannot be used (a denominator may vanish or $q/(a-1)$ may be too large), then other approximations may be used. Let a_m and b_{m+1} be the characteristic values for the given q, between which a lies. Let $\lambda = (a - a_m)/(b_{m+1} - a_m)$. Then λ should be an approximation to β. However, usually λ is too crude; a better approximation may be obtained if it is assumed that, at $q = 0$, the iso-β curve intersects the region between a_m and b_{m+1} in approximately the same ratio, λ. Since, at $q = 0$, $a_m = m^2$, and $b_{m+1} = (m+1)^2$, it follows from (4) and the above assumption as to λ that

$$\beta = m\left\{\left[1 + \lambda\left(\frac{2m+1}{m^2}\right)\right]^{\frac{1}{2}} - 1\right\}. \tag{5}$$

The author gives still another empirical formula for β. Let

$$\varphi_m = (a^{\frac{1}{2}} - a_m^{\frac{1}{2}})/(b_{m+1}^{\frac{1}{2}} - a_m^{\frac{1}{2}}).$$

Replacing λ by φ_m in (5), one obtains

$$\beta = m\left\{\left[1 + \varphi_m\left(\frac{2m+1}{m^2}\right)\right]^{\frac{1}{2}} - 1\right\}. \tag{6}$$

In an appendix, the author gives a table showing the accuracy of the several approximations in a few of the instances in which they were tried. The schedule given below summarizes the author's results; all figures except those in the last column were recalculated by this reviewer; in cases of discrepancy, the recalculated figures are given. The results obtained were close to those of the author, except on line 2 of the schedule, where the author obtained, by formula (4), an amount .57—apparently by neglecting the third term under the radical.

				Formulae Used			
m	a	q	λ	(4)	(5)	(6)	More Accurate Value of β
1	3	2	.48	—	.56	.59	.579
2	8	4	.48	.52	.53	.55	.59
2	6	2	.21	[3].34	.25	.27	.34
5	36	16	.48	[3].62	.50	.51	[3].583
1	2	.0125	.33	.4142	.41	.49	.4142
0	−1.45	2	.52	—	.72	.72	.52

The last two examples were supplied by the reviewer. Except in the last example, formulae (4), (5), and (6) are better than λ, and (4) gives a good approximation in many cases. The author recommends the use of this formula wherever possible. It is to be noted that the author's ingenious method of improving on λ in (5) and (6) is fruitful in its results, especially when (4) cannot be used.

Once an approximation to β has been obtained, the method of improving it by iteration or interpolation, in the process of computing the coefficients c_r, is fairly easy. Thus in the recurrence relation

$$[a - (2r + p + \beta)^2]c_r - q(c_{r+1} + c_{r-1}) = 0, \tag{7}$$

one may neglect $c_{\pm(r+1)}$ for r sufficiently large, and compute in turn $c_{r-1}, \cdots, c_0$ in terms of c_r (hence also $c_r, \cdots, c_1$ in terms of c_0); and again $c_{-r+1}, \cdots, c_0$ in terms of c_{-r} (hence also c_{-r+1}, etc. in terms of c_0). Then setting $r = 0$ in (7), the relation between c_{-1}, c_1, and c_0 will be satisfied only if β is correct, and the divergence of the right-hand side of (7) from zero shows how to correct β and the coefficients c_r. One may of course compute the coefficients in terms of any c_m rather than in terms of c_0. Several variations of the computing technique are given, with methods of checking the computations.

The method of approximation may also be used for the unstable regions, once iso-μ

curves have been plotted. It is recommended that the coefficients c_r be normalized so that $\sum c_r^2 = 1$.

(b) Here are given a great variety of representations for the solutions of Mathieu's equation and of Mathieu's "modified" equation. Quoting from the author: "The number of representations in the guise of series, integral relations, etc., exceeds 300. Of these, about 200 have not been published hitherto. . . . No attempt is made to show the derivation of the new formulae, as this paper would then be much too long."

The first part of the paper deals with solutions of the first and second kind for Mathieu functions of integral order, and includes the very useful Bessel-function-products solutions, previously given in a paper by W. G. BICKLEY & N. W. MCLACHLAN, *MTAC*, v. 2, Jan. 1946, p. 1–11. It may be worth pointing out that Bessel-function-products solutions of Mathieu's differential equation were given by BRUNO SIEGER[4]; and although Sieger's work is again mentioned by STRUTT,[5] the importance of such solutions seems to have been little understood until it was emphasized by Bickley & McLachlan in their paper. (This reviewer learned of Sieger's work from Professor Bickley.) All forms given in the January paper are included in this larger one by McLachlan, now under review. In addition, a great many variations of the Bessel-function-products are given, which may prove useful from a computational standpoint.

Functions of the third kind (analogous to the well-known Hankel functions) are defined in section 11. Except for the normalization factor, these solutions are the same as the ones previously defined by L. J. CHU and J. A. STRATTON (*Jn. Math. Phys.*, Aug., 1941), see *MTAC*, v. 1, p. 157. The relations between the various solutions, when the parameter q is either positive or negative, are also given.

In addition to formulae relating to Mathieu functions of fractional order (stable solutions) the author devotes considerable space to the "unstable" solutions. It is shown that, when the solution is put into the form

$$y_1 = e^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r+p} e^{(2r+p)zi},$$

(with $p = 0$ if (a, q) lies between a_{2m} and b_{2m} and $p = 1$ if (a, q) lies between b_{2m+1} and a_{2m+1}) then c_{2r} and c_{-2r} are conjugate complex numbers, if expressed in terms of c_0, real; furthermore if $p = 1$, then c_{2r+1} and $(c_1/c_{-1})c_{-2r-1}$ are conjugate, if c_1 is taken to be real. It is shown that there exists a real solution of the differential equation which tends to zero as $z \to -\infty$ through real values, if a, q, and $\mu(> 0)$ are real. Solutions denoted by $ce_{m+\mu}(\pm z, q)$ are defined, analogous to the solutions $ce_m(\pm z, q)$ for integral m; similarly for $se_{m+\mu}(\pm z, q)$. Such solutions are neither even nor odd. It is shown how to construct even and odd solutions, but they are less useful than $ce_{m+\mu}(\pm z, q)$ and $se_{m+\mu}(\pm z, q)$.

For the same q and μ, there are *two* values of a. One is such that the solution approaches $ce_m(z, q)$ as $\mu \to 0$; the second a corresponds to the solution which approaches $se_m(z, q)$ as $\mu \to 0$. Solutions of fractional order corresponding to Mathieu's modified equation are also given.

The paper contains a number of asymptotic expansions for Mathieu functions of integral order, both for large z and large q. These expansions involve certain multipliers (resulting from the normalization adopted) which cannot readily be expressed asymptotically. To this extent the solutions for large q, in a practical case, are really never obtainable by the given asymptotic formulae—they are known *except* for those multipliers which are functions of q.

It is this unfortunate property of the normalization adopted by McLachlan (and the English school generally) which is at the crux of the divergence of opinion, regarding the normalization scheme, between the English and American schools.

