Coefficients for Expressing the First Thirty Powers in Terms of the Hermite Polynomials[1]

The present table of coefficients gives the exact expression for x^n, $n = 0(1)30$, in terms of the HERMITE polynomials, $H_m(x)$, where $H_m(x) \equiv (-1)^m e^{x^2}(d^m e^{-x^2}/dx^m)$. The computation followed the formula given in E. FELDHEIM, "Formules d'inversion et autres relations pour les polynomes orthogonaux classiques," Soc. Math. d. France, *Bull.*, v. 68, p. 199–228, namely,

$$x^n = 2^{-n} \sum_{\nu=0}^{[\frac{1}{2}n]} n!/\{\nu!(n-2\nu)!\}H_{n-2\nu}(x).$$

The quantities $n!/\{\nu!(n-2\nu)!\}$ were expressed as $\binom{n}{2\nu}(2\nu)!/\nu!$ After obtaining the factors $(2\nu)!/\nu!$, they were multiplied by $\binom{n}{2\nu}$, which were taken from J. W. L. GLAISHER's table of binomial coefficients (*Mess. Math.*, v. 47, p. 97–107). An overall check on the final manuscript was provided by evaluating the right member of each of the expressions for x^n that are given below, for $x = 1$. The necessary auxiliary quantities $H_m(1)$ were obtained from the recurrence formula $H_{n+1}(1) = 2H_n(1) - 2n\,H_{n-1}(1)$.

Similar to the use for the expression of powers in terms of LEGENDRE polynomials, this present table is useful for approximating a polynomial of high degree by a polynomial of much lower degree, which is best in an important least square sense (interval $-\infty$ to $+\infty$, weight factor e^{-x^2}). Thus when any function $f(x)$, in the interval $[-\infty, \infty]$, is expanded in terms of Hermite polynomials (where the coefficients in that expansion are determined by the condition of orthogonality with weight factor e^{-x^2} in the interval $[-\infty, \infty]$), if the partial sum of degree r is denoted by $q_r(x)$, while any polynomial of degree $\leqslant r$ which is distinct from $q_r(x)$ is denoted by $p_r(x)$, then

$$\int_{-\infty}^{\infty} e^{-x^2}[f(x) - q_r(x)]^2 dx < \int_{-\infty}^{\infty} e^{-x^2}[f(x) - p_r(x)]^2 dx.$$

The analogous set of coefficients for the expansion of x^n in terms of the LAGUERRE polynomials $L_m(x) \equiv (e^x/m!)d^m(e^{-x}x^m)/dx^m$ is useful for best approximation in the least square sense, where the interval is $[0, \infty]$ and the weight factor is e^{-x}. However it is unnecessary to compute them, since from the expression

$$x^n = n! \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{n-\nu} L_\nu(x),$$

it is obvious that all one needs are the binomial coefficients, and Glaisher's readily available table provides the coefficients up to x^{50}. Coefficients up to

[1] The coefficient of H_m in the expression for $(2x)^n$ is identical (except for alternation of + and − sign) with the coefficient of $(2x)^m$ in $H_n(x)$. Thus there are given here also the first thirty Hermite polynomials.

x^{60} may be had in the appendix, by J. PETERS & J. STEIN, to J. Peters, *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922.

