DISCUSSIONS

Statistical Treatment of Values of First 2,000 Decimal Digits of e and of π Calculated on the ENIAC

The first 2,000 decimal digits of e and of π were calculated on the ENIAC by Mr. G. REITWIESNER and several members of the ENIAC Branch of the Ballistic Research Laboratories at Aberdeen, Maryland (*MTAC*, v. 4, p. 11–15). A statistical survey of this material has failed to disclose any significant deviations from randomness for π, but it has indicated quite serious ones for e.

Let $D_n{}^i$ be the number of digits i (where $i = 0, 1, \cdots, 9$) among the first n digits of e or of π. The count begins with the first digit left of the decimal point. If these digits were equidistributed, independent random variables, then the expectation value of each $D_n{}^i$ (with n fixed and $i = 0, 1, \cdots, 9$) would be $n/10$, and the χ^2 would be

$$a_n = \chi_n{}^2 = \sum_{i=0}^{9} \left(D_n{}^i - \frac{n}{10}\right)^2 \Big/ \frac{n}{10}.$$

The system of the $D_n{}^i$'s (where $i = 0, 1, \cdots, 9$) has 9 degrees of freedom. Therefore let

$$p = P^{(k)}(a) = \frac{1}{2^{\frac{1}{2}k}\Gamma(\frac{1}{2}k)} \int_0^a e^{-\frac{1}{2}x} x^{\frac{1}{2}k-1} dx$$

be the cumulative distribution function of $a = \chi^2$ for k degrees of freedom. Then

$$p_n = P^{(9)}(a_n)$$

is a quantity which would be equidistributed in the interval [0, 1], if the underlying digits were equidistributed independent random variables.

Consider $n = 2000$. In this case, the $D_n{}^i$'s for e are

(1) 196, 190, 208, 202, 201, 197, 204, 198, 202, 202.

Hence $a_n = \chi_n{}^2 = 1.11$ and $p_n = .0008$. The $D_n{}^i$'s for π are

(2) 182, 212, 207, 189, 195, 205, 200, 197, 202, 211.

Hence $a_n = \chi_n{}^2 = 4.11$ and $p_n = .096$.
The e-value of p_{2000} is thus very conspicuous; it has a significance level of about 1:1250. The π-value of p_{2000} is hardly conspicuous; it has a significance level of about 1:10.

The relevant fact about the distribution (1) appears upon direct inspection. The values lie too close to their expectation value, 200. Indeed their absolute deviations from it are

4, 10, 8, 2, 1, 3, 4, 2, 2, 2,

and hence their mean-square deviation is $22.2 = 4.71^2$, whereas in the random case the expectation value is $180 = 13.4^2$.

In order to see how this peculiar phenomenon develops as n increases to 2000, $a_n = \chi_n{}^2$ and p_n of e have been determined from $D_n{}^i$ for the following smaller values of n

n	$a_n = \chi_n^2$	p_n
500	6.72	.33
1000	4.82	.15
1100	5.93	.25
1200	4.03	.093
1300	3.83	.080
1400	4.74	.145
1500	3.69	.070
1600	2.47	.019
1700	3.22	.046
1800	2.85	.031
1900	2.22	.013
2000	1.11	.0008

These numbers show that the abnormally low value of p_n which is so conspicuous at $n = 2000$ does not develop gradually, but makes its appearance quite suddenly around $n = 1900$. Up to that point, p_n oscillates considerably and has a decreasing trend, but at $n = 2000$ there is a sudden dip of quite extraordinary proportions.

Thus something number-theoretically significant may be occurring at about $n = 2000$. A calculation of more digits of e would therefore seem to be indicated. A conversion to a simpler base than 10, say 2, may also disclose some interesting facts.

We wish to thank Miss HOMÉ MCALLISTER of the ENIAC Branch of the Ballistic Research Laboratories for sorting the digital material on which the above analyses are based, and Professor J. W. TUKEY, of Princeton University, for discussions of the subject.

Since the above was written (November 9, 1949), the ENIAC Branch of the Ballistic Research Laboratory very obligingly followed our suggestion and calculated the following 500 additional digits[1] of e. These should replace the last 10 digits of the value of e given in *MTAC*, v. 4, p. 15.

55990	06737	64829	22443	75287	18462	45780	36192	98197	13991
47564	48826	26039	03381	44182	32625	15097	48279	87779	96437
30899	70388	86778	22713	83605	77297	88241	25611	90717	66394
65070	63304	52795	46618	55096	66618	56647	09711	34447	40160
70462	62156	80717	48187	78443	71436	98821	85596	70959	10259
68620	02353	71858	87485	69652	20005	03117	34392	07321	13908
03293	63447	97273	55955	27734	90717	83793	42163	70120	50054
51326	38354	40001	86323	99149	07054	79778	05669	78533	58048
96690	62951	19432	47309	95876	55236	81285	90413	83241	16072
26029	98330	53537	08761	38939	63917	79574	54016	13722	36188

This makes it possible to extend the table of $a_n = \chi_n^2$ and p_n up to $n = 2500$

n	$a_n = \chi_n^2$	p_n
2100	1.94	.0075
2200	2.02	.0088
2300	1.65	.0041
2400	1.70	.0046
2500	1.90	.0070

Thus the values of p_n for $2100 \leqslant n \leqslant 2500$ are still significantly low but higher than the value of p_n at $n = 2000$.

Note that the general size and trend of p_n, as well as its sudden deviation at $n = 2000$, indicate a non random character in the digits of e.

More detailed investigations are in progress and will be reported later.

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[1] Both e and $1/e$ were computed somewhat beyond 2500 D and the results checked by actual multiplication.

Notes on Numerical Analysis—2
Note on the Condition of Matrices

1. The object of this note is to establish the following theorem.

THEOREM. *Let A be a real $n \times n$ non-singular matrix and A' be its transpose. Then AA' is more "ill-conditioned" than A.*

This theorem confirms an opinion expressed by Dr. L. FOX[1] based on his practical experience. The term "condition of a matrix" has been used rather vaguely for a long time. The most common measure of the condition of a matrix has been the size of its determinant, ill-conditioned matrices being those with a "small" determinant. With this interpretation imposed, the theorem is clearly correct. More adequate measures of the condition of a matrix have been proposed recently by JOHN VON NEUMANN & H. H. GOLDSTINE[2] and by A. M. TURING.[3] Their definitions concern all matrices, not just the ill-conditioned ones, characterized by very large condition numbers. The following two of these definitions will form a basis for the proof of the above-mentioned theorem:

The P-condition number is $|\lambda_{\max}|/|\lambda_{\min}|$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the characteristic roots of largest and smallest modulus.[2]

The N-condition number is $N(A)N(A^{-1})/n$, where[3]

$$N(A) = (\sum_{i,k} a_{ik}^2)^{\frac{1}{2}}.$$

2. Proof of the theorem in the P case:

Let λ_i be the characteristic roots of A and μ_i those of AA' (which are in general distinct from the squares of the absolute values of λ_i). E. T. BROWNE[4] has shown that

$$\mu_{\min} \leqslant \lambda_i\bar{\lambda}_i \leqslant \mu_{\max}.$$

From this it follows that

$$1 \leqslant \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \leqslant \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|^2 \leqslant \frac{\mu_{\max}}{\mu_{\min}},$$

which implies the required result.

3. Proof of the theorem in the N case:

It is known that $N(A)$ is the square root of the trace of AA' and therefore equal to $(\sum \mu_i)^{\frac{1}{2}}$. The numbers μ_i are all positive since AA' is symmetric and positive definite. Since the characteristic roots of $A'A$ and AA' are the