

RECENT MATHEMATICAL TABLES

553[D].—WADE ELLIS & JOHN E. WALSH, A. *Table: $x \tan x$* . Electronics Research Laboratories, Air Materiel Command, Cambridge, Mass. (*ERL Report* no. E5016). Feb., 1948, iv, 19 p. 21.2 × 27.4 cm. B. *Table: $x \tanh x$* . Cambridge Field Station, Air Materiel Command, Cambridge, Mass. (*CFS Report* no. E5030). June, 1948, iv, 52 p. 21.2 × 27.4 cm.

In several problems considered in the Antenna Laboratory, Cambridge, Mass., the functions $f_1(x) = x \tan x$ and $f_2(x) = x \tanh x$ have arisen. Tables of these functions with the required range, interval of argument, and number of significant figures apparently do not exist; consequently, such tables were computed specifically to satisfy these needs. They were prepared in the form of reports with the belief that they might prove useful on a much wider basis.

A. Values of the function $f_1(x)$ are given for $x = [0(.001)1.570; 8D \text{ or } 8S]$. For the computation, values of $\tan x$ were taken from NBSCL, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*, New York, 1943. The range is immediately extended to $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, by observing the identity $x \tan x = -x \tan(-x)$. The range may be extended without limit by use of the identity

$$x \tan x = [x/(x - n\pi)](x - n\pi) \tan(x - n\pi).$$

However, it is probably more convenient to compute values directly for arguments outside the range $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

It is believed that for many applications the arguments are sufficiently close together to render interpolation unnecessary. In cases where this is not true, reference should be made to the above mentioned table, p. xvii-xxii, or to any standard work on interpolation.

The parts of the table on pages 5 and 16 should be interchanged.

B. Values of the function $f_2(x) = x \tanh x$, which does not seem to have been previously tabulated, are given for $x = [0(.001)5; 8S]$. By virtue of the relation $(-x) \tanh(-x) = x \tanh x$ the range is immediately extended to $-5 \leq x \leq 5$. The computations were carried out at the Center of Analysis, Mass. Inst. of Technology.

For the computations, values of $\tanh x$, $0 \leq x \leq 2$ were taken from NBSCL, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*, New York, 1943. For $2 \leq x \leq 5$, HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen . . .*, Berlin, 1926, furnished the values of $\tanh x$, at interval .01, so that extensive subtabulation was necessary in order to provide values at .001 radian intervals of argument.

Extracts from text

EDITORIAL NOTES: There is a table of $x \tan x$ for $x = [0(.01)3.14, \frac{1}{2}\pi, \pi; 4-5S]$, in E. JAHNKE & F. EMDE, *Tables of Functions*, 1933 (p. 32-35); 1943, and 1945. Addenda, p. 32-35. In F. EMDE, *Tables of Elementary Functions*, Leipzig and Berlin, 1940, p. 125-126, there is a table of $x \tan x$ for $x = [0(10^0)1800^0; 4-5S]$. Up to $x = 900^0$ this table is copied in M. BOLL, *Tables Numériques Universelles*. Paris, 1947, p. 332; on p. 326-329, the JAHNKE & EMDE Table listed above is reprinted.

554[D].—[EMMA GIFFORD (1861-1936)], *Natural Tangents to Eight Decimal Places for Every Second of Arc from 0° to 90°*. First pirated American edition. Parker & Company, publishers, 241 E. Fourth St., Los Angeles, California, 1948, xii, 620 p. 17.2 × 26.7 cm. \$15.00.

Of Mrs. Gifford's three published volumes of 8-place natural values of sines, tangents, secants, for every second of arc, the second, of tangents, appeared in instalments, beginning in 1920, and completed in 1927 (Manchester, viii, 620 p.). Of this edition the leaf (a), p. i-ii, is headed "Some discrepancies in Natural Tangents between Rheticus and Andoyer. The tangents from Rheticus are given in full and those figures in which Andoyer differs from him

are given underneath." (There are 41 10-place values in the list.) Leaves (b) and (c), p. iii–vi, are for half-title, and title pages. Leaf (d), p. vii–viii, contains a preface, and a list of 31 tangents which "have 0 in their 5th, 6th, 7th and 8th places." [As a matter of fact for two of the values thus listed, $36^{\circ}7'46''$ and $48^{\circ}35'44''$, PETERS does not agree with Mrs. Gifford in his similar 1939 volume for the trigonometric functions sin, tan, cot, cos.]

Mrs. Gifford's table for every second of arc is arranged quadrantally, horizontally up to 70° , each line corresponding to $10''$, $10'$ to a page, and then vertically 70° – 90° , 6 minutes to a page. Differences, so called, given up to 84° , would have been much more advantageously replaced by proportional parts so that accuracy to hundredths of a second might have been readily found.

The preparation of the table was based on the *Opus Palatinum*, 1596, of Rheticus. This is a 10D canon, at interval $10''$, with differences for all the tabular results, the six trigonometric functions arranged semiquadrantly.

From 45° to 60° (Mrs. G. remarks) some of the tangents, particularly those in the central columns of the page, read 1 too high in the 8th decimal place. The column headed 0 is correct throughout and so are the other tangents after 60° . Any discrepancies which may be noticed between cotangents in the first part and tangents in the second, are due to the fact that the second part has been corrected by ANDOVER's Table of Natural Tangents (Paris, 1916, at interval $10''$), which was not published when this work was begun. The table fills exactly 620 pages.

In 1940 a second edition, offset print, of this table was prepared by the Scientific Computing Service, London (x, 620 p.), and 55 errors in the table of this second edition are listed on p. ix. Six other listed errors in the first edition (p. 157, 391(2), 617(2), 619) were corrected, but there are some 600 other known errors. This edition has been out of print since 1944.

Let us now turn to the edition under review. The 620 pages of the table differ only from the corresponding pages of the second edition of Gifford in the following respects: (1) the table has been photographically enlarged; (2) the 55 errata referred to above have been corrected in the table; (3) the page-numbers which, in the original, were at the bottom of each page in the center, are now (in a different font of type) at the outer edges of each page; (4) the statement at the bottom of p. 615–620 ("When the whole number has 3 digits, only 7 decimal places are given") has been eliminated, and put in a preface, which is signed by R. B. HUEY. In this preface is given the list referred to above in (d), and the preface then concludes "Every effort has been made to eliminate errors in the preparation of this book but if any discrepancies are discovered we should appreciate being notified so that they may be corrected." From what we have indicated above, the "effort" made by Mr. Huey was confined to ordering the 55 corrections listed as errors in the second edition to be made in the new pirated edition, but hundreds of other errors persist.

The xii introductory pages of this Parker Co. edition are occupied as follows: p. i, title page. Under the title quoted above is the picture of an engineer using a theodolite. Under this picture is "R. B. Huey." Thus the title-page does not actually state that the table was by R. B. Huey. P. ii is blank; on p. iii we find "Copyright 1948 by Parker and Company, Los Angeles" and "Printed in the United States of America." The United States is the only country in the world which requires that a book from another country, to be copyrighted in the United States, must be printed there. Here, then, is a case of a publisher, who appropriated the work of an author, without permission or compensation, and published it as his own, without any reference whatever to the real author of the volume, or to her estate. A casual inspection of the volume would suggest that its author was Huey, but no definite statement to this effect is anywhere made.

On raising certain questions concerning the table Mr. Huey's reply to the reviewer contains the following statements: "The table was not computed by me as this work would have been superfluous. There were several tables of this sort computed in Europe in the past and this was taken from one that has been out of print, I understand, for more than twenty years, and was not available to the public. I would have preferred mentioning this

fact in the book but, because of certain technicalities, was advised not to. The only credit I wish to, or can, claim for the publication of the book is the fact that I spent a great deal of time and money to make it available to every Engineer and Surveyor in the United States at the lowest cost possible—and as soon as possible—as I realized the great need for such a book."

P. iv is blank; p. v, "About R. B. Huey" is signed "Parker and Company." On p. vi-vii are tables "Lengths of the arcs of circles to the radius 1" and "Minutes in decimals of a degree"—the only tables not in Gifford. P. viii is a reproduction of p. (a) referred to above. P. ix-x is an English preface, and p. xi-xii the same preface in Spanish. The tabular part of the strongly bound pirated volume is clearly reproduced on substantial paper.

In *MTAC* we have unfortunately had occasion more than once to refer to other appropriations in this country similar to that indicated above. In *MTAC*, v. 1, p. 112, we noted that a WITTSTEIN book was published in Chicago with the suggestion that a William W. Johnson was its author; on p. 8 and 112 of v. 1 we referred also to a JORDAN volume published in New York, as by a Wm. Chas. Müller.

Has the volume under review any real justification, so far as scientific needs of this country are concerned? Who would pay \$15 for the volume described above, if he knows that at Ann Arbor, Michigan, he can for \$20 purchase a copy of an outstanding, wholly accurate, work containing twice as many functions: J. T. PETERS, *Eight-Place Table of Trigonometric Functions for Every Sexagesimal Second of the Quadrant*, 1943? (See *MTAC*, v. 1, p. 147-148.) Here, in semi-quadrantal arrangements, sine, tangent, cotangent, cosine may be readily read off in separate columns.

R. C. A.

555[F].—A. GLODEN, "Sur les nombres terminaux des cubes dans le système de numération décimal," *Euclides*, v. 7, 1947, p. 393-397. 16.6×24.1 cm.