$$
\begin{aligned}
x^0 &= H_0\\
2\,x^1 &= H_1\\
4\,x^2 &= H_2 + 2H_0\\
8\,x^3 &= H_3 + 6H_1\\
16\,x^4 &= H_4 + 12\,H_2 + 12\,H_0\\
32\,x^5 &= H_5 + 20\,H_3 + 60\,H_1\\
64\,x^6 &= H_6 + 30\,H_4 + 180\,H_2 + 120\,H_0\\
128\,x^7 &= H_7 + 42\,H_5 + 420\,H_3 + 840\,H_1\\
256\,x^8 &= H_8 + 56\,H_6 + 840\,H_4 + 3360\,H_2 + 1680\,H_0\\
512\,x^9 &= H_9 + 72\,H_7 + 1512\,H_5 + 10080\,H_3 + 15120\,H_1\\
1024\,x^{10} &= H_{10} + 90\,H_8 + 2520\,H_6 + 25200\,H_4 + 75600\,H_2 + 30240\,H_0\\
2048\,x^{11} &= H_{11} + 110\,H_9 + 3960\,H_7 + 55440\,H_5 + 2\ 77200\,H_3 + 3\ 32640\,H_1\\
4096\,x^{12} &= H_{12} + 132\,H_{10} + 5940\,H_8 + 1\ 10880\,H_6 + 8\ 31600\,H_4 + 19\ 95840\,H_2\\
&\quad + 6\ 65280\,H_0\\
8192\,x^{13} &= H_{13} + 156\,H_{11} + 8580\,H_9 + 2\ 05920\,H_7 + 21\ 62160\,H_5\\
&\quad + 86\ 48640\,H_3 + 86\ 48640\,H_1\\
16384\,x^{14} &= H_{14} + 182\,H_{12} + 12012\,H_{10} + 3\ 60360\,H_8 + 50\ 45040\,H_6\\
&\quad + 302\ 70240\,H_4 + 605\ 40480\,H_2 + 172\ 97280\,H_0\\
32768\,x^{15} &= H_{15} + 210\,H_{13} + 16380\,H_{11} + 6\ 00600\,H_9 + 108\ 10800\,H_7\\
&\quad + 908\ 10720\,H_5 + 3027\ 02400\,H_3 + 2594\ 59200\,H_1\\
65536\,x^{16} &= H_{16} + 240\,H_{14} + 21840\,H_{12} + 9\ 60960\,H_{10} + 216\ 21600\,H_8\\
&\quad + 2421\ 61920\,H_6 + 12108\ 09600\,H_4 + 20756\ 73600\,H_2\\
&\quad + 5189\ 18400\,H_0\\
1\ 31072\,x^{17} &= H_{17} + 272\,H_{15} + 28560\,H_{13} + 14\ 85120\,H_{11} + 408\ 40800\,H_9\\
&\quad + 5881\ 07520\,H_7 + 41167\ 52640\,H_5 + 1\ 17621\ 50400\,H_3\\
&\quad + 88216\ 12800\,H_1\\
2\ 62144\,x^{18} &= H_{18} + 306\,H_{16} + 36720\,H_{14} + 22\ 27680\,H_{12} + 735\ 13440\,H_{10}\\
&\quad + 13232\ 41920\,H_8 + 1\ 23502\ 57920\,H_6 + 5\ 29296\ 76800\,H_4\\
&\quad + 7\ 93945\ 15200\,H_2 + 1\ 76432\ 25600\,H_0\\
5\ 24288\,x^{19} &= H_{19} + 342\,H_{17} + 46512\,H_{15} + 32\ 55840\,H_{13} + 1269\ 77760\,H_{11}\\
&\quad + 27935\ 10720\,H_9 + 3\ 35221\ 28640\,H_7 + 20\ 11327\ 71840\,H_5\\
&\quad + 50\ 28319\ 29600\,H_3 + 33\ 52212\ 86400\,H_1\\
10\ 48576\,x^{20} &= H_{20} + 380\,H_{18} + 58140\,H_{16} + 46\ 51200\,H_{14} + 2116\ 29600\,H_{12}\\
&\quad + 55870\ 21440\,H_{10} + 8\ 38053\ 21600\,H_8 + 67\ 04425\ 72800\,H_6\\
&\quad + 251\ 41596\ 48000\,H_4 + 335\ 22128\ 64000\,H_2 + 67\ 04425\ 72800\,H_0\\
20\ 97152\,x^{21} &= H_{21} + 420\,H_{19} + 71820\,H_{17} + 65\ 11680\,H_{15} + 3418\ 63200\,H_{13}\\
&\quad + 1\ 06661\ 31840\,H_{11} + 19\ 55457\ 50400\,H_9 + 201\ 13277\ 18400\,H_7\\
&\quad + 1055\ 94705\ 21600\,H_5 + 2346\ 54900\ 48000\,H_3\\
&\quad + 1407\ 92940\ 28800\,H_1\\
41\ 94304\,x^{22} &= H_{22} + 462\,H_{20} + 87780\,H_{18} + 89\ 53560\,H_{16} + 5372\ 13600\,H_{14}\\
&\quad + 1\ 95545\ 75040\,H_{12} + 43\ 02006\ 50880\,H_{10} + 553\ 11512\ 25600\,H_8\\
&\quad + 3871\ 80585\ 79200\,H_6 + 12906\ 01952\ 64000\,H_4\\
&\quad + 15487\ 22343\ 16800\,H_2 + 2815\ 85880\ 57600\,H_0\\
83\ 88608\,x^{23} &= H_{23} + 506\,H_{21} + 1\ 06260\,H_{19} + 121\ 13640\,H_{17} + 8237\ 27520\,H_{15}\\
&\quad + 3\ 45965\ 55840\,H_{13} + 89\ 95104\ 51840\,H_{11} + 1413\ 51642\ 43200\,H_9\\
&\quad + 12721\ 64781\ 88800\,H_7 + 59367\ 68982\ 14400\,H_5\\
&\quad + 1\ 18735\ 37964\ 28800\,H_3 + 64764\ 75253\ 24800\,H_1\\
167\ 77216\,x^{24} &= H_{24} + 552\,H_{22} + 1\ 27512\,H_{20} + 161\ 51520\,H_{18}\\
&\quad + 12355\ 91280\,H_{16} + 5\ 93083\ 81440\,H_{14} + 179\ 90209\ 03680\,H_{12}\\
&\quad + 3392\ 43941\ 83680\,H_{10} + 38164\ 94345\ 66400\,H_8\\
&\quad + 2\ 37470\ 75928\ 57600\,H_6 + 7\ 12412\ 27785\ 72800\,H_4\\
&\quad + 7\ 77177\ 03038\ 97600\,H_2 + 1\ 29529\ 50506\ 49600\,H_0
\end{aligned}
$$

