Proceeding in the manner described in the foregoing sections for the case of a single differential equation, and making use of the abbreviations

$$R^n \equiv D^n f_z, \quad \tau^n = D^n g, \quad \sigma^n \equiv D^n g_y, \quad \rho^n \equiv D^n g_z, \quad \pi \equiv [\tau]^2$$

one obtains for $\epsilon_1 = C_{15} - y^{\text{v}}(x_0, y_{10}, y_{20})/120$ the expression

$$\begin{aligned} 1440\epsilon_1 = (T^4/2) &- 2(f_y T^3 + f_z \tau^3) + 3[T^2(f_y^2 + f_z g_y) + \tau^2(f_y f_z + f_z g_z)] \\ &- 3(ST^2 + R\tau^2) + 12[S(Tf_y + \tau f_z) + R(Tg_y + \tau g_z)] \\ &- 6[T(Sf_y + \sigma f_z) + \tau(Rf_y + \rho f_z)] + 3(TS^2 + \tau R^2) \\ &+ 9(Pf_{yy} + 2T\tau f_{yz} + \pi f_{zz}) \\ &- 12[T(f_y^3 + f_z g_y(2f_y + g_z)) + \tau f_z(f_y^2 + f_y g_z + f_z g_y)] \\ &- 15 f_{yz}(T^2 g + \tau^2 f). \end{aligned}$$

Assumptions similar to (14) now permit one to get a bound for $\epsilon_i h^5$.

If, namely, near $(x_0,, y_{10}, y_{20})$,

$$|f_i| \leq M, \qquad |\partial^{p+q+r} f_i / \partial x^p y_1{}^q y_2{}^r| \leq L^{p+q+r}/M^{q+r-1}$$

for $0 \leq p + q + r \leq 4$, it is found that

$$|E_i| \leq (973/720) M L^4 h^5. \tag{19}$$

For the differential equation $y'' - y = 1$, $y(0) = 0$, $y'(0) = 1$, which is equivalent to the system $y_1' = y_2$, $y_2' = 1 + y_1$, $y_1(0) = 0$, $y_2(0) = 1$, the solution is $y = e^x - 1$. Thus $y(0.1) = .1051709$. With $h = 0.1$, R.K.'s method gives $y(0.1) = .1051707$, whence $E = 2\cdot 10^{-7}$. In the region $0 \leq x \leq 0.1$, $0 \leq y_1 \leq 0.11$, $0 \leq y_2 \leq 1.11$; above estimate (19) asserts that $|E_i| \leq 1.5\cdot 10^{-5}$.

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[1] L. BIEBERBACH, *Theorie der Differentialgleichungen*. New York, 1944, p. 54.
[2] C. RUNGE & H. KÖNIG, *Vorlesungen über numerisches Rechnen*. Berlin, 1924.

RECENT MATHEMATICAL TABLES

878[F].—H. CHATLAND & H. DAVENPORT, "Euclid's algorithm in real quadratic fields," *Canadian Jn. Math.*, v. 2, 1950, p. 289–296.

The tables in this paper are rather special and are used to prove a result of INKERI[1] that the field $k(m^{\frac{1}{2}})$ is not Euclidean for

$$m = 193, 241, 313, 337, 457, 601.$$

There are six tables corresponding to these values of m. Each gives a complete period of a periodic algorithm showing certain inequalities which establish the non-existence of Euclid's algorithm in each field considered.

[1] K. INKERI, "Über den Euklidischen Algorithmus in quadratischen Zahlkörpern," Acad. Sci. Fennicae, *Annales*, v. 41, 1947, p. 5–34.

879[F].—A. GLODEN, "Factorisation de nombres $N^{16} + 1$," *Euclides*, v. 10, 1950, p. 157.

A small table gives the complete factorization of $N^{16} + 1$ for $N = 2(1)13$, 21,27,32 with references to authors. Two additional entries, for $N = 16$ and $N = 64$, might have been given:

$$16^{16} + 1 = 274177 \cdot 67280421310721$$
$$64^{16} + 1 = 641 \cdot 6700417 \cdot 18446744069414584321.$$

The first of these is the historic factorization by LANDRY of the 6-th FERMAT number. For the second see Amer. Math. Soc. *Bull.*, v. 36, 1936, p. 849. A third factorization corresponding to $N = 36$

$$36^{16} + 1 = 2753 \cdot 145601 \cdot 19854979505843329$$

has been announced by HOPPENOT and KRAITCHIK [*Sphinx*, v. 4, 1934, p. 47] but no really satisfactory proof of the primality of the largest factor has been given.

D. H. L.

880[F].—HELMUT HASSE, *Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörper*. Akad. Wissensch., Berlin, *mat. nat. Kl.*, *Abhandlungen*, 1948, n. 2, Berlin, 1950. vii + 95 p., 29.5 × 21 cm. 14 marks.

This paper contains tables for the fundamental units, the cyclotomic units and the class numbers of all cyclic cubic and all real cyclic biquadratic fields with conductor $\leqq 100$.

A systematic method is developed for finding these quantities generally in such fields; it is applicable to the case of non-real cyclic biquadratic fields. The work of H. Bergström[1] is used and generalized.

The analytic expression for the class number depends on the index of the system of cyclotomic units with respect to the system of all units in the maximum real subfield. For real fields this is the field itself. A cyclic cubic field is always real and has two fundamental units, a cyclic biquadratic field has three or one according as it is real or imaginary. It is shown that, for cubic fields, there exists a fundamental unit E such that -1 and the conjugates of E generate all the units. For real biquadratic fields the existence of a relative fundamental unit E is shown which has relative norm ± 1 with respect to the quadratic subfield (which is always real). The conjugates of E, the fundamental unit of the quadratic subfield and -1 generate a subgroup of index 1 or 2 in the complete group of units. In both cases E is characterized by a minimum property of its coordinates with respect to the integral base, just as the fundamental unit of a real quadratic field corresponds to the smallest positive solution of the Pellian equation.

The integral base is chosen in a special way by using ideas of class field theory. This is done in order to determine it in an invariant way. This can be regarded as a generalization of the fact that in the quadratic case the discriminant determines the base.

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NBS

[1] H. BERGSTRÖM, "Die Klassenzahlformel für reelle quadratische Zahlkörper mit zusammengesetzter Diskriminante als Produkt verallgemeinerter Gaussscher Summen," *Jn. reine angew. Math.*, v. 186, 1944, p. 91–115.

881[F].—E. H. NEVILLE, editor, *The Farey Series of Order 1025, Displaying Solutions of the Diophantine Equation $bx - ay = 1$. Designed and compiled by* E. H. N. (*Royal Society Mathematical Tables*, volume 1). Cambridge, University Press; published for the Royal Society, 1950. xxxii, 405 p. 21.5 × 28 cm. There is also a loose card, 20.2 × 27.2 cm., giving the complete factorization, on one side of the numbers 1–549, and on the other of 500–1049. £5 s5. Cambridge University Press, $18.50, New York.

This volume was accepted for publication by the British Association Tables Committee in 1947. In the following year the responsibility which the British Association had exercised since 1871 through that Committee was transferred to the Royal Society which set up a Committee of its own to take over the work [see *MTAC*, v. 3, p. 333–340]. The publication of the volume under the CUNNINGHAM bequest for the production of new tables in the theory of numbers is especially appropriate.

The so-called FAREY series, F_n, of order n, is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n; the numbers 0 and 1 are included in the forms 0/1 and 1/1. Thus, F_7 is

$$\frac{0}{1}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{1}{1}$$

The fraction $\frac{1}{2}$ is the middle term of every Farey Series; if two terms are equidistant from this, their sum is unity; they have the same denominator and this denominator is the sum of the two numerators. To emphasize this property of the series, as well as to economise in presentation, F_7 would be written:

0	1	1	1	1	2	1	2	3	1
1	**7**	**6**	**5**	**4**	**7**	**3**	**5**	**7**	**2**
1	6	5	4	3	5	2	3	4	1

Pages 1–400 of the volume before us are taken up with the presentation of the 319765 terms of F_{1025}, written in this form. Except for the last page there are 400 terms on each page, of 20 lines, each line containing 20 term-pairs, separated as two groups of 10, and the whole arranged in 8 groups of 50 term-pairs. The first three and the last three printed term-pairs of F_{1025} as given are

0	1	1		511	512	1
1	**1025**	**1024**	and	**1023**	**1025**	**2**
1	1024	1023		512	513	1

It is clear that all terms of an F_m, $m < n$, are contained in the series of terms of F_n.

If a_1/b_1, a_2/b_2, a_3/b_3 are any three consecutive terms of F_n

I. $(a_1 + a_3)/(b_1 + b_3) = a_2/b_2$.

II. $(a_1/b_1) - (a_2/b_2) = 1/(b_1b_2)$ or $(a_2/b_2) - (a_3/b_3) = 1/(b_2b_3)$, b_1, b_2, b_3 being relatively prime.

III. Any term between a_s/b_s and a_t/b_t is of the form $(pa_s + qa_t)/(pb_s + qb_t)$, where p and q are integers, and is irreducible if p, q are relatively prime.

IV. The total number of terms in F_n is $\Phi(n) + 1$, where $\Phi(n)$ is the sum function of Euler's $\phi(n)$.

The work under review contains three appendices (which I shall call A, B, and C).

Appendix A (p. 402–403) gives the Farey series of order 50 with decimal equivalents to 5D.

Appendix B (p. 404) displays all the terms of F_{64}, in order to give the reader a general idea of a Farey series which may not be deduced from the 400 pages filled with F_{1025}.

Appendix C (p. 405) exhibits the "Farey integer-series" of order 100, that is, the denominators, in order, of all the terms of the series in F_{100}. The page has to be read twice (once forward and once backward), in order to cover all terms.

Let us now pause to consider some historical facts connected with the series under consideration, insofar as they have bearing on the contents of the work under review.

α. In *Ladies' Diary* for 1747, p. 34, Question 281, J. May, Jr., of Amsterdam, offered the problem: "It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the greatest denominator be less than 100." Since reducible fractions are not excluded the number here would be in excess of the number of terms in F_{99}. Solutions are presented in *Ladies' Diary* for 1748, p. 23, and for 1751, p. 30.