There are three tables giving the 1-, 2-, and 3-digit endings of cubes together with the linear forms of those numbers whose cubes have the given ending. The tables have 10, 63, and 505 lines respectively, and are arranged according to increasing values of the ending. This table, similar to others for squares, is useful for the rapid identification of non-cubes. A manuscript of this table was reported in *MTAC*, v. 2, p. 354, UMT 61.

D. H. L.

556[F].—H. GUPTA, "A table of values of $\tau(n)$," *Nat. Inst. of Sci., India, Proc.*, v. 13, 1947, p. 201-206. 17.5×24.9 cm.

The function $\tau(n)$, usually called Ramanujan's function, may be defined as the coefficient of x^{n-1} in the expansion of the 24th power of the product $(1-x)(1-x^2)(1-x^3) \dots$ and occurs in the arithmetic theory of elliptic functions. This table gives $\tau(n)$ for $n \leq 400$. There is no indication of the methods employed. Previous tables have been given by RAMANUJAN¹ ($n \leq 30$), LEHMER² ($n \leq 300$) and GUPTA³ ($n \leq 130$). A remarkable table of $\tau(n)$ for $n \leq 1000$ by G. N. WATSON has been in press since 1942. Galley proof kindly supplied by its author has been read against the table under review. The agreement is perfect.

Extensive tables of $\tau(n)$ are useful in the study of the many properties and unsolved questions concerning this interesting function.

D. H. L.

¹ S. RAMANUJAN, "On certain arithmetical functions," *Cambridge Phil. Soc., Trans.*, v. 22, 1916, p. 174; *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 153.

² D. H. LEHMER, "Ramanujan's function $\tau(n)$," *Duke Math. Jn.*, v. 10, 1943, p. 483-492; see *MTAC*, v. 1, p. 183-184.

³ H. GUPTA, "Congruence properties of $\tau(n)$," *Benares Math. Soc., Proc.*, s. 2, v. 5, 1943, p. 17-22; see *MTAC*, v. 2, p. 26-27.

557[F].—D. YARDEN & A. KATZ "Goremim chadashim shel mispare Fibonacci" [New factors of Fibonacci numbers], *Riveon Lematematika*, v. 2, Jan. 1948, p. 35. 21.6 × 33.6 cm.

In this short note the factors 1069, 29717 and 27941 are announced for the 89th, 117th and 127th terms, respectively, of the Fibonacci series $u_n = 1, 1, 2, 3, 5, \dots$. Besides these three factors and the ones already given in Yarden's previous table and supplement (*MTAC*, v. 2, p. 343; v. 3, p. 96) there are no further factors of the first 128 terms of the series u_n and $v_n = u_{2n}/u_n$ below 10^6 . Of these 256 numbers 55 are as yet incompletely factored. As indicated above, the rank of apparition of 1069 is 89. In a previous table of the ranks of apparition of primes ≤ 1511 (*MTAC*, v. 2, p. 343) the rank of 1069 is given as 534. This error is noted in conclusion.

D. H. L.

558[H, I, L, S, V].—AKAD. N., MATEMATICHESKIĬ INSTITUT imeni V. A. STEKLOVA, *Trudy*, v. 20, Leningrad & Moscow, 1947, 136 p. 15 × 24.6 cm. Paper covers, 10 roubles. Entirely in Russian.

This is a collection of works completed, during the years 1943–1944, in the Section on approximate calculations of the Mathematical Institute, edited by L. A. LIŪSTERNIK & V. A. DITKIN. The v. contains the following articles: I. ĩa. AKUSHSKIĬ & V. A. DITKIN, "O ĥislenom reshenii uravneniã ĥsirkuliãĥsii kolebliŭshchegosã kryla" [On a numerical solution of the equation of circulation of an oscillating wing], p. 7–38; an illustrative example worked out with details. I. ĩa. AKUSHSKIĬ, "O nekotorykh voprosakh, svãzannykh s primeneniem schetno-analiticheskikh mashin" [On certain questions, connected with the application of analytical computing machines], p. 39–48. L. A. LIŪSTERNIK, "Zamechaniã k ĥislenomu resheniiŭ kraevykh zadach uravneniã Laplasy i vychisleniiŭ sobstvennykh znacheniiŭ metodom setok" [Observations on the numerical solution of the boundary problems of Laplace's equation and on the computation of eigenvalues by the method of nets], p. 49–64. B. I. SEGAL, "Ob odnoi zadache teploprovodnosti" [On a certain problem of heat conduction], p. 65–76; an illustration worked out in detail. V. A. DITKIN, "Reshenie odnoi zadachi teploprovodnosti metodom operatsionnogo ischisleniã" [Solution of a certain problem of heat conduction by the method of operational computation], p. 77–86; this deals with the problem, by SEGAL, in the preceding art., and on p. 85–86 gives a table of values of $u(x)$ and $-v(x)$, defined by the relation $xH_1^{(u)}(x)/H_0^{(u)}(x) = u(x) + iv(x)$, where $H_0^{(u)}$ and $H_1^{(u)}$ are HANKEL's functions, for $x = [0(.02)1; 4D]$ and $[1(.1)9.9; 3D]$. L. ĩa. NEĬSHULER, "O tabulirovanii funkĥsii trech peremennykh" [On the tabulation of functions of three variables], p. 87–108; gives several schedules showing skeletons of tables. L. I. SHATROVSKIĬ, "Primenenie metoda Neishulera k sostavleniiŭ tablĩs vozdušnoi strel'by" [Application of Neishuler's method to the construction of tables for aerial gunnery], p. 109–112. L. ĩa. NEĬSHULER, "Zametki po tabulirovaniũ" [Notes on tabulation], p. 113–116; 1. "Ob odnom variante tablĩsy umnozheniã" [On a certain variant of the multiplication table], p. 113–114; 2. "O tabulirovanii sistem lineinykh funkĥsii mnogikh peremennykh" [On the tabulation of systems of linear functions of many variables], p. 115–116. V. M. PROSHKO, "Elektricheskii pribor dlã resheniã sistemy sovместnykh lineinykh algebraicheskikh uravnenii" [Electric apparatus for the solution of a system of simultaneous linear algebraic equations], p. 117–128. I. S. GRADSHTEIN, "Pribor dlã chercheniã grafika funkĥsii ot funkĥsii" [Apparatus for drawing a graph of a function of a function], p. 129–130. B. KHANOV, "Kinematicheskoe reshenie trekhĥlennoĥ uravneniã" [A kinematic solution of a trinomial equation], p. 131–133. V. A. DITKIN, "Rabota seminarã pri otdelã priblizhennykh vychisleniiŭ za 1943–1944 gg" [The work accomplished in the seminar at the section of approximate computations during the years 1943–1944], p. 134–135; this lists eleven authors, with corresponding titles, some of which are published in the present volume.

We note a few errata which occur in tables: p. 35, l. 12, for B^a , read B_{7b} ; p. 54, l. –7, –5, for 3.95, read 3.85; p. 55, l. 6, for 31.42, read 39.42; p. 55, l. –4, for 22.58, read 22.55; p. 86, l. 8, for 4.91, read .491.

S. A. J.

559[K].—RAYMOND T. BIRGE, "Least squares' fitting of data by means of polynomials," *Rev. Modern Physics*, v. 19, 1947, p. 298–347. "Mathematical appendix," by J. W. WEINBERG, p. 348–360. 19.8×26.7 cm.

The purpose of the present paper is to set forth a critique of methods which have been developed for fitting a polynomial to empirical data by the method of least squares. The author characterizes the different schemes for polynomial approximation as follows: "The least-squares' solution of a polynomial may be given in (A) power-series form, or (B) factorial form. The solution may involve the explicit use of (1) power moments of the observations, (2) factorial moments of the observations, (3) the observations themselves, and (4) the various finite differences of the observations. Finally, the solution may be in (α) orthogonal polynomial form, (β) non-orthogonal polynomial form."

The author contrasts various solutions which have been given to this problem, beginning properly with the classical memoirs of P. L. CHEBYSHEV, which appeared just after the middle of the nineteenth century. The principal object of the paper is to give an exposition of the method of polynomial curve fitting, which was developed by BIRGE in collaboration with J. D. SHEA in 1927 and now slightly modified in the present paper. This method is compared with rival methods for solving the same problem. The paper is somewhat polemical as one may infer from the statement relative to the modified Birge & Shea procedure: "The method about to be presented represents what now appears to the writer the one most advantageous for the general use of physical scientists." There must be at least a dozen different techniques in existence, those, for example, of A. C. AITKEN, R. A. FISHER & F. YATES, S. M. KERAWALA, V. PARETO, E. JÄDERIN, C. JORDAN, F. E. ALLAN, and M. SASULY, to mention only a few of them. All have their merits and also certain defects, which depend in part upon the data which are to be fitted, and the form in which they are given. When the order of approximation is uncertain, then obviously it is better to employ a method which depends upon orthogonal polynomials, since the coefficients of added terms can be determined independently of those already computed. It would seem doubtful that any "best method" exists for all types of data and all problems.

The paper also devotes attention to the important problem of estimating the error of approximation. This is obtained from the sum of the squares of the residuals, a quantity which has been given previously in several places, and which is readily seen to be equal to the defect in Bessel's well-known inequality in the general theory of orthogonal functions.