$$\begin{aligned}
335\ 54432\, x^{25} = {} & H_{25} + 600\, H_{23} + 1\ 51800\, H_{21} + 212\ 52000\, H_{19} \\
& + 18170\ 46000\, H_{17} + 9\ 88473\ 02400\, H_{15} + 345\ 96555\ 84000\, H_{13} \\
& + 7710\ 08958\ 72000\, H_{11} + 1\ 06013\ 73182\ 40000\, H_{9} \\
& + 8\ 48109\ 85459\ 20000\, H_{7} + 35\ 62061\ 38928\ 64000\, H_{5} \\
& + 64\ 76475\ 25324\ 80000\, H_{3} + 32\ 38237\ 62662\ 40000\, H_{1}
\end{aligned}$$

$$\begin{aligned}
671\ 08864\, x^{26} = {} & H_{26} + 650\, H_{24} + 1\ 79400\, H_{22} + 276\ 27600\, H_{20} \\
& + 26246\ 22000\, H_{18} + 16\ 06268\ 66400\, H_{16} + 642\ 50746\ 56000\, H_{14} \\
& + 16705\ 19410\ 56000\, H_{12} + 2\ 75635\ 70274\ 24000\, H_{10} \\
& + 27\ 56357\ 02742\ 40000\, H_{8} + 154\ 35599\ 35357\ 44000\, H_{6} \\
& + 420\ 97089\ 14611\ 20000\, H_{4} + 420\ 97089\ 14611\ 20000\, H_{2} \\
& + 64\ 76475\ 25324\ 80000\, H_{0}
\end{aligned}$$

$$\begin{aligned}
1342\ 17728\, x^{27} = {} & H_{27} + 702\, H_{25} + 2\ 10600\, H_{23} + 355\ 21200\, H_{21} \\
& + 37297\ 26000\, H_{19} + 25\ 51132\ 58400\, H_{17} + 1156\ 51343\ 80800\, H_{15} \\
& + 34695\ 40314\ 24000\, H_{13} + 6\ 76560\ 36127\ 68000\, H_{11} \\
& + 82\ 69071\ 08227\ 20000\, H_{9} + 595\ 37311\ 79235\ 84000\, H_{7} \\
& + 2273\ 24281\ 38900\ 48000\, H_{5} + 3788\ 73802\ 31500\ 80000\, H_{3} \\
& + 1748\ 64831\ 83769\ 60000\, H_{1}
\end{aligned}$$