β. In *Journal de l'École Polyt.*, cahier 11, v. 4, 1802, p. 364–368, Citizen Haros writes on the topic "Tables pour évaluer une fraction ordinaire avec autant de décimales qu'on voudra; et pour trouver la fraction ordinaire la plus simple, et qui approche sensiblement d'une fraction décimale."

Haros here proved the general results I and II stated above.

$\gamma 1$. In 1816 Henry Goodwyn (1745–1824) printed privately and distributed some copies of his *The First Centenary of a Series of Concise and Useful Tables of all the complete Decimal Quotients, which can arise from dividing a unit, or any whole Number less than each Divisor by all Integers from 1 to 1024.* This publication ("specimen" he calls it) contained xiv + 18 pages. Page i is simply the title page quoted above; p. ii is blank; on p. iii is an introductory statement signed by Goodwyn and dated March 5th, 1816. The 18 pages of tables simply contain 6D equivalents of the various fractions with denominators successively 1(1)100. From this series of tables one could get no ready idea of the order of successive terms in a so-called Farey series.

$\gamma 2$. At London in 1818 Goodwyn sent out a publication on the upper part of the title page of which is the italics title quoted above; then follows immediately: *To which is now added a Tabular Series of Complete Decimal Quotients for all the Proper Vulgar Fractions of which when in their lowest terms, neither the Numerator nor the Denominator is greater than 100: with the equivalent vulgar fractions prefixed.* This is repeated as title page (i) of the separately paged addition. viii + 32 p. Here we have, p. 1–14, not only all the terms in the first half of F_{100}, and their 10D equivalent values, but also the means for writing down at once the remaining terms of the series and their decimal equivalents.

Hence, this table offers the means for (a) reading off all the results of Appendix C in the table under review; (b) displaying the Farey series of order 100 rather than of order 64 in Appendix B; (c) reading off the Farey series of order 100 with decimal equivalents rather than the series of order 50 with diminished decimal approximations in Appendix A.

Furthermore, on page v of this 1818 publication of Goodwyn we find in his survey of the series: "every Fraction in the Series (since they are all in their lowest terms), is the *exponent* of the *ratio* of the *sum* of the *numerators* to the sum of the *denominators*, of the Fractions occurring *next before* and *after* it." This is just one of the results found by Haros 16 years before. But in this connection Goodwyn does not mention the name of anyone else.

δ. It will presently appear why we must now turn back in our chronological treatment. In *Phil. Mag.*, v. 47, May 1816, p. 385–386, the geologist John Farey (1766–1826; see *Dict. Nat. Biog.*) contributed a letter to the editor. The title for the heading to this letter is "On a curious property of vulgar fractions." The letter commences: "On examining lately, some very curious and elaborate Tables of 'Complete decimal Quotients' calculated by Henry Goodwyn, Esq. of Blackheath, of which he has printed a copious specimen, for private circulation among curious and practical calculators, preparatory to the printing of the whole of those useful tables, if sufficient encouragement, either public or individual, should appear to warrant such a step: I was fortunate while so doing, to deduce from them the following general property; viz.,"—and then follows the Haros-Goodwyn theorem, with numerical illustrations from F_5 and F_{99}.

What deductions may we now make? Farey was writing about May 1816 and refers to Goodwyn's privately distributed publication of the previous March, and also, to tables of "Complete Decimal Quotients." It is obvious that by 1816 Farey must have seen both tables of the 1818 publication of Goodwyn, to each of which the above quoted title applies, in order to make public his deduction (impossible from the 1816 table alone). We have noted that Goodwyn formulates this same result.[1]

Farey's letter was reported in Soc. Philomathique de Paris, *Bull. d. Sci.*, 1816, p. 112, but Farey's name is not mentioned. This communication attracted the attention of Cauchy, who gave a proof, and extension, in the same volume of the *Bulletin*, p. 133–135. As we have seen these results, I–II, were already published by Haros 14 years earlier.

The result III was proved by EDWARD SANG (R. Soc. Edinb., *Trans.*, v. 28, 1879, p. 287). The name "Farey Series," which, as we have seen, has not the slightest justification historically, seems to have been first used by Sylvester, who also tabulated IV (*Phil. Mag.*, s. 5, v. 15, 1883, p. 251f.). Well knowing all the facts in question Neville would have served scholarship better by entitling his book: *The Haros Series of Order 1025*, or *The Goodwyn Series of Order 1025*.

The table in clockmaker Achille Brocot's *Calcul des Rouages par Approximation. Nouvelle Méthode*, Paris, 1862, p. 49–89, is a display of F_{100}, with each term approximated to 10D; the first half is identical with Goodwyn's 1818 table. (γ2)

In HÜTTE, *Hilfstafeln zur I. Verwandlung von echten Brüchen in Dezimalbrüche* . . . Third newly revised edition, Berlin, 1922, p. 24–61, J. T. PETERS checked and extended "Brocot's" table from 10D to 11D.

In his *Manual of Gear Design, Section One*, New York, 1935, p. 148–169, EARLE BUCKINGHAM proposed to give a table of F_{120} to 8D, but there are numerous omissions and errors; see *MTAC*, v. 1, 1943, p. 92.

On the other hand, the terms of F_{120} in R. M. PAGE, *14000 Gear Ratios*, New York, 1942, are complete and the corresponding 11D values more reliable. [See *MTAC*, v. 1, 1943, p. 21–23; 1944, p. 326–329, 430.]

Until the publication of Neville's volume the most extensive F_n to be published was that for $n = 120$. The statement by D. H. L. in his *Guide to Tables in the Theory of Numbers*, Washington, 1941, p. 8, that Goodwyn gave F_{1000} is therefore incorrect; see *MTAC*, v. 1, p. 372, 1945.

All solutions (x, y) of the equation

$$bx - ay = c, \qquad (a, b, c, \text{ integers}, a, b, \text{ coprimes}) \tag{1}$$

are given by the formulas

$$x = ka + Xc, \qquad y = kb + Yc, \qquad k \text{ any integer},$$

if

$$bX - aY = 1. \tag{2}$$

Hence, a/b and X/Y are consecutive terms of a Farey series, and solutions (X, Y) of (2), and hence of (1) are known if the neighbours of a/b in F_b are known. Neville's Introduction explains, with numerical illustrations, not only the processes by which solutions of any equation (1) may be obtained, however much b exceeds n, but also how to find desired terms in F_{1025}. A "Decimal Index" table on p. xxviii–xxix, and the loose card, are useful in this connection.

Solutions of (1) have important applications in different parts of the theory of numbers.

The two earlier tables for finding solutions of (2) are by A. L. CRELLE (in his *Journal*, v. 42, 1851, p. 304–313), and by A. J. CUNNINGHAM (in his *Quadratic and Linear Tables*, London, 1927, p. 134–157). Crelle gives immediate solutions of (2) for $a < b \leq 120$. The results of Cunningham's table, for $b < 100$, $a < 100$, are therefore included in Crelle's. Within its range the Crelle table would naturally furnish results more readily than Neville's tables would.

To ensure accuracy in this splendid new volume of the British Mathematical Tables Committee more than 40 people collaborated with Professor Neville. Hence we must wonder why neither they nor the printer noticed that on page 399 are ten numbers which should have been in ordinary rather than in blackface type.

This volume is dedicated "To the memory of SRINIVASA RAMANUJAN," Neville's friend, since "to every mathematician of our time Farey's series" recalls his name.

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[1] Not only the publications of GOODWYN to which we refer, but others, have been discussed in thorough fashion by J. W. L. GLAISHER: (i) *Report of the Comm. on Mathematical Tables*, 1873, p. 31–33, 150; (ii) "On circulating decimals with special reference to Henry Goodwyn's 'Table of circles' and 'Tabular series of decimal quotients' (London 1818–1823)," Cambridge Phil. Soc., *Proc.*, v. 3, 1878, p. 185–206; (iii) "On a property of vulgar fractions," *Phil. Mag.*, s. 5, v. 7, 1879, p. 321–336. We have already quoted DEMORGAN's

Statement of 1861 [*MTAC*, v. 2, p. 87] concerning Goodwyn: "His manuscripts, an enormous mass of similar calculations, came into the possession of Dr. OLINTHUS GREGORY, and were purchased by the Royal Society at the sale of his [Gregory's] books in 1842." In the above publications of Glaisher it is stated that no trace of the papers could be found at the Royal Society. Neville's statement, p. xi, "Goodwyn left a mass of papers, no one knows what became of them," is therefore slightly misleading.

882[F].—BENGT PERSSON, "On a Diophantine equation in two unknowns," *Arkiv Mat.*, v. 1, 1949, p. 45–57.

The table on p. 57 gives all solutions of the diophantine equation $x^2 + x + (p + 1)/4 = y^3$ for all values of the prime $p < 100$ of the form $4n + 3$, except the value $p = 3$ which is handled separately on p. 45 of the paper.

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883[F].—B. VAN DER POL & P. SPEZIALI, "The primes in $k(\rho)$," K. Nederl. Akad. Wetensch., *Proc.*, v. 54 A, 1951, p. 9–15, 1 plate.

The primes of the quadratic field $k(e^{2\pi i/3})$ are presented in graphical form. Each integer in the field is assigned to the center of a hexagon. The totality of integers thus fills the plane. Those integers which are primes are indicated by black hexagons, the non-primes by white hexagons. The plate inserted between p. 13 and 14 gives a graphical representation of all primes in the field whose norms are less than 10000. The beautiful symmetrical diagram contains 7416 black hexagons out of approximately 36276 hexagonal cells giving a density of primes of about one out of five.

An enlargement of the central part is given in another figure. The paper contains a table of solutions (c, β) of the quadratic partition

$$p = 6m + 1 = c^2 + 3\beta^2 \qquad \text{for} \qquad p < 10000.$$

There are two misprints, communicated by the author:

for	$p = 1559$	*read*	1549
for	$p = 7459$	*read*	7549

This table was used to construct the diagram. The authors were unaware that previous tables of this extent exist. JACOBI's table[1] for $p \leqslant 12007$ is more than a century old. Other tables by REUSCHLE[2] and CUNNINGHAM[3] extend to $p = 13669$ and $p = 125683$ respectively. [See also UMT **124.**] A comparison of Cunningham's table with the present one reveals no discrepancy.