The method of Birge & Shea, as well as the modified method given in the present paper, is designed to obtain the coefficients in the expression,

$$u_j(\epsilon) = \sum_{i=0}^j a_i T_i(\epsilon),$$

where $T_i(\epsilon)$ are polynomials, which satisfy the recurrence relationship

$$4(4t^2 - 1)T_{t+1}(\epsilon) - 4(4t^2 - 1)\epsilon T_t(\epsilon) + t^2(n^2 - t^2)T_{t-1}(\epsilon) = 0.$$

This difference equation goes back to Chebyshev and the polynomials to $t = 8$ were explicitly given by V. PARETO in 1899 (*Z. f. Schweizerische Statistik*, v. 1, 1899, p. 121–150). Two additional ones were later added by F. E. ALLAN in 1930 (*R. Soc. Edinb., Proc.*, v. 50, 1930, p. 310–320). The explicit formulae for the Chebyshev polynomials through $n = 10$ are reprinted in the present article.

The paper contains 14 tables of which five give illustrative data, one, namely Table III, gives the explicit forms of the polynomials, two give algebraic relationships between the various variables of the theory, and the remainder contain values of the various quantities used in the technique of fitting the polynomials to data. They are of small intrinsic interest, but necessary in the application of the theory. The last table gives the values of certain functions used in the theory of fitting the polynomials to data when finite differences, instead of direct observations, are used.

H. T. D.

560[K].—MARIAN C. JACKMAN, Tables 1, 2, 5, in C. I. BLISS, "Estimation of the mean and its error from incomplete Poisson distributions," Connecticut Agricultural Experiment Station, New Haven, *Bull.* no. 513, Jan. 1948, 12 p. 15.2 × 22.8 cm.

The phrase "random events" refers to points distributed either in time or in space. Typical examples for the latter are spores or bacteria in blood, animal litters in a field, weed-seeds among grass-seeds. The usual definition of complete randomness is that the probability p_k of having exactly k events in an interval, area, or volume of specified size is given by the POISSON distribution

$$p_k = e^{-m} m^k / k!,$$

where m is a positive constant, the so-called mean of the distribution. The problem then arises to estimate m from actual observations and to judge the reliability of the estimate. The usual methods are based on a complete enumeration of all cases where 0, 1, ... events were observed. The author points out that very often it is difficult or impossible to obtain such a complete enumeration. Instead, the data are truncated at a class x so that only cases with 0, 1, ..., x events are enumerated and no information is available concerning cases with more than x events. The author describes a technique of estimating m and its error from such incomplete data. To facilitate the work he gives tables which are obtained from E. C. D. MOLINA, *Poisson's Exponential Binomial Limit*, New York, 1942.

Table 1 (p. 3) gives to 5D the cumulated values $P_x = p_0 + p_1 + \dots + p_x$ for $x = 0, 1, 2, 3$ and $m = .1(1)6$. Table 2 (p. 4) gives $p_x/(1 - P_x)$ within the same range. This table is to 6D or 6S. Table 5 (p. 8) gives values obtained from these tables and useful for the estimation of the error: this table is of interest only to specialized statisticians. Other tables refer to empirical observations.

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561[K].—T. L. KELLEY, *The Kelley Statistical Tables. Revised 1948*. Cambridge, Mass., Harvard Univ. Press. 1948, x, 223 p. 21.4 × 27.8 cm. \$6.00. The first edition was published in 1938; see *MTAC*, v. 1, p. 151–152.

It is not the purpose of this book to provide an extensive selection of many tables used by statisticians. The main part of the book is devoted to tables of the normal distribution arranged in a handy form which will in many cases increase the usefulness and convenience of the tables. All other tables are more or less auxiliary tables designed so that maximum information can be extracted from the main table. A special feature of the book is the attention paid to interpolation errors. Careful instructions for direct and inverse interpolation are given. With the natural exception of the tables of interpolation coefficients all columns in this book indicate at the bottom the corresponding maximum two- and three-point interpolation errors for both direct and inverse interpolation. This feature will be appreciated by many users.

T.I., the main table, extends over p. 38–137 with a supplement on p. 36 and extensive explanations in the first part of the book. Let $z = (2\pi)^{-1/2} e^{-z^2/2}$, and $p = \int_{-\infty}^z z dx$, $q = 1 - p$. The columns of the table are p , x , z , $(pq)^{1/2}$, $(1 - p^2)^{1/2}$, $(1 - q^2)^{1/2}$, q in the order named. The argument $p = .5(.0001).9999$ and $q = .5(-.0001).0001$. Between them the two arguments cover the entire range: if q is used in the meaning of p , then x has to be taken with the negative sign. All tabulated values are to 8D. Linear interpolation is sufficient over almost the entire range of the table. Values at the extreme end of the table are of importance to statisticians and are provided in a supplementary table on p. 36. It gives p , x , z , q for the ten values $p = .9999$ 5, .9999 9, .9999 95, .9999 99, .9999 995, .9999 999, .9999 9995, .9999 9999, .9999 9999 5, .9999 9999 9. The number of significant places for x and z decreases but is never less than 5.

For the computation of the table various methods and checks were used. Values were computed to 14D, and any inaccuracy of the last digit in the table should be due to a rounding error in the 11th place or better. Tables of the familiar Lagrange interpolation coefficients in the usual arrangement are provided as follows. Three-point coefficients in **T. II** (p. 138–187) for $p = [0(.0001).5; 5D]$. Four-point coefficients in **T. III** (p. 188–197) for $p = [0(.001).5; 7D]$. Six-point coefficients in **T. IV** (p. 198–199) for $p = [0(.01).5; 10D]$. Finally, eight-point coefficients in **T. V** (p. 200) for $p = [.1(.1).9; 11D]$.

T. VI is devoted to the chi-square distribution familiar to all statisticians. For pure mathematicians the contents may be described in terms of the incomplete gamma function rather than the statistical form. Define $F_n(x) = 1/\Gamma(\frac{1}{2}n) \int_x^\infty t^{\frac{1}{2}(n-2)} e^{-t} dt$. (This notation is not generally used.) On p. 202–203 we find double-entry tables. The independent variable is called $\chi/n^{\frac{1}{2}}$ and ranges from 0 to 4.1 in steps of .1. The second column is χ^2/n , and contains exact values (2D). In the successive 15 columns $F_n(\chi^2)$ is tabulated to 4D for $n = 1(1)10, 12, 15, 19, 24, 30$. For larger values of n the function can be approximated by the normal distribution of **T. I** and detailed rules are provided how this should be done. Two-way interpolation in **T. VI** is described and maximum errors tabulated at the bottom of the table and on p. 201.

T. VII (p. 204–221) gives $N^{\frac{1}{2}}, (10N)^{\frac{1}{2}}, N^{\frac{1}{3}}, (10N)^{\frac{1}{3}}, (100N)^{\frac{1}{3}}, \ln N$ for $N = [1(.01)10; 8S]$.

T. VIII (p. 222) gives $\theta_i = 3(2)^{-i} - 2^i/(3i)$ for $i = [1(1)100(10)200(20)300(50)500(100)1000, 10000; 7D]$. This function is used to reduce certain distributions of interest to statisticians to the normal distribution.

T. IX (p. 223) collects a few frequently needed constants and their logarithms, to 15D.

WILL FELLER

562[K].—K. L. NIELSEN & L. GOLDSTEIN, "An algorithm for least squares," *Jn. Math. Phys.*, v. 26, 1947, p. 120–132. 17.4 × 25.4 cm.

In this paper the authors make use of an algorithm designed by P. D. CROUT (AIEE, *Trans.*, v. 60, 1941, p. 1235–1241) for the solution of linear systems of equations. The algorithm is adapted to the problem of fitting a polynomial of the form,

$$y = \sum_{i=0}^n a_i x^i,$$

to a set of values of y , $\{y_j\}$, given at the equally spaced abscissa values: $x_j = x_1 + (j-1)h$, $j = 1(1)m$.

Normalizing the data by introducing the variables: $X_j = x_j - \bar{x}$, $Y_j = y_j - \bar{y}$, where we abbreviate,

$$\bar{x} = \frac{1}{m} \sum_{j=1}^m x_j, \quad \bar{y} = \frac{1}{m} \sum_{j=1}^m y_j,$$

the authors reduce the problem to finding the values of b_i in the equation,

$$Y = \sum_{i=0}^n b_i X^i,$$

which will give the best fit to the observed values: X_j and Y_j , in the sense of least squares.

The sum of the squares of the residuals:

$$R_j = \sum_{i=0}^n b_i X_j^i - Y_j,$$

is then minimized by the method of least squares, which leads to the customary normal equations which must be solved for the coefficients, b_i . It is to this problem that the algorithm of Crout is then applied.

Introducing a new variable α_i defined by the equation: $b_i = \alpha_i h^{-i}$, $i = 0(1)n$, and a function, S_2 , defined as follows:

$$S_2 = \sum_{k=1}^{r-1} k^2, \quad \text{if } m = 2r - 1; \quad = \frac{1}{2} \sum_{k=1}^r (2k - 1)^2, \quad \text{if } m = 2r;$$

the authors then compute the following observable quantities:

$$c_1 = 0, \quad c_i = \frac{1}{2} S_2^{-1} h^{-i+1} \sum_{j=1}^m X_j^{i-1} Y_j.$$

The solution of the problem is then finally given in terms of the following iterative expressions:

$$\alpha_i = K_{i+1} - \sum_{j=i+2}^n \alpha_j A_{i+1, i+1, j}, \quad i = 0(1)n,$$

$$K_i = \left[c_i - \sum_{j=1}^{i-1} K_j A_{i, j} \right] / A_{i, i}, \quad i = 1(1)n + 1,$$

where A_{ij} is the general element of the solving matrix for the linear system which defines the α_i . It is observed that $A_{ij} = 0$ if $i + j$ is an odd number.