The author concludes with a section on the zeros of the functions and another on the classification of the various solutions of integral order, fractional order, and the unstable solutions. A useful iso-μ chart, the data for which are credited to Dr. L. J. COMRIE, is also given.

The paper is concisely written and represents a prodigious effort, both as to span covered and the variety of formulae given. It forms a priceless compendium of known results (with a very considerable portion of them due to the author himself). The summary given above by no means covers all topics treated. It is hoped that the book promised by the author, enlarging on the theory covered in this paper, may soon be forthcoming.

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[1] In the author's diagrams, the horizontal lines in the a–q plane are parallel to the "q" axis. Hence it might have been better to refer to the "q–a" plane, and to the parametric point as (q, a). We shall not, however, depart from the author's notation.

[2] Mrs. IDA RHODES of the NBSMTP checked this reviewer's inversion. Dr. McLachlan pointed out that the second term of (4), given incorrectly in (a), is correct in (b).

[3] These three entries are corrections of the printed values, furnished by Dr. McLachlan. EDITOR.

[4] B. SIEGER, "Die Beugung einer ebenen elektrischen Welle an einem Schirm von elliptischem Querschnitt," *Ann. d. Phys.*, s. 4, v. 27, 1908, p. 626–664.

[5] M. J. O. STRUTT, *Lamésche-Mathieusche und verwandte Funktionen der Physik u. Technik*, (*Ergebnisse d. Math.*, v. 1, no. 3), Berlin, 1932, p. 46–48.

383[L].—J. C. P. MILLER, "Two numerical applications of Chebyshev polynomials," R.S. Edinb., *Proc.*, v. 62, no. 22, 1946, p. 204–210. 17.5 × 25.7 cm.

1. The strong convergence of an expansion in Chebyshev polynomials renders them useful for interpolation. Let $f(a + t)$ be expanded into a series of $C_n(4t/h)$. The coefficients a_n of this expansion are expressible in terms of the derivatives $f^{(p)}(a)$. They are also expressible in terms of central differences. The corresponding formulae are derived by operational methods and numerical tables given. Even if high central differences are needed, the smaller number of a_m terms makes interpolation more convenient, particularly in conjunction with a table for the $C_n(x)$. A numerical example illustrates the advantage of the method.

2. Since an expansion in Chebyshev polynomials is merely a modified form of a Fourier series, a table of the Chebyshev polynomials becomes useful for harmonic synthesis whenever the sum of a Fourier series is required for arguments which are convenient numbers in $x = 2\cos\theta$ rather than in θ itself. An application is given, showing how the evaluation of the Mathieu functions can be facilitated by this procedure.

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384[L].—FRANK H. SLAYMAKER, WILLARD F. MEEKER & LYNN L. MERRILL, "The directional characteristics of a free-edge disk mounted in a flat baffle or in a parabolic horn," Acoustical Soc. Amer., *Jn.*, v. 18, Oct. 1946, p. 363–368. 19.4 × 26.6 cm.

There are two tables on p. 367. **T. I** of

$$\phi(\lambda_n r) = \frac{I_1(\lambda_n a)}{J_1(\lambda_n a)} J_0(\lambda_n r) + I_0(\lambda_n r)$$

$r/a = 0(.1)1$; $\lambda_n a \sim 3.01, 6.21, 9.37$ for $n = 1(1)3$; $n = 1, 2$ to 3S, and $n = 3$ mostly to 4S.

$$\begin{aligned} G &= J_1(ka\sin\phi)/ka\sin\phi; \\ P_n &= -\,I_1(\lambda_n a)[\lambda_n a J_1(\lambda_n a)J_0(ka\sin\phi) - ka\sin\phi J_0(\lambda_n a)J_1(ka\sin\phi)] \\ &\qquad \div [J_1(\lambda_n a)(\lambda_n^2a^2 - k^2a^2\sin^2\phi)],\ \text{if } \lambda_n \neq k\sin\phi \\ &= -\,I_1(\lambda_n a)[J_0^2(\lambda_n a) + J_1^2(\lambda_n a)]/[2J_1(\lambda_n a)],\ \text{if } \lambda_n = k\sin\phi; \\ Q_n &= [\lambda_n a I_1(\lambda_n a)J_0(ka\sin\phi) + ka\sin\phi I_0(\lambda_n a)J_1(ka\sin\phi)] \\ &\qquad \div (\lambda_n^2a^2 + k^2a^2\sin^2\phi). \end{aligned}$$

T. II is of G, $P_n + Q_n[n = 1(1)3]$, mostly to 3S, for $ka\sin\phi = 0(.5)10(1)12$. There are graphs of $P_1 + Q_1$, $P_2 + Q_2$, $P_3 + Q_3$ on p. 368.

R. C. A.

385[L].—FRANCESCO TRICOMI, "Generalizzazione di una formula asintotica sui polinomi di Laguerre e sue applicazioni," Accad. delle Scienze di Torino, *Cl. d. sci. fis., mat., e nat., Atti*, v. 76, 1941, p. 288–316. 16.6 × 25 cm.

The Tricomi polynomials of Laguerre[1] are defined by

$$\begin{aligned} L_n(t) &= (n!)^{-1}e^t d^n(e^{-t}t^n)/dt^n = M(-n, 1, t) \\ &= (-1)^n(n!)^{-1}\left[t^n - \frac{n^2}{1!}t^{n-1} + \frac{n^2(n-1)^2}{2!}t^{n-2} - \cdots + (-1)^n n!\right]. \\ &= 1 - \binom{n}{1}\frac{t}{1!} + \binom{n}{2}\frac{t^2}{2!} - \cdots + (-1)^n\frac{t^n}{n!} \end{aligned}$$

This is the case where $a = 0$ in the more general formula

$$L_{n+a}^a(t) = (n!)^{-1}e^t t^{-a} d^n(e^{-t}t^{n+a})/dt^n = M(-n, a+1, t).$$

In Tricomi's paper are the following:

p. 292, values of $e^{-\frac{1}{2}t}L_{10}(t)$, for $t = [.5(.5)3(1)8(2)34; 5D]$, Δ;
p. 302, a 4D table of the roots of the equations $x + \sin x = a$, for $a = 0(.1)3(.02)3.18$;
p. 303, graph of zeros of $L_n(t)$ for $n = 1(1)10$;
p. 315–316, table of $e^{-\frac{1}{2}t}L_n(t)$; $n = 1(1)10$, $t = [.1(.1)1(.25)3(.5)6(1)14(2)34; 4D]$.

The polynomials $L_n(t)$ and their properties were given in E. N. LAGUERRE, "Sur l'intégral $\int_x^\infty e^{-x}dx/x$," *Bull. Sci. Math.*, v. 7, 1879, p. 72–81; *Oeuvres de Laguerre*, Paris, v. 1, 1898, p. 428f ($L_n(t)$, p. 430). $L_{n+a}^a(t)$ seems to have been discussed simultaneously with $L_n(t)$, by N. SONIN, in a memoir dated Aug. 1879, *Math. Annalen*, v. 16, 1880, p. 41 (function $T_m{}^n$). The first one to refer to $L_{n+a}^a(t)$ as generalized Laguerre polynomials appears to have been another Russian, WERA MYLLER-LEBEDEFF, *Math. Annalen*, v. 64, 1927, p. 410.