A similar diagram for the primes of the field $k(i)$, based on squares, was published by van der Pol.[4]

D. H. L.

[1] K. G. J. JACOBI, "Über die Kreistheilung und ihre Anwendung auf die Zahlentheorie," *Jn. reine angew. Math.*, v. 30, 1846, p. 166–182.

[2] K. G. REUSCHLE, *Mathematische Abhandlung, enthaltend: Neue Zahlentheoretische Tabellen*, Stuttgart, 1856.

[3] A. J. C. CUNNINGHAM, *Quadratic Partitions*. London, 1904; *Quadratic and Linear Tables*. London, 1927.

[4] B. VAN DER POL, Verslagen van de Maatschappij Diligentia, The Hague, 1946. A copy of this diagram woven in red and white squares hangs on the wall of the reviewer's study, a gift from the author.

884[H].—W. P. THIJSEN, "Over de geïtereerde machtsverheffing," *Simon Stevin*, v. 27, p. 177–192, 1950.

The table on p. 185 gives approximate solutions of the equations:

$$(1) \qquad t = x^t$$

$$(2) \qquad t = x^{x^t}$$

for given x. Equation (1) has two solutions:[1] $t = \lambda, \mu$; $1 < \lambda < e < \mu < \infty$. Similarly equation (2) has two solutions: $t = \sigma, \tau$; $0 < \sigma < e^{-1} < \tau < 1$. 5D values (mostly) are given for λ, and 4D values for σ, τ, all for $x = 0(.01).06$, $e^{-e} = .06599$, $.1(.1)1$, 1.2, $e^{1/e} = 1.447$. The values of μ corresponding to $x = 1, 1.2, e^{1/e}$ are ∞, 14.77, e respectively.

On p. 184 there are graphs[1] illustrating $\lambda, \mu, \sigma, \tau$ as functions of x.

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[1] The curve $t = x^t$ has been frequently studied before. For example: (a) J. F. C. HESSEL, "Über das merkwürdige Beispiel einer zum Theil punctirt gebildeten Curve das der Gleichung entspricht $y = \sqrt[x]{x}$," *Archiv Math. Phys.*, v. 14, 1850, p. 169–187; H. SCHEFFER, "Über die durch die Gleichung $y = \sqrt[x]{x}$ dargestellten Curven," *Archiv Math. Phys.*, v. 16, 1851, p. 133–137.

The role which $\xi = \omega^\xi$ plays in CANTOR's theory of transfinite numbers will be recalled; see, for example, G. Cantor, *Math. Annalen*, v. 49, 1897, p. 242–246 (also the English translation of P. E. B. JOURDAIN, Open Court, 1915, p. 195–201).

885[I].—H. E. SALZER, "Coefficients for polar complex interpolation," *Jn. Math. Phys.*, v. 29, 1950, p. 96–104.

This article is supplementary to two earlier papers,[1] both of which are devoted to the computation of the coefficients of interpolation formulas for the representation of a function $f(z)$ defined at n successive points z_k, where k ranges from $[\frac{1}{2}n - \frac{1}{2}]$ to $[\frac{1}{2}n]$, the points being spaced equally along the arc of a circle about the origin of coordinates.

In the first of these papers the interpolation formula was written in the form

$$f(z) \sim \sum L_k^{(n)}(P)f(z_k),$$

where the point P was defined to be $P = (z - z_0)/(z_1 - z_0) = p + iq$, and where P^n was represented by $p_n + iq_n$. Explicit formulas were given for the real and imaginary parts of the coefficients $L_k^{(n)}$, which are functions not only of p_n and q_n, but also of θ, since $z = pe^{i\theta}$. The three cases of 3-point, 4-point, and 5-point interpolation were treated.

In the second paper the interpolation formula appeared as follows:

$$f(z) \sim (\sum a_k f_k)/\sum a_k,$$

where we write

$$a_k = [(z_k - z_0) \cdots (z_k - z_{k-1})(z_k - z_{k+1}) \cdots (z_k - z_{n-1})(z - z_k)]^{-1}.$$

If new quantities $A_k^{(n)} = (z - z_k)a_k$ are introduced, and if $z_0 = 0$, $z_1 = 1$, $z_2 = 1 + e^{i\theta}$, $z_3 = 1 + e^{i\theta} + e^{2i\theta}$, $z_{-1} = -e^{-i\theta} - e^{-2i\theta}$, etc., then $A_k^{(n)}$ can be computed as functions of θ. This article gives the explicit values for these functions for 3-point to 9-point interpolation inclusive.

The article under review provides tables of the functions $L_k^{(n)}$ and $A_k^{(n)}$, computed in the first case to 8D and in the second case to 9D, with some doubt as to the ninth place, over the range $\theta = 1°, 5°, 10°, 15°, 20°$, and $30°$.

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[1] H. E. SALZER, "Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs," *Jn. Math. Physics*, v. 24, 1945, p. 141–143, *MTAC*, v. 2, p. 73, and "Alternative formulas for direct interpolation of a complex function tabulated along equidistant circular arcs," *Jn. Math. Physics*, v. 26, 1947, p. 56–61.

886[K].—R. L. ANDERSON & T. W. ANDERSON, "Distribution of the circular serial correlation coefficient for residuals from a fitted Fourier series," *Annals Math. Stat.*, v. 21, 1950, p. 59–81.

The model for observations x_i, $i = 1, 2, \cdots, N$, here used is

$$x_i - \mu_i = \rho(x_{i-1} - \mu_{i-1}) - u_i,$$

with $x_0 \equiv x_n$ and $\mu_0 \equiv \mu_n$, in which the $\{u_i\}$ are random disturbances each independently and normally distributed with zero mean and variance σ^2, and μ_i is a linear form in Fourier terms. The coefficients of the Fourier terms are the usual regressions of the x_i on these terms. The fitted values of the μ_i are denoted by m_i and the serial correlation coefficient

$$R = \sum_{i=1}^{N} (x_i - m_i)(x_{i-1} - m_{i-1}) \Big/ \sum_{i=1}^{N} (x_i - m_i)^2$$

(with lag 1), where $m_0 \equiv m_n$, is used to test hypotheses concerning ρ. Under the null hypothesis of zero serial correlation ($\rho = 0$), Table I, p. 65, gives to 3D the exact 5% and 1% points for R for P periods used in the fitted series for the following cases: $P = 2$, $N = 6(2)60$; $P = 2, 4$, $N = 8(4)100$; $P = 2, 3, 6$, $N = 12(6)150$; $P = 2, 12/5, 3, 4, 6, 12$, $N = 24(12)300$. Table II, p. 66–67, gives exact 5% and 1% significance points for R (the distribution of R is asymmetric in these cases) for single periods for $P = 3$, $N = 6(6)150$; $P = 6$, $N = 12(6)150$; $P = 4$, $N = 8(4)100$, $108(12)144$; $P = 12$, $N = 12(12)300$.

C. C. C.

887[K].—D. G. CHAPMAN, "Some two sample tests," *Annals Math. Stat.*, v. 21, 1950, p. 601–606.

Let s represent the difference of two independent variables, each of which has a t-distribution with $n - 1$ degrees of freedom. Table I contains values of $P(0 \leqq s \leqq s_0)$; the values of n and s_0 considered are: $n = 2(2)12, \infty$; $s_0 = .5(.5)8(2)12, 21, 30, 50, 100$. Table II contains values of s_0 such that $P(|s| \geqq s_0) = .01, .05$ for $n = 2(2)12, \infty$.

Another place where 1 percent and 5 percent points for the absolute difference of two independent t-statistics can be obtained is in the 1948 edition of FISHER & YATES' tables.[1] Let t_1 have a t-distribution with n_1 degrees of freedom while t_2 is independent of t_1 and has a t-distribution with n_2 degrees of freedom. Let $d = (t_1 - t_2)\cos\theta$. Table V_1 (by P. V.

SUKHATME) contains values of d_0 such that $P(|d| \geq d_0) = .01, .05$ for $n_1, n_2 = 6, 8, 12, 24$, and $\theta = 0°(15°)90°$.

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[1] R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural, and Medical Research.* London, 1948.

888[K].—JEROME CORNFIELD & NATHAN MANTEL, "Some new aspects of the application of maximum likelihood to the calculation of the dosage response curve," Amer. Stat. Assn., *Jn.*, v. 45, 1950, p. 181–210.

Table 1 (p. 185–188) gives Z/P, $-Z/Q$, $Z^2/Q^2 - ZY/Q$, and $ZY/P + Z^2/P^2$, all to 5D, for $X = 5(.01)10$, and for $X = 0(.01)5$ by symmetry. $X - 5 = Y$ is a normal deviate, Z being the corresponding ordinate; P is the probability of a deviate less than Y, and Q that of a larger deviate. The purpose of the table is to minimize the arithmetic of the maximum likelihood estimate of the probit regression. There are detailed instructions for calculation.

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889[I, K].—D. B. DELURY, *Values and Integrals of the Orthogonal Polynomials up to n = 26.* Toronto, Published for Ontario Research Foundation by University of Toronto Press. 1950, v + 33 p. 17.5 × 25.2 cm. $1.25.

The polynomials tabulated are those of CHEBYSHEV[1] and GRAM and are the FISHER & YATES[2] polynomials denoted by $\xi_r'(x) = \xi_r'(x, n)$.

The values of $\xi_r'(x, n)$ are tabulated for $r = 0(1)n - 1$ and for $n = 3(1)26$.

The sums of squares of the ξ's are given at the foot of each column. For $r \leq 14$ and $n \leq 26$ values of

$$I_r = \int_0^{n-1} \xi_r'(x, n)\,dx$$

and

$$J_r = \int_{-\frac{1}{2}}^{n-\frac{1}{2}} \xi_r'(x, n)\,dx$$

are tabulated.

The tables presented in the book were prepared for the purpose of furnishing an arithmetical basis for a numerical integration procedure appropriate to situations in which the observed or measured ordinates are subject to a random error.