$$\begin{aligned}
2684\ 35456\, x^{28} = {} & H_{28} + 756\, H_{26} + 2\ 45700\, H_{24} + 452\ 08800\, H_{22} \\
& + 52216\ 16400\, H_{20} + 39\ 68428\ 46400\, H_{18} + 2023\ 89851\ 66400\, H_{16} \\
& + 69390\ 80628\ 48000\, H_{14} + 15\ 78640\ 84297\ 92000\, H_{12} \\
& + 231\ 53399\ 03036\ 16000\, H_{10} + 2083\ 80591\ 27325\ 44000\, H_{8} \\
& + 10608\ 46646\ 48202\ 24000\, H_{6} + 26521\ 16616\ 20505\ 60000\, H_{4} \\
& + 24481\ 07645\ 72774\ 40000\, H_{2} + 3497\ 29663\ 67539\ 20000\, H_{0}
\end{aligned}$$

$$\begin{aligned}
5368\ 70912\, x^{29} = {} & H_{29} + 812\, H_{27} + 2\ 85012\, H_{25} + 570\ 02400\, H_{23} \\
& + 72108\ 03600\, H_{21} + 60\ 57075\ 02400\, H_{19} + 3452\ 53276\ 36800\, H_{17} \\
& + 1\ 34155\ 55881\ 72800\, H_{15} + 35\ 21583\ 41895\ 36000\, H_{13} \\
& + 610\ 40779\ 26186\ 24000\, H_{11} + 6714\ 48571\ 88048\ 64000\, H_{9} \\
& + 43949\ 36106\ 85409\ 28000\, H_{7} + 1\ 53822\ 76373\ 98932\ 48000\, H_{5} \\
& + 2\ 36650\ 40575\ 36819\ 20000\, H_{3} + 1\ 01421\ 60246\ 58636\ 80000\, H_{1}
\end{aligned}$$

$$\begin{aligned}
10737\ 41824\, x^{30} = {} & H_{30} + 870\, H_{28} + 3\ 28860\, H_{26} + 712\ 53000\, H_{24} \\
& + 98329\ 14000\, H_{22} + 90\ 85612\ 53600\, H_{20} + 5754\ 22127\ 28000\, H_{18} \\
& + 2\ 51541\ 67278\ 24000\, H_{16} + 75\ 46250\ 18347\ 20000\, H_{14} \\
& + 1526\ 01948\ 15465\ 60000\, H_{12} + 20143\ 45715\ 64145\ 92000\, H_{10} \\
& + 1\ 64810\ 10400\ 70284\ 80000\, H_{8} + 7\ 69113\ 81869\ 94662\ 40000\, H_{6} \\
& + 17\ 74878\ 04315\ 26144\ 00000\, H_{4} + 15\ 21324\ 03698\ 79552\ 00000\, H_{2} \\
& + 2\ 02843\ 20493\ 17273\ 60000\, H_{0}.
\end{aligned}$$

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RECENT MATHEMATICAL TABLES

519[A].—A. J. SACHS, "Babylonian mathematical texts, I. Reciprocals of regular sexagesimal numbers," *Jn. Cuneiform Studies*, v. 1, 1947, p. 219–240. 21.5 × 27.9 cm.

Regular sexagesimal numbers are those whose reciprocals may be expressed in a finite number of terms. In Old-Babylonian table texts (say, 1700 B.C.) the object of a set of multiplication tables was not only to yield the results of any multiplication, but also to give the multiples of reciprocals commonly used in division. "The existence of a multiplication table for the three-place number 44, 26, 40, for example, makes sense only in the light of the fact that 44, 26, 40 is the reciprocal of 1, 21."

Dr. Sachs has established the standard technique employed in the Old-Babylonian period to find the reciprocal of any regular number which is not contained in the standard reciprocal table (say, 2, 5 or 23, 43, 49, 41, 15).