The so-called polynomials of Laguerre were introduced into mathematical analysis by LAGRANGE,[2] more than 130 years earlier than Laguerre, in his solution of a dynamical problem in which the oscillations of a vertical chain are represented approximately by those of a set of similar weights equally spaced on a light string. (See H. BATEMAN, "Lagrange's compound pendulum," *Amer. Math. Mo.*, v. 38, 1931, p. 1–8.) The polynomials were also considered by ABEL, in 1826, "Sur une espèce particulière de fonctions entières nées du développement de la fonction $(1 - v)^{-1}e^{-xv/(1-v)}$ suivant les puissances de v." [3] This function is equal to $\sum_k L_k(x)v^k/k!$. In H. BETHE, "Quantenmechanik der Ein- und Zwei-Elektronenprobleme," *Handbuch der Physik*, second ed., v. 24_1, Berlin, 1933, p. 289, this result is attributed to E. SCHRÖDINGER,[4] just 100 years later.

Laguerre polynomials are also of use in (a) the theory of hydrogen-like atoms; (b) the problem of numerical integration over the range 0 to $+\infty$ [Gauss's method with Legendre polynomials for ranges -1 to $+1$, or 0 to 1; Hermite's polynomials for the range $-\infty$ to $+\infty$]; (c) the discussion of the mathematical foundations of the electromagnetic theory of the paraboloidal reflector.[5] In Bateman's Bibliography[5] there are 47 references for $L_n(x)$, and 73 for "$L_n{}^a(x)$."

In preparing this RMT I have been indebted for some assistance from Dr. J. C. P. MILLER, and from Dr. ALAN FLETCHER.

R. C. A.

[1] See *MTAC*, v. 1, p. 361, 425; and v. 2, p. 31 [where $L_n(x)$ is defined without the factor $(n!)^{-1}$], 89.

[2] J. L. LAGRANGE, "Solution de différents problèmes de calcul intégral," *Miscellanea Taurinensia*, v. 3, 1762–65; *Oeuvres*, v. 1, Paris, 1867, "Des oscillations d'un fil fixe par une de ses extrémités, et chargé d'un nombre quelconque de poids," p. 534–536; there are four of the polynomials on p. 536.

[3] N. H. ABEL, *Oeuvres Complètes*, Christiania, 1881, v. 2, p. 284.

[4] E. SCHRÖDINGER, *Annalen d. Physik*, v. 385, 1926, p. 485.

[5] PINNEY, *Jn. Math. Phys.*, v. 25, 1946, p. 49f. Harry Bateman's Bibliography, p. 77–79.

386[L, M].—S. A. KHRISTIANOVICH, S. G. MIKHLIN & B. B. DAVISON, *Nekotorye novye voprosy mekhaniki sploshnoĭ sredy* [Some new questions in mechanics of a continuous medium]. Moscow and Leningrad, Akad. N., Matematicheskiĭ Institut imeni V. A. Steklova, 1938. 407 p. 16.7 × 25.3 cm.

We shall list certain tables in this volume p. 274–336 of a section written by DAVISON, and p. 392–395 of an appendix to the work, presumably written by the joint authors.

On p. 274 a table is given of

$$k\pi x/q = \tan^{-1}\sqrt{e^{2\pi ky/q}-1} - (1/\mu)\sqrt{e^{2\pi ky/q}-1}$$

for $k\pi y/q = [0(.2)3; 5S]$, $1/\mu = 1(1)4$.

$$K(t) = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{1-t\sin^2\theta}}, \qquad \tfrac{1}{2}\gamma(t) = \tan^{-1}\frac{K(1-t)}{K(t)}.$$

On p. 334 there is a table of $\frac{1}{2}\gamma(t)$, to 3D, for $t = 0(.00001).0001(.0001).0005(.0002).0015$, $.002(.001).01(.003).016$, $.02(.005).05(.01).1(.02).2(.1).5$.

On p. 335–336 are the following five tables:

$$\text{(a) } z = \int_0^1 \frac{\gamma(t)dt}{t-\lambda}, \qquad \text{(b) } e^{-2z/2\pi}, \qquad \text{to 3–4S,}$$

for $-\lambda = .01, .1, .2, .5(.1).7, 1(1)3, 5, 8, 15, 20$, and $\lambda = 1.05, 1.1, 2(1)8, 12, 15, 20$.

$$\text{(c) } \gamma(\lambda), \qquad \text{(d) } w = R\left[\int_0^1\frac{\gamma(t)dt}{t-\lambda}\right], \qquad \text{(e) } |e^{-2w/2\pi}|,$$

mostly to 3–4S, for $\lambda = .00001, .005, .01, .05(.05).3, .4(.05).6, .7(.05).95, .99, .995, .9999$, $.99999$. In (a) and (d) approximations are given to true results with possible maximum deviations. Before discussing the remaining tables in the volume we may quote some results from B. A. BAKHMETEV, *Hydraulics of Open Channels*, New York, 1932, p. xv–xvi, 308–311: If y is the depth or stage of flow, y_0 the normal depth of flow or the depth of flow in uniform movement, and $\eta = y/y_0$, then the varied flow function $B(\eta) = -\int_0^\eta \frac{d\eta}{\eta^n - 1}$, n being the hydraulic exponent. There are tables of $B(\eta)$ (f) for $\eta > 1$, (g) for $\eta < 1$, $n = 2.8(.2)$ $4.2(.4)5.4$

(f) $\eta = 1.001, 1.005(.005)1.02(.01)1.2(.02)1.5(.05)2(.1)3(.5)5(1)10, 20$
(g) $\eta = 0(.02).6(.01).97(.005).995, .999$.

Now on p. 392–395 are tables of

$$\text{(h) } A(\eta) = \int_{\eta_0}^{\eta}\frac{\eta^{\frac{1}{2}(k-1)}}{\eta^k-1}d\eta \qquad \text{(k) } C(\eta) = \int_{\eta_0}^{\eta}\frac{\eta^{k-\frac{1}{2}}}{\eta^k-1}d\eta,$$

$k = 2n+1$, for $k = 3(n = 1)$, $4(n = 3/2)$, $\eta = [0(.05).6(.01).9(.005)1.05(.01)1.5(.05)2(.1)$ $3(.5)5, 6(2)10; 4D]$, Δ.

By trial we found that in the table of $A(\eta)$, when $k = 3$, $\eta_0 = 0$, for $\eta < 1$, and $\eta_0 = \infty$ for $\eta > 1$. We were unable to determine the values of η_0 leading to the table of $C(\eta)$, or to that of $A(\eta)$, when $k = 4$.

S. A. J. & R. C. A.

387[L, M].—NBSMTP, "Table of the Struve functions $L_\nu(x)$ and $H_\nu(x)$," *Jn. Math. Phys.*, v. 25, Oct. 1946, p. 252–259. 17.3 × 25.3 cm. This is Applied Mathematics Panel, *Report* 59.1, referred to in *MTAC*, v. 2, p. 39.