Of considerable interest is the discussion concerning the fitting of a function in two variables by means of orthogonal polynomials.

Two examples of the use of the tables are given. The first deals with evaluating an integral over a range of equally spaced data. The second example shows how the total number of trees on a given area can be estimated from counts on square plots situated centrally on square blocks within the area. The orthogonal polynomials are used to determine the equation of the surface. The equation is then integrated to yield the desired estimate.

It is of interest to point out that a table of orthogonal polynomials and their derivatives has been prepared by W. F. BROWN, JR. & C. W. DEMPSEY [*MTAC*, v. 4, p. 224–5].

A comparison between the values of the polynomials tabulated by DeLury and by Brown & Dempsey show no discrepancies. It is to be regretted that the values of the ratio of ξ' to ξ are not given in the book.

The book is clearly printed. It should be a welcome addition to the library of statisticians and mathematicians.

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Further values of these orthogonal polynomials have been given by ANDERSON & HOUSEMAN[3] for $n = 3(1)104$, $r = 1(1)\min(5, n)$ and by VAN DER REYDEN[4] for $n = 5(1)13$, $r = 1(1)$ various values less than 9 and for $n = 14(1)52$, $r = 1(1)9$.

The introduction describes the uses of the table in fitting polynomials by least squares, in finding integrals of the resulting polynomials, in carrying out the associated analyses of variances and in determining confidence intervals in one and in two independent variables.

Values of ξ_r' for $r \leq 5$ are said to be reprinted from FISHER & YATES.[2] The author states that "The director of the Onderstepoort Veterinary Research Institute and Dr. D. VAN DER REYDEN[4] also have kindly granted permission to reproduce portions of the tables which appeared in the *Onderstepoort Journal.*" The present tables are, however, correct in two places where minus signs are missing from van der Reyden's table (for $n = 17$, $r = 5$, $\xi_1' = -3$ and $n = 18$, $r = 5$, $\xi_1' = -1$. Van der Reyden gives values of the polynomials for $x \leq \bar{x}$). And a common factor in the values for $n = 25$, $r = 8$ in van der Reyden's table has been removed.

For more complete discussions of these orthogonal polynomials see the review by EISENHART [*MTAC*, v. 1, p. 148–150] and papers by BIRGE,[5] WEINBERG[6] and van der Reyden[4] and the references given in these places. For punched card procedures using such tables as the one under review see a paper by LILA KNUDSEN.[7]

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[1] P. L. CHEBYSHEV, "Sur l'interpolation par la méthode des moindres carrés," Akad. Nauk, Leningrad, *Mémoires*, s. 7, v. 1, no. 15, 1859, p. 1–24. *Oeuvres*, v. 1, p. 471–498.

[2] R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*. Third ed., London, 1948. (First ed., 1938.)

[3] R. L. ANDERSON & E. E. HOUSEMAN, *Tables of Orthogonal Polynomial Values Extended to N = 104*. Ames, Iowa, Iowa State College of Agriculture and Mechanic Arts, 1942. (*Research Bulletin* 297, p. 593–672, Agricultural Experiment Station, Statistical Sections.) [*MTAC*, v. 1, p. 148–150, v. 5, p. 81.]

[4] D. VAN DER REYDEN, "Curve fitting by the orthogonal polynomials of least squares," *Onderstepoort Journal of Veterinary Science and Animal Industry*, v. 28, 1943, p. 355–404.

[5] RAYMOND T. BIRGE, "Least-squares' fitting of data by means of polynomials," *Review of Modern Physics*, v. 19, 1947, p. 298–347.

[6] J. W. WEINBERG, "Mathematical appendix," *ibid.*, p. 348–360.

[7] LILA F. KNUDSEN, "A punched card technique to obtain coefficients of orthogonal polynomials," Am. Stat. Assn., *Jn.*, v. 37, 1942, p. 496–506.

890[K].—A. K. GAYEN, "The distribution of the variance ratio in random samples of any size drawn from non-normal universes," *Biometrika*, v. 37, 1950, p. 236–255.

The author is interested in the non-normal sampling distribution of the two statistics: w = the ratio of two estimates of the variance obtained in a one-way classification for the analysis of variance (homogeneity), v = the variance ratio of two independent samples (compatibility). To introduce non-normal populations the EDGEWORTH series,

$$f(x) = \varphi(x) - \frac{\lambda_3}{6}\varphi^{(3)}(x) + \frac{\lambda_4}{24}\varphi^{(4)}(x) + \frac{\lambda_3^2}{72}\varphi^{(6)}(x),$$

is used. With an Edgeworth series population the author evolves the frequency densities of the two statistics. These distributions are then related to their corresponding "normal" sampling distributions and correction terms for the skewness and excess are determined. The coefficients of λ_3 ($= \sqrt{\beta_1}$) and λ_4 ($= \beta_2 - 3$) are tabulated in Table 4 (p. 252–255) to 4D for the 5 percent points of Fisher's Z for the degrees of freedom $\nu_1 = 1(1)6$, 8, 12, 24, ∞; $\nu_2 = 1(1)6$, 8, 12, 20, 24, 30, 40, 60, 120, ∞. These tables enable one to compute the error one would make in using the ordinary tables associated with the normal population distribution of Z for the 5 percent level if he were aware of the λ_3 and λ_4 of his population and was willing to assume an Edgeworth type distribution. In this connection the author notes that the Edgeworth restriction would not be serious if the size of sample was sufficient to enable one to neglect terms of order N^{-3}. If λ_3 and λ_4 are unknown one might use their sample estimate in the formulas to get an indication of the size of error involved.

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891[K].—N. F. GJEDDEBAEK, "Contribution to the study of grouped observations. Application of the method of maximum likelihood in case of normally distributed observations," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 135–159.

Let

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2), \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy, \qquad \varphi'(x) = \frac{d\varphi(x)}{dx},$$

$$z_1(x, y) = -\frac{\varphi(x+y) - \varphi(x)}{\Phi(x+y) - \Phi(x)}, \qquad z_2 = \frac{\varphi'(x+y) - \varphi'(x)}{\Phi(x+y) - \Phi(x)}.$$

Table 1, p. 151–154, contains 4D values of $z_1(x, y)$ for $y = 0(.1)4$ and $x = -.1h(.1)4$, where h is the largest integer $\leq 5y + 1$. Table 2, p. 155–158, contains 4D values of $z_2(x, y)$ for the same values of y and x in Table 1. Given y, both z_1 and z_2 are symmetrical about the value $x = -y/2$. Table 3, p. 159, contains 4D values of z_1 and z_2 for $y \to \infty$ and $x = -4.5(.1)0$. The functions z_1 and z_2 are used in an iterative procedure for determining the maximum likelihood estimates of the mean and standard deviation of a normal population when the observations from this population have been grouped.

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892[K].—A. HALD, *The Decomposition of a Series of Observations Composed of a Trend, a Periodic Movement and a Stochastic Variable.* Copenhagen, G. E. C. Gads, 1948, 134 p., 17.2 × 25.1 cm. 8 kr.

The author defines polynomials $\xi_r(x_{\nu i})$, of degree $r = 1, \cdots, l$ ($\nu = 1, \cdots, n$; $i = 1, \cdots, k$), by means of the relations $\xi_{\mu\cdot\cdot} = 0$ (a dot in a subscript indicates a mean with respect to the corresponding variable), $\mu = 1, \cdots, l$, and

$$\sum_{\nu=1}^{n} \sum_{i=1}^{k} (\xi_{r\nu i} - \xi_{r\cdot i})(\xi_{\mu\nu k} - \xi_{\mu\cdot i}) = 0 \text{ for } \mu < r, r = 2, \cdots, l; \mu = 1, \cdots, l-1.$$

Here $\xi_{r\nu i}$ is used to denote $\xi_r(x_{\nu i})$.

The leading coefficients are assigned the value unity and then the remaining coefficients are determined by the defining equations. For $k = 1$ these become the ordinary CHEBYSHEV polynomials over equally spaced abscissas. The tables give values generally to 6S, of these polynomials of degrees 1 to 5 for the combinations $k = 3$, $n = 5, 7$; $k = 5$, $n = 5, 7, 8, 9, 10, 11$. For $k = 4$, $n = 25$ the degrees run from 1 to 4 and for $k = 12$, $n = 10$, the degrees run from 1 to 6. In addition explicit formulas for the polynomials tabulated are given (the author was unable to derive the general explicit formula) and the values, again generally to 6S, of the elements of the three matrices $||\xi_{r\cdot i}||$, $||S_{r\mu}||$, $||S'_{r\mu}||$, for the polynomials tabulated in which

$$S_{r\mu} = \sum_{\nu=1}^{n} \sum_{i=1}^{k} (\xi_{r\nu i} - \xi_{r\cdot i})(\xi_{\mu\nu i} - \xi_{\mu\cdot i}), \qquad (S_{r\mu} = 0, r \neq \mu),$$

and

$$S'_{r\mu} = \sum_{\nu=1}^{n} \sum_{i=1}^{k} \xi_{r\nu i}\xi_{\mu\nu i}.$$

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893[K].—H. O. HARTLEY, "The maximum F-ratio as a short-cut test for heterogeneity of variance," *Biometrika*, v. 37, 1950, p. 308–312.

The ratio of the largest to the smallest in a set of k variance estimates is proposed as a short-cut test in place of BARTLETT's test,[1]

$$M = N \log_e \{N^{-1} \sum_t \nu_t s_t^2\} - \sum_t \nu_t \log_e s_t^2$$

where the estimate s_t^2 is based on ν_t degrees of freedom and $N = \sum_t \nu_t$.

