Radix Table for Trigonometric Functions and Their Inverses to High Accuracy

1. Introduction.—By means of two algorithms, based upon the addition theorem for $\tan x$, and a radix table of $\arctan (m \cdot 10^{-n})$, $m = 1(1)9$, $n = 1(1)6$, to 20D, one has the means of calculating $\tan x (x < \pi/4)$ and $\arctan x$ to at least 18D from a one-page table. The other trigonometric functions and their inverses follow easily. All division operations may be done upon an ordinary 10-bank desk calculator, so that no intermediate written work is necessary.

2. To obtain $\tan x$.—If $x < \frac{1}{4}\pi$, subtract from x the largest tabular value of $\arctan a_1$ that leaves a positive remainder x_1 ; from this remainder subtract the largest tabular value of $\arctan a_2$ that leaves a positive remainder x_2 .

Continue in this way until the remainder $x_6 = x - \sum_{i=1}^6 \arctan a_i$ is reached.

The a_i are all exact, each with a single significant digit, unless zero.

By evaluating in turn

$$t_6 = \tan x_6 \doteq x_6, \quad t_{i-1} = \frac{t_i + a_i}{1 - t_i a_i}, \quad i = 6, 5, 4, 3, 2, 1;$$

one can obtain $t_0 = t = \tan x$. An easier alternative is to evaluate in turn

$$\begin{aligned} a &= a_1 + a_2, & b &= a_3 + a_4, & c &= a_5 + a_6; \\ d &= 1 - a_1 a_2, & e &= 1 - a_3 a_4, & f &= 1 - a_5 a_6; \\ A_1 &= ae + bd, & A_2 &= de - ab, \\ B_1 &= c + fx_6, & B_2 &= f - cx_6. \end{aligned}$$

Then $t = \tan x \doteq (A_1 B_2 + A_2 B_1) / (A_2 B_2 - A_1 B_1)$.

If $\frac{1}{4}\pi < x < \frac{1}{2}\pi$, evaluate $x' = \frac{1}{2}\pi - x = 1.57079\ 63267\ 94896\ 61923 - x$; then $\tan x = 1/\tan x'$ to as many significant figures as are known to 18 decimals in x' .

Other functions are given by

$$\begin{aligned} \sin x &= t(1 + t^2)^{-\frac{1}{2}}, & \cos x &= (1 + t^2)^{-\frac{1}{2}}, & \cot x &= 1/t, \\ \sec x &= (1 + t^2)^{\frac{1}{2}}, & \operatorname{cosec} x &= (1 + t^2)^{\frac{1}{2}}/t. \end{aligned}$$

3. To obtain $\arctan t$.—If $t < 1$, evaluate in turn

$t_1 = (t - a_1)/(1 + a_1 t)$ where a_1 is the first decimal digit of t ,
 $t_2 = (t_1 - a_2)/(1 + a_2 t_1)$ where a_2 is the second decimal digit of t_1 , the first being zero, and so on, until t_6 is reached, using

$$t_{i+1} = (t_i - a_{i+1})/(1 + a_{i+1} t_i).$$

Then $\arctan t \doteq \sum_{i=1}^6 \arctan a_i + t_6$.

If desired, two successive stages can be combined, thus

$$t_{i+2} = \frac{(1 - a_{i+1} a_{i+2}) t_i - (a_{i+1} + a_{i+2})}{1 - a_{i+1} a_{i+2} + (a_{i+1} + a_{i+2}) t_i}.$$

If $t > 1$, $\arctan t = \frac{1}{2}\pi - \arctan(1/t) = 1.57079\ 63267\ 94896\ 61923 - \arctan(1/t)$. Other functions are given by

$$\begin{aligned}\operatorname{arccot} x &= \frac{1}{2}\pi - \arctan x, & \arcsin x &= \arctan(x(1-x^2)^{-\frac{1}{2}}), \\ \arccos x &= \arctan((1-x^2)^{\frac{1}{2}}/x), & \operatorname{arcsec} x &= \arctan(x^2-1)^{\frac{1}{2}}, \\ \operatorname{arccsc} x &= \arctan(x^2-1)^{-\frac{1}{2}}.\end{aligned}$$