To facilitate the computation a table is given for the values of A_{ij} , $i = 1(1)5$, $j = 3(1)5$, and for values of $m = 5(1)100$, computed to 6D. A table is also included over the same range of m for values of the function S_2 , which are given exactly.

H. T. D.

563[L].—W. R. BENNETT, "Distribution of the sum of randomly phased components," *Quart. Appl. Math.*, v. 5, Jan. 1948, p. 385-393; table and graph, p. 387-388. 17.7 × 25.4 cm.

Table I gives the values, 4-6D, of $\phi_n(k) = n \int_0^\infty J_0(kz) J_1(z) J_0^{n-1}(z) dz$, for $n = 6(1)10$, $k = 1(1)10$ and a few scattered values, for $n = 6$, $k = 5.8$; $n = 7$, $k = 6.6$; $n = 8$, $k = 7.2$, 7.5 ; $n = 100$, $k = 4(3)10(5)30$. $\phi_n(k)$ is equal to the probability that the envelope of n sine waves of equal amplitude exceeds k times the amplitude of one.

Extracts from text

564[L].—C. J. BOUWKAMP, "Concerning a new transcendent, its tabulation and application in antenna theory," *Quart. Appl. Math.*, v. 5, 1948, p. 394-402.

$$E(z) = C(z) + iS(z) = \int_0^z (1 - e^{-it}) dt/t,$$

where

$$S(z) = Si(z) = \int_0^z \sin t dt/t, \quad C(z) = \gamma + \ln z - Ci(z) = \int_0^z (1 - \cos t) dt/t.$$

$$E_1(z) = \int_0^z E(t) dt/t = \int_0^1 \int_0^1 (1 - e^{-iust}) ds dt/st.$$

The function $E_1(z)$ was encountered in antenna theory but may possibly be of some value in other fields as well. The functions $\alpha_1(x)$ and $\alpha_2(x)$ occurring in Hallén's antenna theory were found to be expressible in terms of the functions $E(x)$ and $E_1(x)$ as follows:

$$\alpha_1(x) = \frac{1}{2} e^{ix} E(4x) - \cos x E(2x)$$

$$\alpha_2(x) = -\alpha_1(x) [\ln 4 + E(2x)] - \frac{1}{2} \cos x E^2(2x) + 2i \sin x E_1(4x) + \cos x [E_1(4x) - 2E_1(2x)].$$

On p. 401 are tables of the real and imaginary parts of $E_1(z)$ for $z = [0(0.2)20; 6D]$, and on p. 402, tables of real and imaginary parts of α_1 and α_2 for $x = [0(0.1)5; 4D]$. The tables of

$\alpha_1(x)$ show only small differences from those of R. KING & F. G. BLAKE in *Inst. Radio Engin., Proc.*, v. 30, 1942, p. 337. The tables for α_2 are in good agreement with the corresponding two-decimal values obtained by graphical integration in C. J. BOUWKAMP, *Physica*, v. 9, 1942, p. 618–619.

Extracts from text

565[L].—ENZO CAMBI, "Complete elliptic integrals of complex Legendrian modulus," *Jn. Math. Phys.*, v. 26, p. 234–245, 1948. 17.4×25.3 cm.

These interesting tables have arisen from a problem in differential equations and not from any problem of quadrature or of elliptic functions as such. For the purpose of this review, it appears simpler to reverse the author's order, by first describing his six tables from an elliptic standpoint, and then briefly outlining their relation to a differential equation.

All six tables may be described as tables of functions of a squared elliptic modulus, $z = k^2$, when $|z| = 1$; say $k^2 = -e^{ix}$, where x is real, and it suffices to consider $0 \leq x \leq \pi$. The functions involved are the usual complete integrals K, K', E, E' or combinations thereof, the definitions being

$$K(z) = \int_0^{\pi/2} (1 - z \sin^2 \phi)^{-1/2} d\phi, \quad K'(z) = K(1 - z),$$

$$E(z) = \int_0^{\pi/2} (1 - z \sin^2 \phi)^{1/2} d\phi, \quad E'(z) = E(1 - z).$$

These four integrals may all be computed either from the scale of arithmetic-geometric means of $(1, k')$, or from the scale of a.g. means of $(1, k)$, or from any expedient combination of the two processes. The fact that k is complex is no hindrance, apart from the usual slight arithmetical complication involved in working with complex numbers as compared with real ones. There is no difficulty in taking suitable determinations of sign in computing the square roots giving geometric means.

In the following descriptions of T. I–VI, it is to be understood that $k^2 = -e^{ix}$.

T. I (p. 238) gives $|K'|$ to 10D with δ^6 for $x = 0(0^{\circ}.1)2^{\circ}$, where $1^{\circ} = 1$ quadrant $= \frac{1}{2}\pi$. It is easily seen that the phase of K' is $\frac{1}{2}(\pi - x)$, which is also tabulated.

T. II (p. 240) gives $|K|$ to 10D with δ^6 and phase K to 7D of 1° without differences for $x = 0(0^{\circ}.1)2^{\circ}$.

Since K is infinite at $x = \pi$ ($k^2 = 1$), **T. III** (p. 241) gives $|K|$ and $|K'|$ to 10D, phase K to 7D of 1° , and phase K' exactly, all without differences, for $x = 1^{\circ}.9(0^{\circ}.01)2^{\circ}$.

T. IV (p. 242) gives $|k'|$, phase k' , $R(K/K')$, $|K/K'|$ and $-\log |q'|$, all to 10D (except that phase k' is exact) and without differences, for $x = 0(0^{\circ}.1)1^{\circ}.9(0^{\circ}.01)2^{\circ}$. $R(K/K')$ denotes the real part of K/K' . $I(K/K')$, the imaginary part of K/K' , does not need tabulation, since it is $\pm \frac{1}{2}$; with K and K' as tabulated in **T. I** and **II**, it is $-\frac{1}{2}$. It follows that $q' = e^{-\pi K/K'}$ is purely imaginary.

T. V (p. 243) gives $v = \frac{1}{2} \tan \frac{1}{2}x$ to 7D, $|k'K|$ to 10D, phase $(k'K)$ to 7D of 1° , and $R(k'K)$, $I(k'K)$ to 7D, all without differences, for $x = 0(0^{\circ}.1)1^{\circ}.9(0^{\circ}.01)2^{\circ}$. The last two columns of this table give solutions of a differential equation, as will be explained below.

T. VI (p. 244) gives to 10D, without differences, the real and imaginary parts of $e^{1/2(\pi-x)}(1 - E'/K')$ for $x = 0(0^{\circ}.1)2^{\circ}$. The function was convenient to calculate, and E' may evidently be derived from **T. I** and **VI** by elementary computation.

E may then be found from Legendre's relation

$$EK' + E'K - KK' = \frac{1}{2}\pi.$$

Cambi gives no table involving E .

The reason for the varying number of decimals given for the various functions is that the accuracy of those functions whose computation requires trigonometrical tables was limited by the use of 7-figure tables.

It is evident that the tables, with the help of some elementary computation if E and E' are required, define with considerable numerical precision the chief functions of the modulus

when k^2 lies on the unit circle centered at the origin. (For other complex values of k^2 , see the diagrams on p. 46, 74, 76 of the 1945 Dover edition of JAHNKE & EMDE.)

It remains to give some idea of the utility of T. V in connection with a differential equation. Cambi's paper should be consulted for further details. $K(z)$ and $K'(z)$ are well known to be independent solutions of the hypergeometric equation

$$z(1 - z)d^2y/dz^2 + (1 - 2z)dy/dz - \frac{1}{2}y = 0.$$

Putting $z = t/(t - 1)$, and transforming the differential equation, we find that $K(t/t - 1)$ and $K'(t/t - 1)$ are independent solutions of

$$t(1 - t)^2d^2y/dt^2 + (1 - t)^2dy/dt + \frac{1}{2}y = 0.$$

Putting further $t = -e^{ix}$, and again transforming, we find that independent solutions of the equation

$$(1) \quad 8(1 + \cos x)d^2y/dx^2 + y = 0$$

are $K(\frac{1}{2} + \frac{1}{2}i \tan \frac{1}{2}x)$ and $K'(\frac{1}{2} + \frac{1}{2}i \tan \frac{1}{2}x) = K(\frac{1}{2} - \frac{1}{2}i \tan \frac{1}{2}x)$. But these are complex conjugates, hence the real and imaginary parts of $K(\frac{1}{2} + \frac{1}{2}i \tan \frac{1}{2}x)$ are independent solutions of (1). These are the quantities tabulated in the last two columns of T. V, since it is easily shown that

$$K(t/t - 1) = \sqrt{1 - t} K(t) = k'K \text{ of Table V.}$$

Since (1), like Hill's equation, belongs to the class of linear differential equations with periodic coefficients, the result is of obvious interest.