Since $\log_e s^2$ is approximately normal with variance $2/(\nu - 1)$,

$$F_{\max}(\alpha) = \exp \{w_k(\alpha)\sqrt{2/(\nu - 1)}\}$$

gives approximately the $100\alpha\%$ point of the ratio when the k estimates are based on the same number, ν, of d.f. and where $w_k(\alpha)$ is the corresponding point of the distribution of the range in samples from normal (PEARSON & HARTLEY[2]). Tables 1 and 2 give comparisons of the approximate and exact 5% points of $F_{\max}$ to 3S for $k = 2$ and $\nu = 2(1)10, 12, 15, 20$, and for $\nu = 2$ and $k = 2(1)12$, respectively. The two sets of exact results were used to obtain second approximations by the adjustment, $F_{\max}(\alpha)$ (second approx.) $= F_{\max}(\alpha)$ (first approx.) $(1 + q_\nu q_k)$, with q_ν and q_k fitted to the exact values. The second approximations, with exact values in the first row and the first column, are given in Table 3, which shows upper 5% points of $F_{\max}$ to 3S in a set of k mean squares all based on ν d.f. for $k = 2(1)12$ and $\nu = 2(1)10, 12, 15, 20, 30, 60, \infty$.

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[1] M. S. BARTLETT, "Properties of sufficiency and statistical tests," R. Soc. London, *Proc.*, v. 160A, 1937, p. 268–282.
[2] E. S. PEARSON & H. O. HARTLEY, "The probability integral of the range in samples of n observations from a normal population," *Biometrika*, v. 32, 1942, p. 301–310.

894[K].—H. O. HARTLEY & E. S. PEARSON, "Tables of the χ^2-integral and of the cumulative Poisson distribution," *Biometrika*, v. 37, 1950, p. 313–325.

Let $$P(\chi^2, v) = 2^{-\frac{1}{2}v}\{\Gamma(\tfrac{1}{2}v)\}^{-1} \int_{\chi^2}^{\infty} e^{-\frac{1}{2}x} x^{\frac{1}{2}v-1} dx.$$

The authors have tabulated (p. 318–325) $P(\chi^2, v)$ to 5 places of decimals for $v = 1(1)20(2)70$ degrees of freedom and $\chi^2 = 0(.001).01(.01).1(.1)2(.2)$-$10(.5)20(1)40(2)134$. At the same time the table provides the values of the cumulative Poisson distribution since $P(\chi^2, v) = \sum_{i=0}^{c-1} e^{-m} m^i/i!$ with $m = \chi^2/2$, $c = v/2$. Hence for $m \leqq 15$ the complete Poisson distribution is provided and for $m > 15$, only the truncated Poisson sum up to $c = 35$ ($v = 70$). Methods of interpolation in the table are thoroughly discussed. The authors point out that Pearson's table of the incomplete Γ-function[1] and MOLINA's table of the Poisson exponential limit[2] are more extensive, the former for χ^2 and the latter for the Poisson distribution. But Pearson's table involves a troublesome transformation and Molina's tables do not give $P(\chi^2, v)$ for odd values of v. In addition it may be noted that Salvosa's table of the Type III function, both areas, ordinates and the first six derivatives[3] may be used to evaluate $P(\chi^2, v)$ by setting t in SALVOSA's table $= (\chi^2 - v)(2v)^{\frac{1}{2}}$, and $\alpha_3{}^2 = 8/v$. A comparison by the reviewer with the tables of $P(\chi^2, v)$ given in the Elements of Statistics by DAVIS & NELSON[4] for small values of χ^2 in a few instances revealed no divergencies, but for $v = 2$, $\chi^2 = 5$ Davis and Nelson give .08208 49986, while the present table gives .08209. The same phenomenon was noted for $v = 9$, $\chi^2 = 3.0$. No doubt the table will be found useful in statistical investigations.

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[1] KARL PEARSON, *Tables of the Incomplete Γ-Function*. London, 1922.
[2] E. C. MOLINA, *Poisson's Exponential Binomial Limit*. New York, 1945.

[3] L. R. SALVOSA, "Tables of Pearson's type III function," *Annals Math. Stat.*, v. 1, 1930, p. 191–198. Appendix, p. 1–187.
[4] H. T. DAVIS & W. F. C. NELSON, *Elements of Statistics with Applications to Economic Data*. Bloomington, Ind., 1935.

895[K].—J. M. HOWELL, "Errata to 'Control chart for largest and smallest values,'" *Annals Math. Stat.*, v. 21, 1950, p. 615–616.

In the paper cited in the title [*MTAC*, v. 4, p. 207–208] the author notes that numerical errors exist in Table I (as was pointed out by *MTAC*'s reviewer). Using values appearing in a paper by GODWIN,[1] HOWELL now gives a corrected version of Table I to 4D.

C. C. C.

[1] H. J. GODWIN, "Some low moments of order statistics," *Annals Math. Stat.*, v. 20, 1949, p. 279–285. [See *MTAC*, v. 4, p. 20.]

896[K].—R. F. LINK, "The sampling distribution of the ratio of two ranges from independent samples," *Annals Math. Stat.*, v. 21, 1950, p. 112–116.

Consideration is given to a sample of n ordered observations drawn from a population with variance σ^2. Let $w = (x_n - x_1)/\sigma$. Then for two samples of sizes n_1 and n_2 drawn from populations of the same variance, the ratio w_1/w_2 becomes the ratio of the two ranges. Table II gives values of R for all combinations of n_1 and $n_2 \leqq 10$ and for $\alpha = .005, .01, .025, .05$, and $.10$, such that

$$Pr\ (w_1/w_2 < R) = \alpha$$

where the samples are drawn from populations with the same variance. These values may be used for testing the hypothesis that two independent samples were drawn from normal populations with the same variance. This test is therefore comparable to the F test.

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897[K].—E. LORD, "Power of the modified t-test (u-test) based on range," *Biometrika*, v. 37, 1950, p. 64–77.

For a sample of n independent observations from a normal population with unknown standard deviation, σ, the u-test is the ratio of the normal deviate to an independent range estimate of the standard error. The power of a two-tailed test, $\beta(\rho)$, against an alternative $\rho = \xi/\sigma$, is the sum of the two power components, $\beta'(\rho) = Pr\ \{\text{lower tail} | \rho\}$ and $\beta''(\rho) = Pr\ \{\text{upper tail} | \rho\}$. The standardized error of NEYMAN & TOKARSKA[1] is ρ_α such that $1 - \beta(\rho_\alpha) = \alpha$ for any significance level α. Six tables are given to compare the standardized errors of the u-test and the t-test for $\alpha = .05$ and $\alpha = .01$.

Table I, p. 67, shows to 5D the power components $\beta'(\rho)$ for $\rho = 1(1)10$ and $\beta''(\rho)$ for $\rho = 1(1)3$ and $\nu = 1(1)20, \infty$ of the t-test, where ν is the number of degrees of freedom for estimating variance. Table II, p. 68, gives the standardized errors for the t-test, $\rho_{.05}$ to 3D and $\rho_{.01}$ to 2D for $\nu = 1(1)20, 30, 60, 120, \infty$.

Table III, p. 70, gives the power components for the u-test to 5D for $n = 2(1)20, \infty$; $\beta'(\rho)$ for $\rho = 1(1)10$ and $\beta''(\rho)$ for $\rho = 1(1)3$. Table IV,

p. 71, shows the standardized errors for the u-test, $\rho_{.05}$ to 3D and $\rho_{.01}$ to 2D for $n = 2(1)20, \infty$.

In Table V, p. 72, there are given values of $\rho_{.05}$ to 2D for the u-test based on the average range in m samples of n independent observations each, for $n = 2(1)20$ and $m = 1(1)10(5)20, 30, 60, 120$. From Table V, it appears that the optimum grouping of N $(= mn)$ observations is one such that $6 \leqq n \leqq 10$ in general.

Table VI, p. 76, gives the ratio of the standardized errors at 5% and 1% levels of the u-test to the t-test for 3D for $n = 2(1)20, \infty$. The ratio is $\leqq 1.024$ for the 5% level and $\leqq 1.045^+$ for the 1% level.

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[1] J. NEYMAN & B. TOKARSKA, "Errors of the second kind in testing 'Student's' hypothesis," Am. Stat. Assn. *Jn.*, v. 31, 1936, p. 318–326.

898[K].—P. B. PATNAIK, "The use of mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.

Let $x_1, x_2, \cdots, x_n$ be a random sample of n observations, arranged in ascending order of magnitude, drawn from a normal population with mean μ and variance σ^2. Define the range by $w_n = x_n - x_1$, and if there are m such independent samples with n observations each, the mean range by $\bar{w}_{m,n}$. Further d_n is defined by $E(w_n) = \sigma d_n$. The author approximates the distribution of $\bar{w}_{m,n}$ by means of the first two moments of $\bar{w}_{m,n}$ and the χ^2 distribution with v degrees of freedom and scale factor c. In Table 1, needed for the distribution of $\bar{w}_{m,n}$, are given the values of d_n for $n = 3(1)10$ to 4D, v (the degrees of freedom which are not integers) to 3D, c, the scale factor to 4D, for $m = 1(1)5$, and $n = 3(1)10$. Table 2 gives the comparisons of the exact and approximate percentage points of $u = d_n x/\bar{w}_{m,n}$ for $n = 5, 8, 10$, $m = 1, 3, 5, 10$, $\alpha = .1, .05, .02, .01$, where x is distributed normally with mean zero, and variance σ^2 and α denotes the percentage points. The exact results are from LORD's computations,[1] while the approximate results are based on the author's approximation to the u distribution. Table 3 compares the exact power function of u with the author's approximate Student's non-central t, for $\mu/\sigma = 1, 2, 3$, $\mu \neq 0$, $n = 5, 10$, $m = 3, 5$ for a one-tailed region, $\alpha = .05$, to 3D. For the symmetrical or double-tailed test, comparisons of the power are made for $m = 1$, $n = 10$; $m = 2$, $n = 10$; $m = 4$, $n = 5$; $m = 5$, $n = 3$, and $m = 2$, $n = 5$ usually to 3D. The results are particularly useful in quality control and wherever it is desirable to reduce computation time by use of the mean range in place of the sample standard deviation as an estimator of the population standard deviation.