The functions in question are, in Watson's notation, in effect,

$$L_n(x) = \sum_{m=0}^{\infty} a_m(x); \qquad H_n(x) = \sum_{m=0}^{\infty}(-1)^m a_m(x),$$

where

$$a_m(x) = (\tfrac{1}{2}x)^{2m+1+n}/\Gamma(m+\tfrac{3}{2})\Gamma(m+n+\tfrac{3}{2}).$$

Also

$$L_n(x) = N\int_0^{\frac{1}{2}\pi} \sinh(x\cos\theta)\sin^{2n}\theta d\theta, \qquad N = 2(\tfrac{1}{2}x)^n/\Gamma(n+\tfrac{1}{2})\Gamma(\tfrac{1}{2}),$$

$$H_n(x) = N\int_0^{\frac{1}{2}\pi} \sin(x\cos\theta)\sin^{2n}\theta d\theta, \qquad \text{provided that } R(n) > -\tfrac{1}{2}.$$

But for all n

$$L_n(x) = A_n(x) + B_n(x), \qquad H_n(x) = A_n(x) - B_n(x),$$

where

$$A_n(x) = \sum_{m=0}^{\infty} a_{2m}(x), \qquad B_n(x) = \sum_{m=0}^{\infty} a_{2m+1}(x).$$

Tables are given of $L_n(x)$ and $H_n(x)$, $n = 0, -1, -2$, $x = [0(.1)10; 7\text{–}10S]$. $L_0(x)$, $L_{-1}(x)$, are also with δ^2 and $*\delta^4$ modified throughout the range; $L_{-2}(x)$ with δ^2 and $*\delta^4$ for $x=2.1(.1)10$ For $x = 0(.1)2$, $L_{-2}(x)$, $xL_{-2}(x)$, $\delta^2(xL_{-2})$, $*\delta^4(xL_{-2})$ are given. In the case of $H_0(x)$ and $H_{-1}(x)$, and $H_{-2}(x)$, for $x = 2(.1)10$, $*\delta^2$ is given; $H_{-2}(x)$ for $x = 0(.1)2$, also $xH_{-2}(x)$, has $*\delta^2$.

$$H_1(x) = 2/\pi - H_{-1}(x), \qquad L_1(x) = -2/\pi + L_{-1}(x).$$

WATSON[1] has tabulated $H_0(x)$ and $H_1(x)$ $x = [0(.02)16; 7D]$, no Δ; hence linear interpolation is here correct to about 5D. $H_{-1}(x)$ is readily obtained from $H_1(x)$, and has been tabulated before by AIREY,[2] for $x = [0(.02)16; 6D]$, and by JAHNKE & EMDE,[3] for $x = [0(.01)14.99; 4D]$. $H_{-2}(x)$, $L_{-1}(x)$, and $L_{-2}(x)$ seem to be here independently tabulated for the first time.

KARL HERMANN STRUVE investigated[4] only the special functions $H_0(x)$ and $H_1(x)$, but properties of the general function were later extensively developed by SIEMON[5] and WALKER.[6] The function $L_n(x)$ bears the same relation to Struve's function $H_n(x)$, as $I_n(x)$ bears to $J_n(x)$; $L_n(x) = i^{-n-1}H_n(ix)$, $L_0(x) = -iH_0(ix)$; $L_0'(x) = 2/\pi - H_1(ix)$. Tables of $L_0(x)$ and $L_0'(x)$ for $x = [.02, .1, .5, 1(1)12; 6\text{–}7S]$ were given by OWEN.[7] $H_n(x) = (-i)^{n+1}L_n(ix)$, and

$$H_n(xi^{\frac{1}{2}}) = i^{n+1}L_n(xi^{\frac{1}{2}}) = \text{ster}_n x + i\,\text{stei}_n x,$$

a notation due to MCLACHLAN & MEYERS[8] (see *MTAC*, v. 1, p. 252, 460). For tables of $y = \frac{1}{2}\pi[I_0(x) - \mathbf{L}_0(x)]$, and $-y' = 1 - \frac{1}{2}\pi[I_1(x) - \mathbf{L}_1(x)]$ by R. ZURMÜHL and R. MÜLLER, see *MTAC*, v. 2, p. 59; and on p. 39 a table by GREAT BRITAIN, Nautical Almanac Office, of $f(x) = \pi e^{-x}(2x)^{-1}[I_1(x) + L_{-1}(x) - 2/\pi]$.

When n is half an odd positive integer $H_n(x)$ is expressible in terms of elementary functions. For example, $H_{\frac{1}{2}}(x) = B(1 - \cos x) = B - J_{-\frac{1}{2}}(x)$, where $B = [2/(\pi x)]^{\frac{1}{2}}$. For various tables of $J_{\frac{1}{2}}(x) = H_{-\frac{1}{2}}(x)$, see *MTAC*, v. 1, p. 233.

R. C. A.

[1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, p. 328–329; tables, p. 666–697.

[2] J. R. AIREY, BAAS, *Report, 1924*, p. 280f, $-H_{-1}(x)$ is also tabulated here for the same range.

[3] JAHNKE & EMDE, *Tables of Functions*, fourth ed., New York, 1945, p. 219f. $H_0(x)$ is also tabulated here for the same range, p. 212f, and 218f. There are also two other tables of $H_0(x)$ and $H_1(x)$, to 4D, by S. P. GLAZENAP, *Matematicheskie i Astronomicheskie Tablitsy*, Leningrad, 1932, p. 110f, $x = 0(.02)16$, an abridgment of Watson; and by N. W. MCLACHLAN, *Bessel Functions for Engineers*, 1934, p. 176, $x = 0(.1)15.9$.

[4] See *MTAC*, v. 1, p. 305.

[5] P. SIEMON, *Ueber die Integrale einer nicht homogenen Differentialgleichung zweiter Ordnung*. Progr. Luisenschule. Berlin, 1890; see *Jahrb. Fort. d. Math.*, 1890, p. 340f.

[6] J. WALKER, *The Analytical Theory of Light*, Cambridge, 1904, p. 392f.

[7] S. P. OWEN, "Table of values of the integral $\int_0^x K_0(t)dt$," *Phil. Mag.*, s. 6, v. 47, 1924, p. 736; see also *MTAC*, v. 1, p. 245, 247, 301.

[8] See also N. W. MCLACHLAN & A. L. MEYERS, (a) "The ster and stei functions" (b) "Integrals involving Bessel and Struve functions," *Phil. Mag.*, s. 7, v. 21, 1936, p. 425–436, 437–448.

388[L, M].—S. SKOLEM, "En del bestemte integraler av formen $\int_0^\infty f(x)\cos akdx$ og $\int_0^\infty f(x)\sin(ax)dx$," *Norsk Matem. Tids.*, v. 27, 1945, p. 65–75; tables, p. 70–71. 15.5 × 23.2 cm.

$$S(a,1) = \int_0^\infty \frac{\sin ax}{1+x^2}dx = \tfrac{1}{2}[e^{-a}Ei(a) - e^aEi(-a)]$$
$$= \cosh a \operatorname{shi} a - \sinh a \operatorname{chi} a$$
$$= 1/a + 2!/a^3 + 4!/a^5 + \cdots + (2n)!/a^{2n+1}$$
$$S'(a,1) = -\tfrac{1}{2}[e^{-a}Ei(a) + e^aEi(-a)] = \sinh a \operatorname{shi} a - \cosh a \operatorname{chi} a$$
$$= -[1/a^2 + 3!/a^4 + 5!/a^6 + \cdots].$$

T. I, $S(a, 1)$, for $a = [0(.01).1(.1)1(1)10(10)100; 5D]$; maximum value at $a \sim .8791$ is approximately .64996.

T. II, $S'(a, 1)$, for the same range of a as in **T. I**; zero value at $a \sim .8791$, and minimum value at $a \sim 1.8594$ is approximately $-.15583$.