A. FLETCHER

566[L].—H. B. DWIGHT, "Table of roots for natural frequencies in coaxial type cavities," *Jn. Math. Phys.*, v. 27, p. 84-89, 1948, 17.4×25.3 cm.

On p. 85-87 are tables relating to the roots, $x_n^{(s)}$, of

$$(1) \quad J_n(x)N_n(kx) - J_n(kx)N_n(x) = 0.$$

$(k - 1)x_n^{(s)}$ is given to 2-6S for $n = 0(1)3$, $s = 1(1)5$, and for values of k always extending from $k = 1$ to some point in the sequence

$$1(.1)1.6(.2)2(.5)4(1)6(2)20(5)50.$$

For $n = 0, 1, 2, 3$, the value of k does not for any s exceed 50, 50, 2, 4 respectively. $(k - 1)x_n^{(s)}$ is tabulated instead of $x_n^{(s)}$ for convenience in interpolation.

On p. 88-89 are tables relating to the roots, $x_n'^{(s)}$, of

$$(2) \quad J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0;$$

a heading to this effect would have been useful. $x_n'^{(1)}$ and $(k - 1)x_n'^{(s)}$, $s = 2(1)6$, are given to 2-6S for $n = 1(1)3$ and for values of k always extending from $k = 1$ to some point in the sequence given above. For $n = 1, 2, 3$, the value of k never exceeds 50, 4, 4 respectively. Values of $x_0'^{(s)}$ are not tabulated, since $x_0'^{(s)} = x_1^{(s)}$. The tables include roots of (2) not derivable at all from the MCMAYON asymptotic formulae.

Some tabulated values of (1) coincide with values in the well-known tables of KALÄHNE (reproduced in all editions of JAHNKE & EMDE) and others; see FMR, *Index*, p. 268, and *MTAC*, v. 2, p. 37.

As Dwight points out, numerical values of the roots of the prime equation (2) do not seem to have been published previously, though graphs for $s = 1$ were given in R. TRUPELL, "Concerning the roots of $J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0$," *Jn. Appl. Phys.*, v. 14, 1943, p. 350-352.

It so happens that the same number of *Jn. Math. Phys.* has (v. 27, p. 37-48) a paper by M. KLING, "Some Bessel equations and their application to guide and cavity theory," which contains graphs of roots of

$$J_n(x)N_n(\rho x) - J_n(\rho x)N_n(x) = 0$$

and

$$J_n'(x)N_n'(\rho x) - J_n'(\rho x)N_n'(x) = 0.$$

Kline takes $0 \leq \rho \leq 1$, but the reader will have no difficulty in following the correspondence between Dwight's tables and Kline's graphs; the latter cover some ranges not included in Dwight's tables.

A. FLETCHER

567[L].—MARION C. GRAY & S. A. SCHELKUNOFF, "The approximate solution of linear differential equations," *Bell System Tech. Jn.*, v. 27, Apr. 1948, p. 350–364. 15×22.9 cm.

For the differential equation $y'' + x^{-1}y' + y = 0$, $0 < x \leq \infty$, p. 362, $y(x) = J_0(x) - IY_0(x)$, and at infinity $\bar{W}_0 = (1 + i)e^{-ix}(\pi x)^{-\frac{1}{2}}$, $\bar{W}_1 = \frac{1}{2}(1 + i)e^{ix}(\pi x)^{-\frac{1}{2}}[Ci(2x) - i(Si(2x) - \frac{1}{2}\pi)]$. There is a table for $x = .6(2)1(1)10$, or $y(x)$, \bar{W}_0 , \bar{W}_1 , and for $x = .6(2)1$, of $\bar{W}_0 + \bar{W}_1 + \bar{W}_2$. Graphs of \bar{W}_0 and $\bar{W}_0 + \bar{W}_1$, p. 360.

For the differential equation $y'' + x^{-1}y' = (x^2 - 1)y$, $1 \leq x \leq 3$, $y(1) = 1$, $y'(1) = 0$, p. 363, a solution using Picard's method and the integragraph was described by T. C. FRY, Intern. Congr. Mathems., Toronto, *Proc.*, v. 2, 1928, p. 405–428. For the accurate solution $y = 1.4034J_1(x) - 0.3251Y_1(x)$, $x^{-1}W_0 = \cos \beta(x - 1) + (2\beta)^{-1} \sin \beta(x - 1)$, and $x^{\frac{1}{2}}W_1 = (4\beta)^{-1} \int_1^x (3u^2 - 1)[\cos \beta(u - 1) + (2\beta)^{-1} \sin \beta(u - 1)] \sin \beta(x - u) du$, $\beta = \frac{1}{2}\sqrt{3}$. There are tables of y , W_0 , and $W_0 + W_1$, and of the third and eighth Picard approximations y_3 and y_8 taken from Fry's paper, p. 410, for $x = [1(.2)3; 3D]$.

Extracts from text

568[L].—E. L. KAPLAN, "Auxiliary table for the incomplete elliptic integrals," *Jn. Math. Phys.*, v. 27, p. 11–36, 1948. 17.5×25.4 cm.

It is well known that interpolation in LEGENDRE'S classic tables of the incomplete elliptic integrals $F(\theta, \phi)$ and $E(\theta, \phi)$ is difficult or impossible when both the modular angle $\theta = \sin^{-1} k$ and the amplitude ϕ are near 90° . Kaplan's important paper gives (i) tables of auxiliary functions for the determination of $F(k, \phi)$ and $E(k, \phi)$ in the case mentioned, (ii) lists of some of the errata in Legendre, based partly on the results of previous workers and partly on a certain amount of original checking and of recomputation using (i).

Kaplan puts $r = k'/k = \cot \theta$, $x = \cos \phi$, and uses auxiliary functions f and e which may be defined by

$$K - F(k, \phi) = 2\pi^{-1}K' \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}}f,$$

$$E - E(k, \phi) = 2\pi^{-1}(K' - E') \sinh^{-1}(x/r) + x(r^2 + x^2)^{\frac{1}{2}}e.$$

His paper should be consulted for the series giving $f(r, x)$ and $e(r, x)$ and for further algebraical developments. On p. 20–35 he tabulates f and e to 10D for $x^2 = \cos^2 \phi = -.005(.005) + .160$ and $r^2 = k'^2/k^2 = -.005(.005) + .160$, the negative arguments $-.005$ being included to facilitate interpolation. He remarks that fourth differences may be neglected, and that linear and quadratic interpolation is valid to about 6D and 8D respectively. On p. 36 Kaplan tabulates $2K'/\pi$ and $2(K' - E')/\pi$ to 12D, with modified second differences, for $r^2 = -.005(.005) + .160$. Kaplan's tables appear to the reviewer to be of very great value.

With respect to errors in Legendre, Kaplan gives two main lists. The first, on p. 17, relates to errors in PEARSON'S Table I, which photographically reproduces Legendre's Table I of 1826, and gives $\log K$ and $\log E$ to 12–14D for $\theta = 0(0.1)90^\circ$. Kaplan gives 2 errors in $\log K$, 6 in $\log E$, 14 in the differences, of various orders, of $\log K$, and 11 in the differences of $\log E$. The 8 errors in the function values include 4 not previously published (but known to the reviewer). Two cases of slightly defective type in Pearson's reprint are noted. The errors in differences will form useful material, but such as relate to final digits are of little importance, since the reviewer shows elsewhere in this number that the function

values require many more corrections than Kaplan gives. Until the values of $\log K$ and $\log E$ have been satisfactorily corrected, one is not in a position to consider any but quite gross errors in the differences. In each case Kaplan indicates whether or not the errors in his list occur in Legendre's Table I of 1816.

Kaplan's second list of Legendre errata, on p. 18-19, is far more important. It relates to Pearson's Table II which photographically reprints Legendre's Table IX of 1826, and gives $F(\theta, \phi)$ and $E(\theta, \phi)$ to 9-10D for $\theta = 0(1^\circ)90^\circ$, $\phi = 0(1^\circ)90^\circ$. Some supplementary information is given on p. 15. Kaplan appears to the reviewer to list all previously known errors in Pearson's Table II, and he gives many more found by himself, partly through recomputation of some values by his own tables. Kaplan indicates in all cases whether or not the error is also present in the 1816 edition of Legendre's Table IX (photographically reprinted by POTIN and by EMDE). It need only be noted that there are three known errors in Legendre's 1816 values of $E(\theta, \phi)$ which were corrected in 1826, and therefore do not appear in Kaplan's list. These are:

θ	ϕ	For	Read
51°	23°	0.33502	0.39502
82	14	0.24195	0.24196
86	2	0.03489 9351	0.03489 9531.

On account of the great importance of Kaplan's list of errors in $F(\theta, \phi)$ and $E(\theta, \phi)$, it is reproduced by kind permission in the Guide to Tables of Elliptic Functions elsewhere in this number.

A complete list of the errors in Legendre's double-entry tables has never been given, and remains a desideratum, but the list under review goes further than any previous list.¹

A. FLETCHER

¹ Errors in Legendre have previously been pointed out in:

K. BOHLIN, *Tables des Fonctions Elliptiques*, Stockholm, 1900, p. 3.

C. J. MERFIELD, "Traité des Fonctions Elliptiques (Legendre). Errors in Tome II," *Astron. Jn.*, v. 30, p. 190, 1917.

S. P. GLAZENAP, *Matematicheskie i Astronomicheskie Tabliŕsy* [Mathematical and Astronomical Tables], Leningrad, 1932, p. 214.