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[1] E. LORD, "The use of range in place of standard deviation in the t-test," *Biometrika*, v. 34, 1947, p. 41–67; "Power of the modified t-test (u-test) based on the range," *Biometrika*, v. 37, 1950, p. 64–77. [RMT **897**]

899[K].—L. W. POLLAK, assisted by U. N. EGAN, *All Term Guide for Harmonic Analysis and Synthesis Using 3 to 24; 26, 28, 30, 34, 36, 42, 44, 46, 52, 60, 68, 76, 84 and 92 Equidistant Values.* (Department of Industry and Commerce, Meteorological Service, *Geophysical Publications*, v. 2.) Dublin, Stationery Office, 1949, xx, 185 p. 24.5 × 30.2 cm. £2 s2.

The purpose of the present work is to facilitate the work of fitting the Fourier series

$$y = p_0 + p_1 \cos x + p_2 \cos 2x + \cdots + q_1 \sin x + q_2 \sin 2x + \cdots$$

to a set of n equidistant values, where n is any one of the numbers given above in the title.

The tables are supplementary to those published by the author in 1947 and described in *MTAC*, v. 2, p. 306. The tables of the "Schedule," which comprises most of the work (185 p.), give the values to 5D of the functions

$$\cos \frac{2\pi km}{n}, \qquad \sin \frac{2\pi km}{n},$$

where n is any one of the numbers given above in the title, $m < n$, and $k \leq \frac{1}{2}n$.

In a typical entry n is printed in large blackface type, let us say $n = 20$, and for this number there are then 10 tables corresponding to $k = 1$ to $10 = \frac{1}{2}n$. The value of k is printed in smaller blackface type at the top of each table. Two arguments are given, one for m, denoted by i by the author, which ranges over the positive integers, and the other for $i' = km - sn$, where s is chosen so that $km - sn$ is positive and less than n.

Two other entries are included, which refer to tables in the author's *Rechentafeln zur harmonischen Analyse*, Leipzig, 1926, by means of which the multiplication of the harmonic terms is facilitated. The choice of n given above was guided by the choice of values found in the *Rechentafeln*.

The present work is a valuable addition to other similar tables which have been computed and which are listed in *MTAC*, v. 1, p. 193.

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900[K].—E. M. SCHROCK, *Quality Control and Statistical Methods.* New York, Reinhold Publishing Corporation, 1950. xi + 213 p., 17.1 × 24.1 cm. $5.00.

Tables 9.7a, p. 69–71, and 9.7b, p. 73–122, of this book were prepared with the apparent purpose of routinizing the construction of control charts for industrial applications covering most situations which are likely to be encountered in practice, in so far as problems dealing with the per cent defective are concerned.

Table 9.7a labeled, "Table of Width of 3-Sigma Band in Per Cent," gives the amounts to be added to and subtracted from the average per cent defective or process level, $\bar{p}$, in order to obtain 3-sigma upper and lower control limits for sample per cent defectives. Thus Table 9.7a gives values of $3[\bar{p}(1 - \bar{p})/N]^{\frac{1}{2}}$ to 4D, where N is the sample size, for $N = 10(10)100$-

(100)4000(200)5000(500)10000, and $\bar{p}$ = .001, .0025, .005, .01, .015, .02(.01)-.1(.02).2(.05).5. For non-tabulated $\bar{p}$'s and N's, the author states, "although the relationships are curvilinear, straight line interpolation between rows and columns may be used for all practical purposes."

Table 9.7b, Section One, gives 3-sigma upper and lower control limits for the number of defectives to the nearest integer and for the per cent defective to 2D and also gives the expected number of defectives for ranges of $\bar{p}$ and N as above. GRANT[1] gives a comparable table of upper and lower control limits alone. Section II of this table is a rearrangement of the same information for numbers of defectives, giving separately for each N = 100-(100)4000(200)5000(500)10000, the values for the same $\bar{p}$'s as above up to .35. Thus Section One of Table 9.7b with different sample sizes on each page is convenient for testing control with respect to an established or standard process level of per cent defectives, whereas Section II of Table 9.7b "is extended for use in situations where several different quality characteristics, each with a different quality level, will be checked on samples of the same size."

Tables 9.7a and 9.7b occupy some 54 pages of the book and because of the wide ranges of sample sizes and values of per cent defective should be of considerable value as a ready reference to the quality control engineer for industrial problems involving per cent defective. Figures 9.7a and 9.7b are included in the book for obtaining limits for sample sizes other than those given in Tables 9.7a and 9.7b or for values of per cent defective larger than 35%. These figures can also be used for 3-sigma confidence intervals for the per cent defective of the lot or material samples. Examples are given for the use of Tables 9.7a, 9.7b and Figures 9.7a, 9.7b.

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[1] E. L. GRANT, *Statistical Quality Control.* New York, 1946.

901[K].—C. A. WILLIAMS, JR., "On the choice of the number and width of classes for the chi-square test of goodness of fit," Am. Stat. Assn. *Jn.*, v. 45, 1950, p. 77–86.

This paper contains a presentation and discussion, intended mainly for the non-mathematical user of tests of goodness of fit, of results of a paper by MANN and WALD[1] which make it possible to optimize, in a certain sense, the number of classes and the choice of class-boundaries for the chi-square test. Tables are given showing, for the 1% and the 5% significance level and the sample sizes N = 200(50)1000(100)1500, 2000, the optimum number k of classes, as well as some information related to the power of the test. The author suggests that the tabulated values of k may be halved with little loss of power.

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[1] H. B. MANN & A. WALD, "On the choice of the number of class intervals in the application of the chi-square test," *Annals Math. Stat.*, v. 13, 1942, p. 306–317.

902[L].—AKADEMIIA NAUK, SSSR. Matematicheskiĭ Institut im. V. A. Steklova, Institut Mekhaniki, *Tablitsy znacheniĭ Funktsiĭ Besselia ot mnimogo Argumenta*. Edited by I. M. VINOGRADOV & N. G. CHETAEV, Moscow, Leningrad, 1950, xii, 404 p. 39 roubles.

Here we have finally the work which was started in 1930 and ready for publication in 1941 [*MTAC*, v. 3, p. 66 and v. 5, p. 112], after collaboration of various individuals and academies, as set forth in the preface. The world war delayed publication, but after further revision by the present editors, an edition of 2,500 copies was finally printed last year.

After the introductory matter there are two sections (p. 1–201, 203–403) each containing four tables. These tables are of (1) $I_0(x)$ to 8D; (2) $I_1(x)$ to 9D; (3) $2\pi^{-1}K_0(x)$ and (4) $2\pi^{-1}K_1(x)$ to 8S or 8–12D; (5) $I_{\frac{1}{3}}(x)$ and (6) $I_{-\frac{1}{3}}(x)$ to 8S; (7) $K_0(x)$ and (8) $K_1(x)$ to 8S or 8–12D; all for $x = 0(.001)10$, Δ.

The only publications referred to in the literature list are: WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., 1944 and the Russian translation of 1949. GRAY-MATHEWS–MACROBERT, *A Treatise on Bessel Functions*. The date given is 1931, but presumably 1936 was intended. *A Guide to Tables of Bessel Functions* as in *MTAC*, v. 1, no. 7, 1944. FMR, *Index*, 1946. And finally there is a reference to tables of $I_0(x)$ and $I_1(x)$ in BAAS, *Report*, 1893 and 1896, but these tables are attributed to ALDIS, and not to the real author A. LODGE; these 9D tables were for $x = 0(.001)$-5.1, Δ. This mention of the name Aldis, where there was no reason for it, and general considerations, suggest very definitely that other unmentioned important tables were well known to the compilers and computers. Nevertheless, it is undoubtedly true that most of the values given in the volume before us are new.

Let us recall just what was available before, apart from the tables of Lodge. In R. Soc. London, *Proc.*, v. 64, 1899, p. 218–223, ALDIS gave tables of $I_0(x)$, $I_1(x)$, for x = [.1(.1)6; 21D], [6(1)11; 18D]. BAASMTC, *Math. Tables*, v. 6, 1937, contained tables of $I_0(x)$, $I_1(x)$, for x = [0(.001)4; 8D], [4(.001)5; 7D], δ^2 and of $K_0(x)$, $K_1(x)$, for x = [0(.01)2; 8D], [2(.01)4; 9D], [4(.01)5; 10D], δ^2. In the case of I_0, I_1 it was found that the Lodge tables were not dependable in final digits, and hence the calculations were derived by subtabulation in the 21D values of Aldis. In NBSMTP, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, 1943, second edition 1947, there are tables of $I_0(x)$, $I_1(x)$, for x = [0(.01)10; 10D]. These tables occur again in NBSCL, *Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments*, New York, 1950, published almost simultaneously with the volume under review. In this volume for x = [0(.01)10; 10D] are also tables of (3) and (4) $2\pi^{-1}K_0(x)$, $2\pi^{-1}K_1(x)$. The only previous tables of $I_{\pm\frac{1}{3}}(x)$ are those of A. N. DINNIK, 1915 [*MTAC*, v. 1, p. 287], $I_{\frac{1}{3}}(x)$, for x = [0(.1)15; 4–6S]; $I_{-\frac{1}{3}}(x)$, for x = [0(.1)7; 5–6S]. But Dinnik's other tables have been found notoriously unreliable [*MTAC*, v. 2, p. 379]. Finally there is NBSAMS 1, *Tables of the Bessel Functions, $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$*, 1948 [*MTAC*, v. 3, p. 187–188], $K_0(x)$ and $K_1(x)$ for x = [0-(.0001).033(.001)1; 7S], Δ^2.

From these statements the exact relation of the Russian volume to what had appeared before (which excludes NBSCL, 1950) is clear. The only exact duplication is in $I_0(x)$, $I_1(x)$, for $x = 0(.001)4$ in the BAASMTC volume, and in the Lodge $I_1(x)$ table.

Thus, if the tables in this new volume are found to be accurate,[1] a notable addition has been made to the large amount of tabular material dealing with Bessel functions.

NOTE ADDED IN PROOF: A friend (who desires that his name be withheld) has drawn my attention to ten additional errata, each of 3 to 10000 units, in final decimal places:

Function	x	for	read
$I_0(x)$	1.009	175	075
	2.46	8582	7582
$I_1(x)$	4.06	6	3
	4.67	6	3
	4.87	4	1
	4.88	5	2
	5.07	3	0
	5.08	6	3
$I_0(x)$	6.79	9981	1981
	9.73	46664	56664

Because of these, and many other less serious errors already observed these Russian tables of the Academy of Science must be characterized as decidedly unreliable.