TABLE OF ARCTAN a_i .

a_1	$\arctan a_1$	a_4	$\arctan a_4$
.1	.09966 86524 91162 02738	.0001	.00009 99999 99666 66667
.2	.19739 55598 49880 75837	.0002	.00019 99999 97333 33340
.3	.29145 67944 77867 09200	.0003	.00029 99999 91000 00049
.4	.38050 63771 12364 88630	.0004	.00039 99999 78666 66871
.5	.46364 76090 00806 11621	.0005	.00049 99999 58333 33958
.6	.54041 95002 70584 15544	.0006	.00059 99999 28000 01555
.7	.61072 59643 89208 61654	.0007	.00069 99998 85666 70028
.8	.67474 09422 23552 66306	.0008	.00079 99998 29333 39887
.9	.73281 51017 86506 59164	.0009	.00089 99997 57000 11810
<hr/>			
a_2	$\arctan a_2$	a_5	$\arctan a_5$
.01	.00999 96666 86665 23821	.00001	.00000 99999 99999 66667
.02	.01999 73339 73150 53306	.00002	.00001 99999 99997 33333
.03	.02999 10048 56877 89968	.00003	.00002 99999 99991 00000
.04	.03997 86871 23290 04141	.00004	.00003 99999 99978 66667
.05	.04995 83957 21942 76141	.00005	.00004 99999 99958 33333
.06	.05992 81551 21207 88443	.00006	.00005 99999 99928 00000
.07	.06988 60016 34642 49929	.00007	.00006 99999 99885 66667
.08	.07982 99857 12237 31589	.00008	.00007 99999 99829 33333
.09	.08975 81741 89950 52315	.00009	.00008 99999 99757 00000
<hr/>			
a_3	$\arctan a_3$	a_6	$\arctan a_6$
.001	.00099 99996 66666 86667	.000001	.00000 09999 99999 99967
.002	.00199 99973 33339 73332	.000002	.00000 19999 99999 99733
.003	.00299 99910 00048 59969	.000003	.00000 29999 99999 99100
.004	.00399 99786 66871 46433	.000004	.00000 39999 99999 97867
.005	.00499 99583 33958 32217	.000005	.00000 49999 99999 95833
.006	.00599 99280 01555 16001	.000006	.00000 59999 99999 92800
.007	.00699 98856 70027 94902	.000007	.00000 69999 99999 88567
.008	.00799 98293 39886 63376	.000008	.00000 79999 99999 82933
.009	.00899 97570 11809 11676	.000009	.00000 89999 99999 75700

4. Examples.

A. To obtain $\tan x$, for $x = .56548\ 66776\ 46162\ 78292$:

Here $\arctan .6$ is seen to be the largest value of $\arctan a_1$ such that $x - \arctan a_1$ is positive. From

$x - \arctan .6 = x_1 = .02506\ 71773\ 75578\ 62748$, a_2 is seen to be .02. From

$x_1 - \arctan .02 = x_2 = .00506\ 98434\ 02428\ 09442$, a_3 is seen to be .005. From

$x_2 - \arctan .005 = x_3 = .00006\ 98850\ 68469\ 77225$, a_4 is seen to be 0, $x_3 = x_4$, and a_5 is seen to be .00006. From