389[L, M].—EDMUND C. STONER, "The demagnetizing factors for ellipsoids," *Phil. Mag.*, s. 7, v. 36, Dec. 1945 (publ. Sept. 1946; note added in proof 28 May 1946), p. 803–821. 17 × 25.4 cm.

J. A. OSBORN, "Demagnetizing factors of the general ellipsoid," *Phys. Rev.*, v. 67, 1945, p. 351–357. 19.2 × 26 cm.

The formulae for the demagnetizing factors, in terms of F and E, are

$$D_a = L/4\pi = \frac{abc}{(a^2-c^2)^{\frac{1}{2}}(a^2-b^2)}[F(k,\phi) - E(k,\phi)]$$
$$= \frac{\cos\theta\cos\phi}{\sin^3\phi\sin^2\alpha}[F(k,\phi) - E(k,\phi)]$$

$$D_b = M/4\pi = -\frac{abc}{(a^2-c^2)^{\frac{1}{2}}(a^2-b^2)}[F(k,\phi) - E(k,\phi)]$$
$$+ \frac{abc}{(a^2-c^2)^{\frac{1}{2}}(b^2-c^2)}E(k,\phi) - \frac{c^2}{b^2-c^2}$$
$$= \frac{\cos\theta\cos\phi}{\sin^3\phi\sin^2\alpha\cos^2\alpha}\left[E(k,\phi) - \cos^2\alpha F(k,\phi) - \frac{\sin^2\alpha\sin\phi\cos\phi}{\cos\theta}\right]$$

$$D_c = N/4\pi = -\frac{abc}{(a^2-c^2)^{\frac{1}{2}}(b^2-c^2)}E(k,\phi)$$
$$+ \frac{b^2}{b^2-c^2} = \frac{\cos\theta\cos\phi}{\sin^3\phi\cos^2\alpha}\left[\frac{\sin\phi\cos\theta}{\cos\phi} - E(k,\phi)\right],$$

where $k^2 = (a^2-b^2)/(a^2-c^2) = \sin^2\alpha$, $\cos\theta = b/a$, $\cos\phi = c/a$; $a, b, c (a \geqslant b \geqslant c)$ are the semi-axes of the ellipsoid. $D_a + D_b + D_c = 1$.

Consider first, ellipsoids of revolution: a polar semi-axis, b equatorial semi-axis, $m = a/b$, $\mu = b/a$; $m < 1$ and $\mu > 1$ an oblate spheroid; $m > 1$ and $\mu < 1$ a prolate spheroid. Then

$$D_a = \tfrac{1}{2}ab^2\int_0^\infty \frac{ds}{(a^2+s)^{\frac{3}{2}}(b^2+s)} = \frac{1}{(m^2-1)}\left[\frac{m}{(m^2-1)^{\frac{1}{2}}}\cosh^{-1}m - 1\right], \quad m > 1,$$
$$= \frac{1}{(1-m^2)}\left[1 - \frac{m}{(1-m^2)^{\frac{1}{2}}}\cos^{-1}m\right], \quad m < 1;$$
$$D_b = \tfrac{1}{2}(1 - D_a).$$

Stoner gives tables (p. 816–817) of D_a, m or $\mu = [0(.1)5(.5)10(1)25(5)50(10)150(50)400(100)1300; 6D]$. In general the m-table will be appropriate for prolate spheroids and the μ-table for oblate spheroids. To ensure accuracy to the sixth place, the calculations were carried out so as to give unit accuracy in the seventh place, and rounded six-place values are presented in the tables.

Osborn gives two tables (p. 353–354) of demagnetizing factors of the general ellipsoid, $L/4\pi$, $M/4\pi$, $N/4\pi$, for (T. I) $\cos\phi (= c/a)$, $\phi = 10°(10°)70°(5°)85°$, $88°$, $89°$, and for $\cos\theta$ $(= b/a)$, $\theta = [0(10°)90°; 5D]$. Also (T. II) $\cos\theta = .1(.1)1$, $\cos\phi$ = various values. There are three large-scale graphs of $L/4\pi$, $M/4\pi$, $N/4\pi$, for $0 \leqslant c/a \leqslant 1$, $b/a = 0(.1)1$. In T. II the values are accurate to 3D and are probably in error several units in the fourth place.

Stoner has a single graph of these same functions for $b/a = .2(.2)1$.

R. C. A.

390[L, P].—N. W. McLachlan, *Bessel Functions for Engineers*, Oxford Univ. Press, London, Geoffrey Cumberleye, 1946. xii, 192 p. Reprinted photographically., 15.3 × 23.3 cm. 18 shillings.

This very useful volume of the Oxford Engineering Science Series first appeared in 1934. In the corrected photographic reprint published in London in 1941, two pages were added; the new material included an introductory "Note," a page of "Additional formulae," and 20 (instead of 6) "Additional references." In the Note it is remarked that "The omission of contour integral representation of Bessel functions and its technical applications has been rectified through publication [by the author] in 1939 of *Complex Variable and Operational Calculus with Technical Applications*," and a correction of an error on p. 300 of this volume is noted.

We have already referred to various tables in the volume under review (see *MTAC*, v. 1, p. 212, 216, 220, 246, 247, 254, 255, 257, 258, 297). In the right-hand member of formula 147, p. 167 one error still persists; the sign − should be changed to +. The 1946 edition is an exact reprint of that of 1941, except for the correction of four signs, two in each of the lines − 3 and − 5, p. xi, "Additional Formulae."

R. C. A.

391[L, S].—C. Strachey & P. J. Wallis, "Hahn's functions $S_m(\alpha)$ and $U_m(\alpha)$," *Phil. Mag.*, s. 7, v. 37, Feb. 1946 [publ. Nov. 1946], p. 87–94. 16.8 × 15.1 cm.

"In a paper[1] on the calculation of fields in certain resonators, Hahn introduced two new functions:

$$-S_m(\alpha) = \sum_{n=1}^{\infty} \frac{m^2 \sin^2 n\pi\alpha}{n(m^2 - n^2\alpha^2)}, \quad \text{and}$$

$$U_m(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^2 m^2 n^2 \sin^2 n\pi\alpha}{(m^2 - n^2\alpha^2)^2}, \quad \text{with} \quad 0 < \alpha < 1,$$

and used these functions to shorten his calculations. Since this time, Hahn's method has been used for certain similarly-shaped resonators and Hahn's two functions usually help to shorten the solution considerably. Hahn himself only gave a small table of $S_m(\alpha)$ and a few values of $U_m(\alpha)$.[1]

"In this report closed expressions are derived for the case of α rational, and are used to produce a much more comprehensive table of $S_m(\alpha)$ and a slightly smaller table of $U_m(\alpha)/m$. In a concluding section integral expressions, power series in α, and asymptotic series in m are given which together facilitate the calculation for values of α not given in the tables."