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[1] On spot-checking the 9D table of $I_1(x)$ in the work under review with the 10D table of NBSMTP, 1943, one difference of 4 units in the end-figure was found at $x = .5$, and 49 differences of one or two units in end-figures were found for the following values of x: .03, .05, .07, .09, .7, .75, .8, 1.02, 1.03, 1.11, 1.15, 1.16, 1.17, 1.18, 1.25, 1.3, 1.35, 1.36, 1.37, 1.39, 1.5, 1.55, 1.57, 1.59, 1.7, 1.85, 1.87, 1.94, 1.97, 1.98, 2.67, 2.68, 2.69, 2.7, 2.85, 3.03, 3.4, 3.95, 4.35, 4.55, 4.65, 4.71, 4.72, 4.76, 4.77, 4.79, 4.85, 5.05.

At every one of these 50 places, except for $x = .5$ (because of the very weakly printed left part of Lodge's ending 4, it was copied in the Russian table as a 1), the values of the Russian table were identical with the values of the Lodge (1893) 9D table of $I_1(x)$. Hence, it seems evident that the Lodge table was copied completely without any checking whatever. Since the British Committee has told us that the 9D tables of $I_0(x)$ and $I_1(x)$ "were subject to errors of two or three units in the last decimal, so that rounding off would not yield 8-figure values of the standard of accuracy set for the Committee's tables," we can assert that the Russian tables are probably incorrect at each of the places indicated above. Furthermore, spot-checking the 8D table of $I_0(x)$, differences were found at $x = .07$, 3.3, 4.4, where there was evident rounding off of Lodge values. Since correct 8 D values are given in the BAASMTC tables it is definitely suggested that the Russians also copied their table of $I_0(x)$, up to $x = 5.1$, rounded off from Lodge, and that they did not have the $I_0(x)$ table in BAASMTC. Since the Lodge table of $I_1(x)$ abridged to interval .01 was reprinted in GRAY & MATHEWS, 1895, 1922, 1936, all the probable errors noted above are also to be found in G. & M. and G. M. & Mac R.

In a brief spot checking beyond $x = 5.1$ up to $x = 7$, unit differences in end figures were found at $x = 5.3$, 5.45, 5.57, 6.15, 6.35, 6.45, 6.75.

903[L].—P. A. CARRUS & C. G. TREUENFELS, "Tables of roots of incomplete integrals of associated Legendre functions of fractional orders," *Jn. Math. Phys.*, v. 29, 1951, p. 292–299.

$P_\mu{}^1(\cos\theta)$ is the associated Legendre function of order one and fractional index μ. Tabulated are the zeros of

$$P_\mu{}^1(\cos\theta) = 0, \qquad \frac{d}{d\theta} P_\nu{}^1(\cos\theta) = 0$$

treated as a function of the index.

Table I gives the first 50 values of μ for $\theta = 90°(5°)175°$, while Table II gives the first 50 values of ν for $\theta = 90°(5°)130°$. The k-th root is denoted by μ_k, ν_k. In both tables entries are mostly to 5S. Values were derived from power series, for the smaller roots, and from asymptotic expansions due to PAL[1] for the larger roots. Convergence difficulties prevented extension of Table II beyond 130°.

No indication as to accuracy of the tables is given, though it is stated that interpolation of the roots θ-wise is practicable with five point Lagrangean interpolation. The reviewer has spot checked the tables by differencing and in a number of cases found that third differences are not smooth, indicating that some errors are present. For example, in Table I, entry for $k = 1$, $\theta = 130°$ is undoubtedly in error. The entry 1.3001 should probably read 1.2975. In Table II, there is a typographical error for the entry $k = 49$, $\theta = 90°$: *for* 88.000 *read* 98.000. In Table II, for k large, third differences are not smooth. The reviewer has not differenced all the entries. In absence of error, interpolation with four point interpolation should be sufficient to insure full accuracy of Tables I and II.

In Tables III and IV, values of the integral $\int_{x_0}^{0}\{P_n{}^1(x)\}^2dx$ are presented. Here $x = \cos\theta$, n stands for μ or ν, and tabulations are given for each value of the index found in Tables I and II. However, in Table III, $\theta = 90°(5°)145°$, in Table IV, $\theta = 90°(5°)130°$. Entries are to 4S and 5S. Finally, Table V gives values of $\int_0^1\{P_n(x)\}^2dx$ to 5D for $n = 0(0.05)1.0$ together with second differences.

All tables are new, though PAL[1] had previously determined some μ and ν values of the functions considered for various orders and values of θ in the first quadrant.

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[1] B. PAL, "On the numerical calculation of the roots of the equations $P_n{}^m(\mu) = 0$ and $\frac{d}{d\mu}P_n{}^m(\mu) = 0$ regarded as equations in n," Calcutta Math. Soc., *Bull.*, v. 9, 1918, p. 85–95 and v. 10, 1919, p. 187–194.

904[L].—A. H. CARTER & O. A. WILLIAMS, JR., "A new expansion for the velocity potential of a piston source," Acoustical Soc. Amer. *Jn.*, v. 23, 1951, p. 179–184.

A small table is given (p. 182) of the values of the hypergeometric function $F(-(m-1), -m; 1; b^2)$ which the authors denote by f_{2m}. These functions are actually polynomials of degree $m-1$ in b^2. Thus

$$f_4 = 1 + 2b^2, \qquad f_6 = 1 + 6b^2 + 3b^4.$$

The 6D tables give $f_{2m}(b)$ for

$$m = 1(1)8 \qquad \text{and} \qquad b = 0(.1)1.$$

There is also a table of the coefficients of f_{2m} for $m = 1(1)8$. These functions occur as factors of coefficients in the expansion of the velocity potential of a piston source in terms of HANKEL functions.

D. H. L.

905[L, P].—R. A. CLARK, "On the theory of thin elastic toroidal shells," *Jn. Math. Physics*, v. 29, 1950, p. 146–178.

Table I, p. 156, gives 4D values of

$$\int_0^\phi \{\sin t/(1 + \lambda \sin t)\}^{\frac{1}{2}} dt$$

for $\lambda = 0(.1).5$ and $\phi = 0^0(2^0)90^0$.

Table II, p. 167, gives 3D values of the real and imaginary parts of $T(x)$ and $dT(x)/dx$ for $x = 0(.05)5$ where $T(x)$ is that solution of

$$\frac{d^2T}{dx^2} - ixT = 1$$

which is asymptotic to i/x for large (positive) x. In terms of LOMMEL's function

$$T(x) = -\tfrac{2}{3}(ix)^{\frac{1}{2}} S_{0,\frac{1}{3}}[\tfrac{2}{3}(ix)^{\frac{3}{2}}].$$

A. E.

906[L].—R. A. FISHER, "Gene frequencies in a cline determined by selection and diffusion," *Biometrics*, v. 6, 1950, p. 353–361.

This paper contains a table of the solution $q(x)$ of the non-linear differential equation $q'' = 4xq(1 - q)$ subject to the two point boundary condition $q(0) = \frac{1}{2}$, $q(\infty) = 0$. The table gives $q(x)$ to 8D for $x = 0(.02)5.7$. Beyond this range $q(x) < 10^{-8}$. The table was computed on the EDSAC [*MTAC*, v. 4, p. 61–65].

D. H. L.

907[L].—CARL-ERIK FRÖBERG, "On determination of proton-proton interaction from scattering experiments," *Arkiv för Fysik*, v. 3, 1951, p. 1–23.

The COULOMB wave equation containing an additional nuclear interaction potential $V(r)$ is

$$d^2y/dr^2 + [k^2 - 2\alpha k r^{-1} - l(l + 1)r^{-2}]\, y = -\, V(r)y. \tag{1}$$

If $V(r) \equiv 0$ a system of two linearly independent solutions can be given by u_k and v_k, such that $u_k(0) = 0$, $u_k \sim \sin(kr + \eta)$ as $r \to \infty$ and $v_k \sim \cos(kr + \eta)$ as $r \to \infty$, with $\eta = -\alpha \log(2kr) - \frac{1}{2}l\pi + \arg\Gamma(i\alpha + l + 1)$. If $V(r)$ is different from zero an additional phase-shift $\delta(k)$ appears in the asymptotic expression for the regular solution y of (1), so that $y(0) = 0$ and $y \sim \sin[kr + \eta + \delta(k)]$ as $r \to \infty$. The author discusses in this paper the determination of $V(r)$ if $\delta(k)$ has been determined experimentally for every positive k. The relation between V and δ is given by

$$V(r) = -\frac{2}{\pi}(2\alpha k)^2 \int_0^\infty \alpha^{-3} \sin\delta(\alpha)\varphi(\alpha, \lambda)\, d\alpha \tag{2}$$

where

$$\varphi(\alpha, \lambda) = \left(\frac{d}{d\rho} u_k(\rho) v_k(\rho)\right)_{\rho=\lambda/(2\alpha)}$$

and

$$\lambda = 2\alpha k r.$$

To determine $V(r)$, a knowledge of the function $\varphi(\alpha, \lambda)$ is necessary. Therefore the author gives asymptotic expansions for u and v. The main part of the paper consists of two numerical tables. The first table is for $v_1(\rho)$ and $v_1'(\rho)$ with $l = 0$ for $\alpha = .01$ and $0 \leqslant \rho \leqslant 6$, $\alpha = .02, .03$ and $0 \leqslant \rho \leqslant 4$, $\alpha = .04$ and $0 \leqslant \rho \leqslant 2$, $\alpha = .06(.02).12(.04).2(.05).3(.1)1$ and $0 \leqslant \rho \leqslant 1$: the intervals in ρ vary. The second table is for $\varphi(\alpha, \lambda)$, α ranging from .01 to ∞ and λ from .012 to .120, with various intervals. The last section of the paper gives the computation of $V(r)$ for two cases, one where $\sin \delta = \text{const. } \alpha^2$ and $\alpha = .02$, the other where $\sin \delta$ is approximately constant (.79) and $.006 < \alpha < .02$. Three smaller numerical tables are included. The tables were computed on the Swedish automatic computing machine BARK in Stockholm [*MTAC*, v. 5, p. 29–34].