N. S. SAMOÏLOVA-ÏAKHONTOVA, *Tabliŕsy Ellipticheskikh Integralov* [Tables of Elliptic Integrals], Moscow and Leningrad, 1935, p. 6.

C. HEUMAN, "Tables of complete elliptic integrals," *Jn. Math. Phys.*, v. 20, 1941, p. 127-206, 336; also sheet of remarks, Stockholm, 1941.

G. WITT, "Sechs Berichtigungen . . .," *Z. angew. Math. Mech.*, v. 21, 1941, p. 254; reported by F. EMDE.

R. C. ARCHIBALD, *MTAC*, v. 2, p. 136-137, 181, 1946.

569[L].—V. KOURGANOFF, "Sur les fonctions $K_n(x) = \int_1^\infty e^{-x^t} dt/t^n$ et certaines intégrales qui s'y rattachent," *Annales d'Astrophysique*, v. 10, 1947, p. 282-299. 21.8 × 27.3 cm.

$$I_{psn}(a) = s!F(n, 1, n + s + 1, -p/a)/[(n + s)(p + a)^a]$$

On p. 295-296 are given the values of $I_{1,0,n}(1)$ and $I_{2,0,n}(1)$ for $n = 1(1)8$; also of $I_{1,1,n}(1)$, $I_{2,1,n}(1)$, $I_{1,2,n}(1)$, $I_{1,3,n}(1)$, $I_{1,4,n}(1)$, for $n = 1(1)7$.

On p. 298-299 are values of $K_2(x)$, $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $x = [0.(02)2;4D]$, except $\phi_2(x)$, 4S, $1 < x \leq 2$; $\phi_0(x) = 2K_2(x)$, $\phi_1(x) = \frac{2}{3} - 2K_4(x)$, $\phi_2(x) = \frac{2}{3}x + 4K_6(x)$.

570[L].—S. O. RICE, "Reflections from circular bends in rectangular wave guides—matrix theory," *Bell System Techn. Jn.*, v. 27, Apr. 1948, p. 305-349, tables p. 343-344. 15 × 22.9 cm.

$$\begin{aligned}
 I_s &= \pi^{-1}\omega^2[u^{-1} - (-1)^s v^{-1} - \pi^{-1} s J_s], \\
 J_s &= \pi[Si(sv) - Si(su)] \cos su - \pi[Ci(sv) - Ci(su)] \sin su, \\
 K_s &= (\rho_1/a) \{ [Ci(sv) - Ci(su)] \cos su + [Si(sv) - Si(su)] \sin su \}, \\
 I_0 &= [(2\rho_1/a)^2 - 1]^{-1}, K_0 = (\rho_1/a) \log(1 + \pi/u), \\
 u &= \rho_1\pi/a - \frac{1}{2}\pi, v = \rho_1\pi/a + \frac{1}{2}\pi.
 \end{aligned}$$

There are tables of I_s , J_s , K_s for $s = 0(1)6$, $\rho_1/a = [.5(1)1.5(.5)2.5; 5D]$, except for $\rho_1/a = 2, 2.5$, the values of I_6 , I_6 , J_6 , J_6 , K_6 , K_6 are omitted; also values of J_s , $s = 1(1)6$, for $\rho_1/a = .5$.

Extracts from text.

571[L, M].—H. TIETZE, "Zusammenstellung einiger Werte des Integral-logarithmus," Akad. d. Wiss., Munich, *math.-naturw. Kl., Sitzb.*, 1947, p. 47–50. 14.5 × 22.6 cm.

Two small tables are given of the logarithmic integral $Li(x)$. One is a table to 8S for

$$x = [2(1)10(10)100(100)1000(1000)10000(10000)100000]$$

and is due to HANS CARL HAMMER. This table, with 25 of its 90 values quoted, is described as UMT 54 in *MTAC*, v. 2, p. 280, where also will be found references to previous tables of this function. The other table is 5–6S for the above values of x , and also for $x = 200000$ and 300000, and is due to LEONHARD WEIGAND. The last half of the table was produced by actual quadrature of the integral. The agreement of the two tables is very good. The reviewer has noted two errata:

Hammer, $x = 7$ for 4.7570508, read 4.7570518

Weigand, $x = 3 \cdot 10^6$ for 26086.4, read 26086.7.

More accurately $Li(300000) = 26086.6920$.

The first of these errors shows that the corresponding entries in DEMORGAN's table, 1842, p. 663, and SOLDNER's table, 1809, p. 48, are also in error.

D. H. L.

572[M].—S. O. RICE, "Statistical properties of a sine wave plus random noise," *Bell System Techn. Jn.*, v. 27, Jan. 1948, p. 109–157. 15 × 22.9 cm.

On p. 152–153 is a table of $Ie(k, x) = \int_0^x e^{-u} I_0(ku) du$ for $x = [0(.2)5(.4)9(1)15, \infty; 4D]$, $k = 0(.2)1$; and for $x = [15(1)20, \infty; 4D]$, $k = .86, .96, 1$. Also for $k = .9$, values for $x = [10(1)20, \infty; 4D]$.

573[M].—D. E. THOMAS, "Tables associated with a semi-infinite unit slope of attenuation," *Bell System Tech. Jn.*, v. 26, 1947, p. 870–899.

In this paper are tables of $B(x) = \pi^{-1} \int_0^x \ln \left| \frac{1+t}{1-t} \right| dt/t = 2\pi^{-1} \int_0^x \tanh^{-1} t dt/t$, and of $(180/\pi)B(x)$, for the following values of x and $1/x$: $[0(.001).996(.0003).998(.0001)1; 5S]$. These tables appreciably extend those of CORRINGTON, *MTAC*, v. 2, p. 218.

574[S].—HAROLD J. PEAKE, *An Aid to the Calculation of Electric Field Strengths in the Interference Region*. N. R. L. Report no. R-3222. Washington, D. C., Naval Research Laboratory, March 1948, vi, 140 p. 20.3 × 26.7 cm.

If E = electric field strength, E_0 = free space field strength at unit distance, d = distance from transmitting antenna, D = divergence factor, $|R'|$ = magnitude of reflection coefficient, $\alpha = \theta - C$ (where θ = phase difference between direct and reflected waves due to difference in path length, C = phase shift upon reflection), then

$$E = (E_0/d)I^{\frac{1}{2}}, \text{ where } I = 1 + (D|R'|)^2 - 2D|R'|\cos \alpha.$$

If db_i = attenuation of electric wave in decibels, λ_m = wavelength in meters, and d_{nm} = distance in nautical miles, then

$$db_i = 10 \log I + 20 \log \lambda_m - 20 \log d_{nm} - 83.04.$$

The tables, p. 5-140, are of I and $\log I$ to at least 3S(4-5D) for $D|R'| = .001(.001)1$, for $\cos \alpha = -1, 0, +1$; and $D|R'| = .01(.01)1$, for $\cos \alpha = .01(.01).99$.

Extracts from introductory text.

575[U].—FRANCE, SERVICE TECHNIQUE AÉRONAUTIQUE (prepared by M. L. CHAMALEIX), *Table de Hauteur et d'Azimut, A.A.F. Type 10*; an official publication of the Armée de l'Air Française (A.A.F.), not available for general sale. Eleven volumes have so far been issued and three more are in the press. "Édition 1946," distinguished by silver stars on the outside cover, consists of seven volumes, 16.5×23.5 cm., bound; Vol. 0, 223 p.; 1, 239 p.; 2, 3, 4 and 5, each 231 p.; 6, 223 p. The other volumes are the same size and contain: Vol. 2, 303 p.; 3, 319 p.; 4, 323 p.; 5, 311 p.

The French air almanac, *Ephémérides Aéronautiques*, was redesigned as from January 1948 to become almost identical with the *American Air Almanac*. The list of the 72 brightest stars on the cover was divided into groups similar to those in the British and American air almanacs; but this grouping is a consequence of the use of the *Astronomical Navigation Tables* (reprinted in U.S.A. as H.O. 218; see RMT 106 and 448), in which separate tabulations are given for the 22 brightest stars, and would be inappropriate unless similar tables were in use. Enquiries through the office of the *Connaissance des Temps* resulted in the verification of the existence of these volumes. I am indebted to le Directeur du Service Technique Aéronautique (STA) of the French Ministère des Forces Armées not only for a complete set of the tables so far issued but also for permission to include here the short account of these tables, prepared by M. J. JACQ of the STA.

In general appearance the tables are very similar to the *Astronomical Navigation Tables* (ANT); they differ in that the tabulations for the stars are entirely separated from those for integral degrees of declination, that ten degrees of latitude (instead of five) are included in each volume, and that the altitudes are not corrected for refraction. M. Jacq's note gives the basis of the tables and the method of preparation. It will suffice here therefore to mention that preliminary matter and explanations have been reduced to a minimum, and that most of the auxiliary tables have been taken directly from the ANT, the interpolation tables on the end pages being reproduced photographically. The new matter consists of star identification tables which give the declination and hour angle for every 3° of latitude and 4° of altitude and azimuth; these are clearly taken from H.O. 214.