Tables: $-S_m(\alpha)$, for $m = 1(1)10$, $\alpha = [0(.1)1, .25, .75, \frac{1}{3}, \frac{2}{3}; 5D]$;
$U_m(\alpha)/m$, for $m = 1(1)10$, $\alpha = [0(.25)1, \frac{1}{3}, \frac{2}{3}; 5D]$.

Extracts from text

[1] W. C. Hahn, "A new method for the calculation of cavity resonators," *Jn. Appl. Phys.*, v. 12, 1941, p. 62–68. There are 2D values of $-S_m(\alpha)$ for $m = 1(1)9$, $\alpha = 0(\frac{1}{8})\frac{3}{8}(\frac{1}{16})\frac{5}{8}(\frac{1}{8}), 1$; also $-S_0(\alpha)$ for $\alpha = 0(\frac{1}{8})\frac{1}{2}, \frac{7}{16}$; and of $U_m(\frac{1}{2})$, $m = 1(1)4$. See *MTAC*, RMT 208 and MTE 69, v. 1, p. 425, 451.—Editors.

392[M].—NATIONAL RESEARCH COUNCIL OF CANADA, Division of Atomic Energy. Report no. MT-1 dated Chalk River, Ontario, December 2, 1946, *The Functions* $E_n(x) = \int_1^\infty e^{-xu}u^{-n}du$, 39 leaves mimeographed on one side, with covers. Introductory material, p. 1–7 by G. PLACZEK; Appendix A, an asymptolic expansion for $E_n(x)$, by Dr. GERTRUDE BLANCH, p. 8; Tables, by NBSMTP, p. 9–39. 20.3 × 27.4 cm. This edition contains corrections of one which appeared in July–August 1946.

The functions $E_n(x)$ play an important role in diffusion theory. The discussion of certain integral equations can be simplified by their use; expansions in terms of these functions are also often found convenient for the numerical evaluation of integrals occurring in connection with transport problems. The functions have been defined by SCHLÖMILCH,[1] and have been extensively used by SCHWARZSCHILD,[2] EDDINGTON,[3] HOPF,[4] and others. In spite of this no systematic effort for their tabulation seems to have been made up to the present. An attempt by MIAN & CHAPMAN[5] to approximate the functions by "index sums" was not accurate enough for our purposes.

$E_n(x)$ is here tabulated, for $n = 0(1)20$, $x = [0(.01)2; 7D]$, $[2(.1)10; 7\text{–}10D]$. On p. 39 are tables of $E_2(x) - x\ln x$, for $x = [0(.01).5; 7D]$, and of $E_3(x) + \frac{1}{2}x^2\ln x$ for $x = [0(.01).1; 7D]$, for use in interpolation. Since $E_1(x) = -Ei(-x) = \int_x^\infty e^{-u}u^{-1}du$, there are extensive tables of this function in NBSMTP, *Tables of Sine, Cosine, and Exponential Integrals*, v. 1–2, 1940, for $x = [0(.0001)2; 9D]$, $[0(.001)10; 9S]$; $[10(.1)15; 14D]$.

Extracts from introductory text

EDITORIAL NOTES: In FMR, *Index*, p. 207, are given details of 8 tables, $E_{-n}(x) = \int_1^\infty e^{-xu}u^n du = x^{-(n+1)}\int_x^\infty e^{-u}u^n du$, five of them including negative values of n. $E_0(x) = e^{-x}/x$, of which values for $x = [.1(.001)1(.01)2; 9D]$ are given by W. L. MILLER & T. R. ROSEBRUGH, "Numerical values of certain functions involving e^{-x}," R. Soc. Canada, *Trans.*, s. 2, v. 9, 1903, sect. III, p. 102–107. See also TAKEO AKAHIRA, "Tables of e^{-x}/x and $\int_x^\infty e^{-u}du/u$, from $x = 20$ to $x = 50$," Inst. Phys. Chem. Research, Tokyo, *Sci. Papers*, Table no. 3, 1929, p. 180–215; the interval of the table is .02, to 5–6S, Δ^2.

[1] O. SCHLÖMILCH, "Ueber Facultätreihen," *Z. Math. u. Phys.*, v. 4, 1859, p. 390f.

[2] K. SCHWARZSCHILD, (a) "Ueber das Gleichgewicht der Sonnenatmosphäre," Gesell. d. Wissen., Göttingen, *Nach.*, *Math-phys. Kl.*, 1906: p. 41f; (b) "Über Diffusion und Absorption in der Sonnenatmosphäre," Akad. d. Wissen., Berlin, *Sitzb.*, 1914, p. 1183f.

[3] A. S. EDDINGTON, *The Internal Constitution of the Stars*, Cambridge, 1926, p. 333.

[4] E. HOPF, *Mathematical Problems of Radiative Equilibrium* (*Cambridge Tracts* . . ., no. 31), 1934, p. 21, etc.

[5] A. M. MIAN & S. CHAPMAN, "Approximate formulae for functions expressed as definite integrals," *Phil. Mag.*, s. 7, v. 33, 1942, p. 115f. It is noted that $E_n(x)$ arises in the theory of absorption of radiation in an exponential atmosphere. There is a table on p. 119 of approximate values of $E_n(x)$, for $n = 2(1)8$, for $x = 0(.5)3(1)6$, also .01, .05, .1, .25.

393[M].—W. SOKOLOVSKY, "Plastic plane stressed states according to Mises," Akad. N., USSR, Leningrad, (*Dok.*), *C. R.*, n.s. 1946, v. 51, p. 177. 16.8 × 26 cm.

There is here a table of

$$-\Omega(x) = \frac{1}{2}\int_{\frac{1}{2}\pi}^{x}\frac{R(t)dt}{\sin t}$$

$= \frac{1}{4}\pi - \sin^{-1}(2\cos x/3^{\frac{1}{2}}) + \frac{1}{4}\tan^{-1}[(4\cos x + 3)/R(x)] + \frac{1}{4}\tan^{-1}[(4\cos x - 3)/R(x)]$, where $R(x) = (3\text{-}4\cos^2 x)^{\frac{1}{2}}$, for $x = \frac{1}{6}\pi$ to $\frac{5}{6}\pi$, mostly at interval $\frac{1}{18}\pi$, to 3D.

394[M].—A. J. C. WILSON, "The integral breadths of Debye-Scherrer lines produced by divergent X rays," Phys. Soc., London, *Proc.*, v. 58, July 1946, p. 407. 18 × 26 cm.

There is given here a table of $D(u) = 4u^{-4}\int_0^u[C^2(u) + S^2(u)]udu$, for $u = [0(.1)5; 4D]$, 5D for $u < 1$. For $u < 2$ the values were calculated from the series for $D(u)$, those for

$u > 2$ by numerical integration of four-place tables of $C(u)$ and $S(u)$. In the range .5 to 2 the greatest difference between the values calculated by the two methods is .0003; the mean difference is about .0001.