$x_4 - \arctan .00006 = x_5 = .00000\ 98850\ 68541\ 77225$, a_6 is seen to be .000009. Finally

$$x_5 - \arctan .000009 = x_6 = .00000\ 08850\ 68542\ 01525.$$

$$a = a_1 + a_2 = .62, b = a_3 + a_4 = .0050, c = a_5 + a_6 = .000069,$$

$$d = 1 - a_1a_2 = .988, e = 1 - a_3a_4 = 1, f = 1 - a_5a_6 = .99999\ 999946;$$

$$A_1 = ae + bd = .62494, A_2 = de - ab = .9849,$$

$$\begin{aligned}
B_1 &= c + fx_6 = .00006 \ 98850 \ 68541 \ 53731, \\
B_2 &= f - cx_6 = .99999 \ 99993 \ 98930 \ 27060. \\
A_1B_2 &= .62493 \ 99996 \ 24367 \ 48331, A_2B_1 = .00006 \ 88298 \ 04006 \ 56010, \\
A_2B_2 &= .98489 \ 99994 \ 08006 \ 42351, A_1B_1 = .00004 \ 36739 \ 74734 \ 34833, \\
A_1B_2 + A_2B_1 &= .62500 \ 88294 \ 28374 \ 04341, \\
A_2B_2 - A_1B_1 &= .98485 \ 63254 \ 33272 \ 07518, \\
t = \tan x &\doteq \frac{A_1B_2 + A_2B_1}{A_2B_2 - A_1B_1} = .63461 \ 92975 \ 44148 \ 10040,
\end{aligned}$$

which is correct to about 3 units in the 19th decimal place.

B. To obtain $\arctan t$, for $t = .59139 \ 83513 \ 99471 \ 09817$:

Since $a_1 = .5$, we have

$$\begin{aligned}
t_1 &= (t - a_1)/(1 + a_1t) = \frac{.09139 \ 83513 \ 99471 \ 09817}{1.2956 \ 99175 \ 69973 \ 55491}, \\
\text{or } t_1 &= .07053 \ 97928 \ 11176 \ 17252, a_2 = .07; \\
t_2 &= (t_1 - a_2)/(1 + a_2t_1) = \frac{.00053 \ 97928 \ 11176 \ 17252}{1.0049 \ 37785 \ 49678 \ 23321}, \\
\text{or } t_2 &= .00053 \ 71405 \ 26474 \ 81117, a_3 = 0; \\
t_3 &= t_2, a_4 = .0005; \\
t_4 &= (t_3 - a_4)/(1 + a_4t_3) = \frac{.00003 \ 71405 \ 26474 \ 81117}{1.0000 \ 00268 \ 57026 \ 324}, \\
\text{or } t_4 &= .00003 \ 71405 \ 16499 \ 97288, a_5 = .00003; \\
t_5 &= (t_4 - a_5)/(1 + a_5t_4) = \frac{.00000 \ 71405 \ 16499 \ 97288}{1.0000 \ 00001 \ 11421 \ 55}, \\
\text{or } t_5 &= .00000 \ 71405 \ 16492 \ 01681, a_6 = .000007; \\
t_6 &= (t_5 - a_6)/(1 + a_6t_5) = \frac{.00000 \ 01405 \ 16492 \ 01681}{1.0000 \ 00000 \ 04998 \ 36}, \\
\text{or } t_6 &= .00000 \ 01405 \ 16492 \ 00979.
\end{aligned}$$

$$\begin{aligned}
\arctan t &= \arctan .5 + \arctan .07 + \arctan .0005 + \arctan .00003 \\
&\quad + \arctan .000007 + .00000 \ 01405 \ 16492 \ 00979, \\
\text{or } \arctan t &= .53407 \ 07511 \ 10264 \ 85054,
\end{aligned}$$

which happens to be correct to 20 decimals.

H. E. SALZER

NBSCL

RECENT MATHEMATICAL TABLES

834[E].—L. PRANDTL & F. VANDREY, "Fließgesetze normal-zäher Stoffe im Rohr. Ein Beitrag zur Rheologie," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 169–174.

This article gives two tables of functions having to do with viscous flow. The function $\phi(a)$ is defined by

$$\phi(a) = 8 \sum_{n=1}^{\infty} n(2n+1)a^{2n-2}/(2n+2)!$$