The tables are reasonably well printed (though using modern face figures instead of the head and tail figures in the ANT), bound and indexed. No gross errors have been found, though no systematic check has been made. However, there are many end-figure errors in the star volumes due to the method of correcting the altitudes in the ANT. Examination shows that the refraction was subtracted according to a simple critical table, which could not of course agree with that used in constructing the tables; in some cases this leads to inconsistent values for the meridian altitudes in consecutive latitudes. For the integral degrees of declination, all quantities ending in 5 in H.O. 214 have been rounded-down.

Note sur les Tables de Hauteur et d'Azimut, A.A.F. Type 10, May, 1948, by M. J. JACQ.

Historique: Peu de temps avant la guerre de 1939, l'emploi de tables d'éléments précalculés avait paru très intéressant à quelques services utilisateurs, pour la résolution du problème du point astronomique par la méthode de la droite de hauteur de Marcq St.-Hilaire. En particulier les *Tables of Computed Altitude and Azimuth*, publication américaine H.O. no. 214, et surtout les *Astronomical Navigation Tables* anglaises avaient retenu notre attention comme étant les plus agréables d'emploi.

Aussi, en 1942, le Service Technique Aéronautique décida l'édition de tables de navigation astronomique analogues du point de vue présentation aux ANT, mais avec quelques

modifications. A cette époque, il n'était prévu que l'édition de tables relatives aux astres de déclinaison comprise entre 29° Nord et 29° Sud, d'une part par raison d'économie, et d'autre part par ce qu'elles étaient complétées par des albums de courbes d'étoiles STA dont l'édition était également entrepris. Ces albums, relatifs aux 6 étoiles suivants: Alioth, Deneb, Dubhé, la Chèvre, Schédir, Véga, analogues aux albums américains Weems, permettent, d'une part de déterminer les coordonnées géographiques de la position de l'aéronef d'après les mesures de hauteur de deux ou trois étoiles choisies, et d'autre part de déterminer la hauteur et l'azimut de l'une de ces 6 étoiles. Chaque album couvre 30 degrés de latitude.

Après la libération, l'État-Major de l'Armée de l'Air demanda l'édition complémentaire des tables relatives aux étoiles principales, tout en conservant la disposition des premières tables.

À la date du 1er janvier 1948, les volumes suivant ont été édités: Type normal: 2,3,4,5; Type étoiles: 0,1,2,3,4,5,6.

Sont en cours d'édition les volumes 0,1 et 6 du type normal (déclinaisons comprises entre $+$ et $- 29^\circ$.)

Aucune extension de ces tables n'est prévue actuellement.

Réalisation: Suivant les directives du STA les manuscrits ont été exécutés par un calculateur privé français, M. L. CHAMALEIX, de la manière suivante:

I. Tables pour les astres de déclinaison comprise entre $+$ et $- 29^\circ$.

D'après les données des *Tables of Computed Altitude and Azimuth*, H.O. 214 Américaines (édition 1940) auxquelles les corrections suivantes ont été apportées:¹

- (a). Hauteurs H : valeurs des tables H.O. 214 arrondies à la minute d'arc la plus voisine.
- (b). Corrections d en fonction de la déclinaison: valeurs des tables H.O. 214 transformées en minutes d'arc (au lieu de $100^{\text{èmes}}$ de degrés) et arrondies à la minute d'arc la plus voisine. Adoption de la table de correction des tables ANT.
- (c). Azimut Z : valeurs des tables H.O. 214 arrondies au degré le plus voisin.

II. Tables pour les 22 étoiles principales.

D'après les données des Tables anglaises ANT, mais en ramenant ces données aux valeurs non corrigées de la réfraction.

Remarques sur la disposition des tables: 1. Nous avons estimé, au moment où fut prise la décision de faire éditer des tables d'éléments précalculés, qu'il était plus intéressant que chaque volume couvre une bande de latitude de 10 degrés, au lieu de 5 degrés, comme les tables ANT, les voyages aériens projetés à l'époque comportant surtout des déplacements importants en latitude (possessions françaises d'Afrique). D'autre part il nous avait paru que les tables relatives aux planètes, au Soleil et aux étoiles de déclinaison comprise entre $+ 29$ degrés et $- 29$ degrés, suffiraient à la pratique de la navigation astronomique, étant donné que ces Tables permettent l'utilisation de la plupart des étoiles principales les plus connues et les plus souvent utilisées, à part quelques unes telles que Véga, la Chèvre, la Croix du Sud et Canopus.

Du point de vue instruction cette disposition conduisait à une seule méthode en fonction des arguments d'entrée dans les tables.

2. La correction de la réfraction atmosphérique pour une altitude de 5000 pieds, appliquée aux hauteurs données par les tables ne présente pas d'intérêt à notre avis. En effet, étant donné les altitudes de vol courantes il y a lieu malgré cette correction d'apporter une correction supplémentaire, aussi il nous a paru préférable de n'en pas tenir compte pour l'établissement des tables et de ne rien changer aux méthodes habituelles françaises d'application des diverses corrections aux hauteurs mesurées au sextant.

La correction de réfraction pour 5.000 pieds a été prise égale à la correction de réfraction pour 1500 m. calculée d'après les *Ephémérides Aéronautiques* françaises.

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¹ Au cours de l'établissement des Manuscrits il n'a été trouvé que deux erreurs grossières sur les Tables H.O. 214 et un certain nombre d'autres erreurs, mais malheureusement elles n'ont pas été relevées systématiquement.

576[U].—JAPAN, HYDROGRAPHIC OFFICE, *Publication* no. 685. *Skew Angle Celestial Navigation Method (A Method of Determining Latitude from Solar Observations made near Meridian Passage)*. Tokyo Hydrographic Office, March, 1944, 10, ix p. 18.2×25.7 cm.

This small volume is intended for use in surface navigation when the altitude of the celestial body is greater than 70° . The methods offered are limited to latitudes between 35°S and 35°N , to declinations -25° to $+25^\circ$ and to meridian angles less than 15° . It contains five tables and four nomograms. Table 1 is the usual one for time-to-arc conversion. Tables 2 and 5 yield the corrections to be applied to observed altitudes greater than 70° for dip, refraction, semi-diameter, parallax and differences between air and water temperatures, for sun, moon and stars.

The nomograms are to be used with angular distances of the celestial body measured from the cardinal points of the horizon. Nomogram 1 allows one to determine the latitude corresponding to the dead reckoning longitude, using a single measure of the celestial body's angular distance from the south (or north) point of the horizon. Using as arguments, declination ($0, \pm 25^\circ$), dead reckoning latitude ($35^\circ\text{S}-35^\circ\text{N}$), and hour angle ($0, 60^m$), this nomogram provides the correction ($0, 130'$) to be applied to the measured angular distance to the cardinal point to obtain the meridian altitude. This can in turn be used with the declination to obtain the latitude.

Nomogram 2 permits the determination of the longitude corresponding to the dead reckoning latitude, using a single measure of the angular distance of a celestial body from the east (or west) point of the horizon. With arguments, declination ($0, \pm 25^\circ$) and the measured angle ($75^\circ, 105^\circ$) to the east or west point of the horizon, one obtains the correction to the measured angle needed to give the local hour angle of the body and hence the longitude.

Nomogram 3 provides the difference of azimuth ($0, 20^\circ$) of two positions of the same celestial body, when one uses the difference of the two times of observation ($0, 60^m$) and the difference in the angles measured to the north (or south) point of the horizon ($0, 30'$).

This difference of azimuth ($0, 10^\circ$) is used as one of the arguments in nomogram 4; the difference, latitude minus declination ($0, 15^\circ$), is the other argument. With them, one obtains the correction ($0, 13'$), to be applied to the average of the two angles measured to the north (or south) point of the horizon to obtain the meridian altitude and therefrom the latitude.

Like all graphical methods, those presented in this volume are rapid but limited in accuracy. No account is taken of the motion of the vessel in the method of nomograms 3 and 4; since only a single observation is used with nomogram 1 or nomogram 2, this difficulty does not enter with them.

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577[V].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 17, *Tables for the Design of Missiles*. By the staff of the Laboratory, Professor H. H. AIKEN, technical director. Cambridge, Mass., Harvard University Press, 1948, iv, 226 p. 19.5×26.7 cm. Offset print. \$9.00.

These tables relate primarily to the problem of designing a missile such that the longitudinal axis, about which it has circular symmetry, shall maintain a direction which at any time is nearly tangent to the path of the missile. This is necessary in the first place to minimize the aerodynamic drag of the missile, and thus to maximize its range and average ve-

locity. It is also necessary to make the trajectory of the missile accurately predictable. Finally, the stable orientation of the missile may be essential to its proper functioning, i.e., penetration, fuze actuation, etc., when it arrives at the target.

The usual criterion for the stability of a spin-stabilized projectile (fired from a rifled gun) is $A^2\omega^2 > 4BFp$, where A and B are the axial and transverse moments of inertia, ω is the rate of spin in radians, F is the drag, and p is the distance from the center of pressure back to the center of gravity. The present tables are not concerned with the drag and the center of pressure, which ultimately depend on experimental data, but do provide valuable assistance in the often tedious work of determining the moments of inertia and the center of gravity, as well as the mass of the missile.