Extracts from text

395[N].—ERICH MICHALUP, "Beitrag zur Amortisationsrechnung," *Skandinavisk Aktuarietidskrift*, 1946, p. 80–84. 15.5 × 23.5 cm.

With references to earlier discussions by E. LINDELÖF, K. A. POUKKA, A. BERGER, R. PALMQVIST, H. HOLME, and E. FRANCKX, the author considers the following five formulae and gives tables for each of them to 7D for half-yearly, quarterly, monthly rates ($p = 2, 4, 12$), for $i = 1\%(1\%)9\%$:

$$a_p \sim \frac{1}{p}\left(1 - \frac{p-1}{2p}i\right), \quad a_p \sim \frac{1}{p}\left(1 - \frac{p-1}{2p}i + \frac{(p-1)(2p-1)}{6p^2}i^2\right),$$

$$a_p \sim 1/\left(p + \frac{p-1}{2}i\right), \quad a_p = [(1+i)^{1/p} - 1]/i, \quad a_p \sim \frac{1}{p}\left[\frac{6p + i(1+p)}{6p + 2i(2p-1)}\right].$$

396[Q].—ENRIQUE VIDAL ABASCAL, *El Próblema de la Órbita Aparente en las Estrellas Dobles Visuales*: Diss. Spain, Consejo Superior de Investigaciones Cientificas, Instituto Nacional de Geofisica, no. 6, Observatorio de Santiago, *Publicaciones*, II, Santiago de Compostela, 1944. xvi, 62 p. 21.2 × 27 cm.

Consider ellipses with common major semi-axis, $OA = 1$, and eccentricities $e =$ the length of $OF_i = .1(.1).9$; then the foci F_i, $i = 1(1)9$, divide OA into tenths. Suppose that a unit circle, with center at O, has been drawn, and P is any point of the circumference, then F_iP and F_iA are the sides of circular sectors, F_iAPF_i, whose angle α may increase from 0 to 360°. A table, p. 53–62, gives the area of such sectors, to 4D, $e = .1(.1).9$, $\alpha = 0(1°)360°$.

R. C. A.

397[U].—FRANCISCO RADLER DE AQUINO, "*Universal*" *Nautical and Aeronautical Tables. Uniform and Universal Solutions Ultra-simplified.* Rio de Janeiro, Imprensa Naval, 1943, 18, 247 p. 17.5 × 24.5 cm. Copies of this volume may be had from Weems System of Navigation, Annapolis, Md. at $9.00.

The author of these tables, a captain in the Brazilian Navy, is well known to navigators around the world, having published more than fifty papers on navigation in the past forty-eight years. Not so well known is the fact that he was born in New York City on January 23, 1878; his mother was an American, his father a Brazilian. He moved to Brazil at the age of 13 and entered the Brazilian Naval Academy at 15.

This volume is the second Brazilian edition of a book which was first published in Rio de Janeiro in 1903. Editions were published in London in 1910, 1912 with reprintings in 1917 and 1918, and in 1924; and in Annapolis, Md. in 1927 and 1938. The title and content of the tables have changed slightly from edition to edition. For those familiar with the earlier editions, it may be said that the principal change in this edition is the reduction of the interval of the argument, latitude, from 1° to 10′. The method and the tables continue to be universal in that they allow the determination of the altitude and azimuth whatever the values of latitude, hour angle, declination and altitude.

The first eighteen pages in this volume include the title page and explanation of the tables in English; the next sixteen pages (numbered 1 to 16 also) present similar but not identical material in Portuguese. The principal table was designed to be used in a solution of the astronomical triangle in which a perpendicular is dropped from the zenith upon the hour circle through the celestial body. The length of the perpendicular and the declination of its foot, are called a and b respectively. The angle at the zenith between the perpendicular and the meridian (toward the elevated pole) is called α; that between the perpendicular and the great circle from the zenith to the celestial body is called β. L, t, and d denote the latitude of the observer, the local hour angle and the declination of the celestial body respectively. A is the angle of the astronomical triangle at the celestial body. C is the angular distance from the celestial body to the foot of the perpendicular.

The basic equations for the solution of the two right triangles are obtained by Napier's rules; they are:

$$\begin{aligned} \csc a &= \sec L \csc t; & \tan b &= \tan L \sec t \\ \tan \alpha &= \csc L \operatorname{ctn} t; & \csc h &= \sec a \sec C \\ \tan A &= \tan a \csc C; & \tan \beta &= \tan C \csc a \end{aligned}$$

The rules necessary for the use of the equations and the tables are given on the pages numbered 8 in the explanations in English and Portuguese. At first sight, they appear to be of the same order of complexity as those in H.O. 208, Dreisonstok (RMT **103**), but use proves them to be somewhat simpler.

The principal table (p. 36–215) has as vertical argument the local hour angle, t, 0(10)90°, and as horizontal argument the latitude, L, 0(10′)89° 50′. The tabulated quantities are a and b as defined above, each to the nearest minute of arc, and α. The values of a are given in heavy type to distinguish them from those of b. On the right-hand side of the page, α is given to the nearest tenth of a degree for each degree of local hour angle and for the middle of the degree of latitude.

One enters the table with the dead-reckoning latitude rounded off to the nearest ten minutes of arc and with the local hour angle of the body to the nearest degree; a and b are copied out for these arguments and α is taken for the nearest half degree of latitude. C is formed using the equation, $C = |b - d|$. One re-enters the table looking for a (rounded off to the nearest 10′) at the *bottom of the page* and C (to the nearest degree) along the *right-hand side of the page*. With these arguments, the dark-faced column yields the altitude, h, and β is found in the right-hand column opposite it. The azimuth angle is found by adding α and β.

In the explanation, the author indicates that the determination of β by this method is weak and proceeds to give three other methods of finding it. The first two involve the substitution of arguments; one can look for $90° - C$ in the β column and interpolate the value of β in the left-hand column above B, or one can interchange the values of a and C and interpolate the value of β in the column footed A. The third method is to use the last equation above with a table of log tangents and log secants. Actually the table given is one of values of $10^5 \log \tan x$ and $10^5 \log \sec x$ to the nearest integer for argument x, 0(1′)89° 59′.

To allow for the minutes discarded in a and C, Aquino suggests the use of:

$$\Delta h_1 = \Delta a \cos \beta, \qquad \Delta h_2 = \Delta C \cos A.$$

He provides a "difference of latitude and departure" table to simplify their use; the table will also be useful in dead reckoning. To allow for the minutes of latitude discarded, another table is provided on the inside of the back cover and the page facing it. This same table is offered on the two sides of a separate sheet of cardboard and again on one side of a separate folded sheet of heavy paper. The corrections, Δh_1 and Δh_2, may be avoided by the use of the equation for $\csc h$, and the log tangent − log secant table.

The author states that the tables were computed by means of Callet's and Bagay's seven-place logarithms, with many values determined by Vlacq's ten-place logarithms; Vega's ten-place table, based on Vlacq, is much more accurate. He further states that if the declination be taken to the nearest tenth of a minute, and Δh_1 and Δh_2 used, the maximum possible error in h will be 1.6′ but that the actual error in practise will hardly ever be over 0.5′. If only Δh_2 is used, the altitude obtained is always within 5′ of the true calculated altitude. Although the volume under review is dated 1943 there is in it a yellow sheet dated 9 April 1946, listing 32 corrigenda.

Other tables contained in the volume are four-place logarithms and antilogarithms with convenient proportional parts tables, distance to the horizon and dip of the horizon for different elevations, combined corrections for refraction, dip of the horizon and, where they are significant, semidiameter and parallax for planets and stars, upper and lower limb of the sun and lower limb of the moon, for altitudes 8° to 90° and for elevations 0 to 15 meters. A similar table for the upper limb of the moon would make a worth-while addition. A small auxiliary table allowing one to correct altitudes less than 8° for refraction is a valuable item, not often found in navigation tables and especially needed in the polar latitudes.

A table that remains in use almost half a century while other tables come and go, can reasonably be said to have a strong appeal to the average navigator. To appreciate Aquino's great contribution to navigation, one needs only to compare the first edition of this table which appeared in 1903 with other tables and methods then in use. It is unlikely that a person who has been trained in the use of H.O. 214 or H.O. 218 will change to this table, but a person who learned Aquino's method first might continue to prefer it because of its beautiful simplicity, its universality and the small bulk of the tables.

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398[U].—W. MYERSCOUGH & W. HAMILTON, *Rapid Navigation Tables*. London, Pitman, 1939. ii, 109 p. 16.4 × 26.6 cm. 10s. 6d.

These tables are designed for the solution of the astronomical triangle for altitude, h, and azimuth, Z, when latitude, L, hour angle, t, and declination, d, are known, and for similar problems. In the procedure for the determination of h, Myerscough & Hamilton follow such tables as those by SOUILLAGOUET, OGURA, WEEMS (RMT **315**), DREISONSTOK (RMT **103**), and HUGHES-COMRIE (RMT **115**), each of which divides the astronomical triangle at the zenith into two right triangles. The first triangle is solved for tabular values of L and t by a table which gives the remaining parts, as angles or as logarithmic functions of angles, without interpolation or other calculation. The second triangle is solved for h by logarithmic processes. The several tables differ only in notation and in that one of the auxiliary angles used in some of the tables is the complement of that used in the others.

In the determination of Z the several tables cited show a pleasing variety in method. Dreisonstok and Comrie follow BERTIN in deriving the two component parts of Z from the same auxiliary triangles as are used in the determination of h. Souillagouet utilizes another division of the astronomical triangle in order to get Z in one piece. Weems uses the graphical "Rust diagram," and in his *New Line of Position Tables* (RMT **315**) provides also an interesting variation on the Bertin procedure. Myerscough and Hamilton, however, follow Ogura in using the equation,

$$\cot Z = \cos L\,(\tan d \csc t - \tan L \cot t)$$

The most interesting and original feature of Myerscough & Hamilton is the inclusion of all data in one table of 91 pages (0 to 90°). At the first entry the page is selected for t, and for the left-hand argument L or d, as the case may be, the following quantities are extracted:

P = length of side of first auxiliary triangle opposite zenith, deg. & min.,
$Q = 10^5 \log \sec$ (side of same triangle opposite pole),
$X = 10 \tan L \cot t$, $\quad Y = 10 \tan d \csc t$.

For the second entry P is combined with d according to rules typical of such tables to give a side of the second auxiliary triangle. The page being for the degrees of this argument, and the entry by the minutes (using the same figures as were previously used for L and d), the following quantity is taken from the R-column of the table: $R = 10^5 \log \csc (P \sim d)$. For the third entry the R-column is searched for $Q + R = 10^5 \log \csc h$, and h is obtained by reading degrees at the top of the selected page and minutes in the left-hand column. The equivalence of this procedure to those of the other tables cited is easily recognized. For the fourth entry into the tables the page is selected for the latitude, and the Z-column is searched for $Y - X = 10 \cot Z \csc L$, and the azimuth is read opposite the nearest value. The four necessary openings equal those required by the other tables cited, so that the prospective user must seek grounds for preference in the arrangement of the tables, which is entitled at least to study by other table makers.

The table would be easier to use if it gave the four values of t for which a given page is used and not merely the one in the first quadrant. It would be improved also if the L and d argument went from 0 to 90° instead of stopping at 70°. (With these changes and two other minor ones the Y-column might be used in the fourth step, in order to eliminate the Z-column.) Since the tables are entered with L in the fourth step, there would be nothing gained by rearranging the tables for entry with L in the first, as there is in Hughes-Comrie and the new Weems. No data on the accuracy of the table are available.

The tables of Myerscough & Hamilton and of the other authors cited above seek to avoid interpolation in the second auxiliary triangle by the use of logarithmic trigonometric functions. While there may be some historical justification for such a treatment, it should be pointed out that H.O. 214 has accustomed navigators to interpolation. There is, accordingly, good reason to reexamine the possibilities of such methods as that of Bertin, in which both triangles are solved by a single table. The four openings of the various tables cited above are reduced to two, with two interpolations of about the same magnitude as those customary with H.O. 214 (RMT **399**). The Bertin method is possible with the well-known *Sea and Air Navigation Tables* of Captain RADLER DE AQUINO (RMT **397**), and with a new *Spheric Tabulations* of R. C. DOVE, R. F. D. No. 1, Collegeville, Penna.

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399[U].—U. S. HYDROGRAPHIC OFFICE, *Publication No. 214, Tables of Computed Altitude and Azimuth, Latitudes 80° to 89°, inclusive, Vol. 9.* Washington, D. C., U. S. Government Printing Office, 1946, 3, xxiv, 263 p. 22.6 × 29.1 cm. This is the last of nine uniform v. of H.O. 214, each v. devoted to 10° of latitude. For sale by the Hydrographic Office and by the Superintendent of Documents, Washington, D. C., $2.25 per v.; foreign price, postage extra.

This review will be limited primarily to a discussion of the differences between v. 9 and the other 8 v. of H.O. 214 which were reviewed earlier (RMT **105**). This volume, like the others, was prepared by the Work Projects Administration, (Philadelphia Project No. 24831), and presumably is of the same order of accuracy (see v. 2, p. 182f). The interval of argument for hour angle is 1° as in the other v.; it might well be 2° or perhaps even 5°, save for the loss of uniformity, since the tabulated altitude and azimuth change slowly and in a relatively linear fashion.

The description of the tables and their use is almost entirely new and occupies some ten pages more than that in the other v. The use of the pole as an assumed position is explained as well as the use of gnomonic, stereographic, azimuthal equidistant and inverse Mercator projections. A brief description of grid navigation is given.

Two ways in which this volume could be improved may be mentioned. The computed altitudes might be carried down to 0° or at least to 2°, since in the polar regions, the sun, moon, and planets spend a considerable fraction of the time at altitudes less than 5°; there are blank spaces available for these data. The second change would be to replace the refraction tables given in the front by others especially prepared for the conditions of temperature and barometric pressure commonly found in the polar regions.

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