The tables thus have a purely geometric basis, and deal with the several forms which are important in missile design, namely ogives, frustums of right circular cones, and fillets, with the greatest emphasis on the first. The distinction between the terms ogive and fillet is one of convenience rather than logic. Since it offers difficulty it will be described in some detail. Both involve the surface generated by the rotation of a circular arc about an arbitrary axis lying in the plane of the arc. The general term is ogive, but the "standard" ogive is restricted to circular arcs which are convex (as viewed from the side away from the axis), and have the axis lying between the arc and its center. In addition there is one class of non-standard convex ogives and two classes of concave ogives. All are terminated by planes normal to the axis. A convex ogive is called tangent if terminated at its maximum cross-section. Any convex ogive is expressible as the difference of two tangent ogives. A segment of a sphere is a convex ogive, the limiting case between the standard and non-standard types.

Fillets are non-standard ogives which are assumed to be small and are therefore given a different sort of treatment. There is a change of terminology in that they are called inside and outside instead of convex and concave, apparently with reference to the associated conical frustum. The generating area is a mere sliver bounded by the circular arc and the intersecting portions of the tangents to the arc at its ends. Furthermore, one of these tangents is assumed to be normal to the axis.

It is stated that the notation conforms with current practice in the Research and Development Division of the Bureau of Ordnance. We shall require the following:

- l = axial length of an ogive or frustum.
- R = radius of the circular arc generating an ogive.
- d = maximum diameter of an ogive ($= 2R \pm 2D$) or frustum.
- D = distance from the center of the generating arc to the axis of an ogive.
- $a = \sin \phi = l/R$.
- $m = D/R$.
- d_1 = diameter of smaller base of a frustum.
- $x = d_1/d$.
- h = axial length of fillet.
- b = radial dimension of fillet measured from the (blunt) corner.
- θ = angular measure of the circular arc of a fillet.

T. 1 gives the functions

$$\begin{aligned}
 H_1 &= 20a^{-5}[2a - \sin^{-1} a - a(1 - a^2)^{\frac{1}{2}} - \frac{1}{3}a^3], \\
 H_2 &= 24a^{-6}[a^2 - \frac{1}{4}a^4 - \frac{2}{3}[1 - (1 - a^2)^{\frac{1}{2}}]], \\
 H_3 &= 28a^{-7}[\frac{2}{3}a^3 - \frac{1}{4}\sin^{-1} a + \frac{1}{4}a(1 - a^2)^{\frac{1}{2}} - \frac{1}{2}a^2(1 - a^2)^{\frac{1}{2}} - \frac{1}{3}a^5], \\
 H_4 &= 144a^{-9}[8a - \frac{8}{3}a^3 + \frac{1}{3}a^5 - \frac{7}{2}\sin^{-1} a - \frac{2}{3}a(1 - a^2)^{\frac{1}{2}} + a^2(1 - a^2)^{\frac{1}{2}}]
 \end{aligned}$$

and their first differences to $6D$ for $\sin \phi = a = 0(.001).999$. These are auxiliary functions which with considerable additional calculation (see T. 3) yield the volume, the center of gravity, and the transverse and axial (or polar) moments of inertia of an ogive of any type.

T. 2 gives $S_0 = 3a^2H_1/20$ and its first differences to $8D$ for $a = 0(.001).999$. It is used in getting the volume of any type of ogive.

T. 3, which is bivariate, gives the functions

$$s = 1 - \frac{a^2}{3(1-m)} + \frac{ma^4}{20(1-m)^2} H_1$$

$$t = s^{-1} \left[1 - \frac{a^2}{2(1-m)} + \frac{ma^4}{12(1-m)^2} H_2 \right],$$

$$u = s^{-1} \left[1 - \frac{2a^2}{3(1-m)} + \frac{a^4}{5(1-m)^2} + \frac{ma^4}{10(1-m)^2} H_1 - \frac{ma^6}{14(1-m)^3} H_3 + \frac{m^2a^8}{144(1-m)^4} H_4 \right],$$

$$v = 4s^{-1} \left[1 - \frac{3a^2}{5(1-m)} + \frac{3ma^4}{28(1-m)^2} H_2 \right] - 3t^2$$

without differences to 6D for $0 \leq m \leq .995$ and $0 \leq a < (1 - m^2)^{1/2}$. The intervals in m and a are as follows:

Range of m	Δm	Δa
0-.600	.020	.020
.610-.710	.010	.020
.720-.790	.010	.010
.800-.895	.005	.010
.900-.995	.005	.005

In equation (17), H_3 should read $H_3/28$, and $-a^2/(1 - m)$ should read $-\frac{1}{2}a^2/(1 - m)$; in equation (68a), K should read mK . These functions apply to standard ogives as follows:

- $V = \text{volume} = \frac{1}{4}\pi d^2 l s;$
- $\bar{x} = \text{distance from base to center of gravity} = \frac{1}{2}l t;$
- $I_p = \text{polar moment} = \frac{1}{4}V d^2 u;$
- $I'_{TCG} = \text{transverse moment} = \frac{1}{12}V l^2 v.$

(Other variants of the transverse moment are also used in the volume.)

T. 4 gives the foregoing functions s, t, u, v without differences to 6D for $a = 0(.001).999$, and $m = (1 - a^2)^{1/2}$. It applies to pointed standard ogives.

T. 5 gives the function $s = \frac{1}{3}(x^2 + x + 1)$ with first differences to 8D for $x = 0(.001).999$. The volume of a conical frustum is then obtainable as $V = \frac{1}{4}\pi d^2 l s$.

T. 6 gives the functions s (as in **T. 5**),

$$t = (3x^3 + 2x + 1)/2(x^2 + x + 1),$$

$$u = 3(x^4 + x^3 + x^2 + x + 1)/5(x^2 + x + 1),$$

$$v = 9[(1 + x)^4 + 4x^2]/20(1 + x + x^2)^2$$

without differences to 6D for $x = 0(.01).99$. These are used in the same formulae as the corresponding functions of **T. 3**, but refer to conical frustums instead of ogives.

T. 7 gives the functions

$$s = (\tan \frac{1}{2}\theta - \frac{1}{2}\theta)/(1 - \cos \theta)(\tan \frac{1}{2}\theta),$$

$$t_{1A} = 1 - 1/6s, \quad t_{2A} = (1 - 1/6s)/(1 - \cos \theta)$$

without differences to 4D for $\theta = 30^\circ(1^\circ)120^\circ$. Then the area which generates the convex fillet by revolution has the measure bhs , and a center of gravity (for the area, not the solid) at a distance ht_{2A} axially and bt_{1A} radially from the "corner" of the area.

T. A, B, and C (in the Introduction) are bivariate tables, totalling only 24 entries each, which relate to the center of gravity and moments of inertia of convex fillets. **T. D** gives 6 values of each of three similar functions for concave fillets for which $\theta = 90^\circ$.

The volume has an excellent introduction, with sections entitled Definitions of the Tabulated Functions, Method of Computation, Interpolation in the Tables, The Use of the Tables, and Non-Standard Ogives.

The tables were designed in collaboration with the Navy Department, Bureau of Ordnance, and the computations carried out on the Automatic Sequence Controlled Calculator, under contract with the Bureau of Ordnance.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in "Guide to Tables in Elliptic Functions" (Airey, Bertrand, Dale, Dwight, FMR, Gauss, Glaisher, Gossot, Greenhill, Hancock, Hayashi, Heuman, Hippisley, Innes, Jahnke & Emde, Kaplan, Legendre, Lévy, Meissel, Merfield, Moore, Nagaoka & Sakurai, Pidduck, Plana, Potin, Rosenbach, Whitman & Moskovitz, Runkle, Samořlova-ĭakhontova, Schlömilch, Spenceley, Verhulst, Wayne), and in RMT 554 (Gifford), 557 (Yarden & Katz), 558 (Akushskii & Ditkin, Ditkin, Lřusternik), 568 (Legendre), 571 (De Morgan, Hammer, Soldner, Weigand), 575 (France), 577 (Harvard).

138.—E. P. ADAMS, *Smithsonian Mathematical Formulae* . . . First reprint, Washington, D. C., 1939. See also *MTAC*, v. 1, p. 191; v. 2, p. 46, 353.

P. 10, under 1.272, for $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, read $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$.

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139. L. J. COMRIE, *Chambers' Four-Figure Mathematical Tables*. 1947. See *MTAC*, v. 3, p. 86–87.

The following error was found during the reading of the proofs for the new Chambers' six-figure table.

Page 64, left column, line 29. Equivalent for 1 calorie, for 1.363×10^{-6} K. W. H., read 1.163×10^{-6} K. W. H.

L. J. C.

140. G. H. HARDY & E. M. WRIGHT, *An Introduction to the Theory of Numbers*. Oxford, 1938.

On p. 71–72 the authors state that the direct converse of Fermat's theorem is false; it is not true that, if a is a prime and $a^{m-1} \equiv 1 \pmod{m}$, then m is necessarily a prime. To illustrate that the cases in which this converse is false are "rather rare," they list what they believe to be all "composite values of m below 2000 for which $2^{m-1} \equiv 1 \pmod{m}$," as "341 = 11·31, 561 = 3·11·17, 645 = 3·5·43, 1387 = 19·73, 1729 = 7·13·19, 1905 = 3·5·127."

Another value of m , not here listed, is $1105 = 5·13·17$.

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141. E. JAHNKE & F. EMDE, *Tables of Functions*, 1933 (fig. 92, p. 192), all later editions (fig. 67, p. 126). See *MTAC*, v. 3, p. 41.