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908[L].—R. GRAMMEL, "Tafeln der verallgemeinerten Kreisfunktionen Sin(4)v, Cos(4)v, Sin(6)v, Cos(6)v, Sin(8)v, Cos(8)v," *Ing.-Arch.*, v. 18, 1950, p. 251–254.

The functions $x = \cos(n)v$ and $y = \sin(n)v$ have been defined in an earlier paper[1] as the inversion of the integrals

$$v = \int_x^1 f_n(t)\,dt = \int_0^v f_n(t)\,dt \tag{1}$$

where

$$f_n(t) = (1 - t^n)^{(1-n)/n}.$$

For $n = 2$ these functions reduce to trigonometric functions. For any even integer n, they provide a parametric representation of the curve

$$x^n + y^n = 1, \tag{2}$$

and they have certain periodic and symmetry properties with half-period

$$\pi_n = \int_0^1 f_n(t)\,dt.$$

In the present paper 6D numerical tables of $\cos(n)v$ and $\sin(n)v$ are given for $n = 4, 6, 8$ and $v = 0(.01)1(.1)4$ and for $2v/\pi_n = 0(.05)1$. The earlier paper[1] gives 3D values of π_n for $n = 2(1)10, \infty$, and the present paper 6D values of $\frac{1}{2}\pi_n$ for $n = 4, 6, 8$. For $n > 2$ one has $\pi < \pi_n \leq 4$.

The infinite series of the earlier paper[1] have been used for the computation of the functions for $0 < v \leq \frac{1}{4}\pi_n$, symmetry and periodic properties combined with interpolation, for $v > \frac{1}{4}\pi_n$. The original computations were carried to 7 or 8D, and the author states that all six decimals printed are correct.

Differencing of the tables was used as a check, the relation (2) was checked for every fifth entry, and for $n = 4$ the connection with Jacobian elliptic functions was used to check the tables.

The computations were carried out by F. JINDRA.

A. E.

[1] R. GRAMMEL, "Eine Verallgemeinerung der Kreis- und Hyperbelfunktionen." *Ing.-Arch.*, v. 16, 1948, p. 188–200.

909[L].—E. KREYSZIG, "Über den allgemeinen Integralsinus Si(z, α)," *Acta Math.*, v. 85, 1951, p. 117–181.

The function

$$\text{(1)} \qquad \mathrm{Si}\,(z, \mu) = \int_0^z t^{-\mu} \sin t \, dt$$

was introduced by A. WALTHER.[1] It is related to the incomplete gamma function in the same manner as the ordinary sine integral, Si (z, 1), is related to the exponential integral function. Together with the generalized cosine integral,

$$\text{(2)} \qquad \mathrm{Ci}\,(z, \mu) = \int_0^z t^{-\mu} \cos t \, dt,$$

the integrals investigated in this paper account for the incomplete gamma function, sine-, cosine-, and exponential integrals, and Fresnel's integrals.

The author investigates Si and Ci in the complex domain. $z = x + iy$, $\mu = \alpha + i\beta$, $0 < \alpha < 2$ in (1), and $0 < \alpha < 1$ in (2). In the theoretical part he discusses power series expansions in ascending powers of z, the connection with other functions including confluent hypergeometric functions, asymptotic expansions as $z \to \infty$, and a convergent expansion in incomplete gamma functions. He investigates the zeros and the product representation of the functions in question and comments on the numerical computation of the zeros.

Table 1 (p. 155–6) gives 3D values of Si (x, α) for $x = 0(.2)4(.5)20$, $\alpha = .25(.25)1.75$.

Table 2 (p. 157–163) gives 2D (sometimes 3S) values of the real and imaginary parts of Si($x + iy$, α) for $x = 0(1)20$, $y = 0(1)5$, $\alpha = .25(.25)1.75$.

Table 3 (p. 164) gives 2D values for the real and imaginary parts of the first three zeros of Si (z, α) for $\alpha = 0(.25)1.75$.

Table 4 (p. 165) gives 5D values of Si (∞, α) for $\alpha = .05(.05)1.95$, 1.99, 1.995, 1.999, 2, and 5D values of Ci (∞, α) for $\alpha = .05(.05)1$.

Table 5 (p. 166) gives 8D values of

$$P(n + 1, z) = \int_0^z e^{-t} t^n \, dt$$

for $z = 1, -1, i$ (in the last case real and imaginary parts) and $n = 0(1)10$.

There are relief diagrams and charts illustrating the functions tabulated, other diagrams; and an extensive bibliography of papers on both theory and applications, and also of numerical tables of the sine integral and FRESNEL'S integral.

The tables are said to be accurate to within a unit of the last decimal. Although the relation

$$\mathrm{Ci}\,(z, \mu) = z^{-\mu} \sin z - \mu\, \mathrm{Si}\,(z, \mu + 1), \qquad 0 < \alpha < 1,$$

allows one to compute Ci from Si, tables of Ci (z, μ) would be a useful addition. Together with the tables of the present paper they could serve to compute the incomplete gamma function in the complex domain.

A. E.

[1] A. WALTHER, "Anschauliches zur Gibbsschen Erscheinung und zur Annäherung durch arithmetische Mittel," *Math. Zeit.*, v. 42, 1937, p. 355–364.

910[L].—L. M. MILNE-THOMSON, *Jacobian Elliptic Function Tables.* Dover Publications Inc., 1950. xii, 132 p., 12.7 × 18.9 cm. $2.45.

The purpose of this book is explained in the opening sentence of its preface. "The widespread belief that calculations involving elliptic functions are difficult is due not to the nature of the calculations themselves but to the lack of suitable numerical tables wherewith to perform them." In fact, a glance at A. FLETCHER's "Guide to Tables of Elliptic Functions" [*MTAC*, v. 3, p. 229–281] suffices to show that while there is a considerable body of numerical material for the use of a professional computer, and also for some special purposes, yet there is no handy and easily available book of tables which could be used by the general practitioner of the various sciences as easily, and as safely as, say, a table of logarithms. The present book attempts to fill this gap.

The standard notations are used, except that the parameter $m = k^2$ is used rather than the modulus k, $\mathrm{sn}\,(u|m)$ is written for $\mathrm{sn}\,(u, k)$, and similarly for cn, dn. The *complementary parameter* is $m_1 = 1 - m$.

The first 39 pages are devoted to the definition of Jacobian elliptic functions and a description of their principal properties, examples showing the use of the tables, complete elliptic integrals, Fourier series and power series expansions of elliptic functions, periods, zeros, poles and residues, formulas for special values of the argument, change of argument (including double and half arguments and addition theorems), change of parameter (transformations), approximations, complex arguments, differentiation, WEIERSTRASS's $\wp$-function, integrals of Jacobian elliptic functions, elliptic integrals and the zeta function, conformal mapping, factorisation of cubic and quartic polynomials, and application to the pendulum problem. In brief, these 39 pages give practically all the information constantly needed when using elliptic functions.

The following tables and graphs form the main body of the book under review.

$\mathrm{sn}\,(u|m)$. Graphs for $m = 0, \frac{1}{2}, 1$ and $0 \leq u \leq 2\pi$ (p. 41). 5D tables with first differences for $m = 0(.1).5$ and $u = 0(.01)2$, $m = .6, .7, .8$ and $u = 0(.01)2.5$, $m = .9$ and $u = 0(.01)3$, and $m = 1$ and $u = 0(.01)3(.1)6.5$, with values of the quarter-period K at the foot of each column (p. 42–61).

$\mathrm{cn}\,(u|m)$. Graph (p. 63) and tables (p. 64–83) in the same range as for sn, except that for technical reasons the values for $m = 1$ (which are common to cn and dn) and $u \geq 3.5$ are distributed over various pages of the dn-table.

$\mathrm{dn}\,(u|m)$. Graph (p. 85) and tables (p. 86–105) again for the same values of m and u as for sn, except that $\mathrm{dn}\,(u|0) = 1$ for all u and need not be given, and the column thus saved is used to accommodate tables for $\mathrm{cn}\,(10u|1) = \mathrm{dn}\,(10u|1)$ up to the point ($u = 1.29$) where these functions vanish (to 5D accuracy).

Complete elliptic integrals and the nome (p. 106–109). 7D tables of K, K', E, E', and 8D tables of $q = \exp(-\pi K'/K)$, and $q_1 = \exp(-\pi K/K')$ for $m = 0(.01).5$, with values of m_1 printed on the right margin and an arrangement like that of trigonometric tables to enable the user to read off values for $.5 \leq m \leq 1$.

$Z(u)$. 7D table (p. 112–123) for $m = 0(.1).6$ and $u = 0(.01)2$, $m = .7, .8$ and $u = 0(.01)2.5$, and $m = .9, 1$ and $u = 0(.01)3$, all with Δ''.

The book is very well printed, and the legibility of the tables is excellent, except that the table for $Z(u)$ seems to have been reproduced photographically from the *Proceedings* of the Royal Society of Edinburgh (v. 52, 1931, p. 239–250) and the considerable reduction in size makes for small figures and a somewhat crowded page.

A. E.

911[L, V].—M. MUNK & G. RAWLING, *Tables of Chaplygin Functions*. Naval Ordnance Laboratory Memorandum 10816, 57 pp. (1950). [Available to government contractors only.]

This paper consists of three tables of certain functions designated by the generic title of CHAPLYGIN functions, which arise as particular solutions of Chaplygin's differential equation. They are closely related to the physical plane stream function or to the LEGENDRE reciprocal potential of the hodograph plane in the computation of plane two-dimensional compressible flows. The adiabatic index is taken as 1.4.

This differential equation is:

$$\tau(\tau - 1)\frac{d^2Z_m}{d\tau^2} + [(m + 1 + \beta)\tau - (m + 1)]\frac{dZ_m}{d\tau} - \tfrac{1}{2}\beta m(m - 1)Z_m = 0$$

in which $\beta = (\gamma - 1)^{-1} = 5/2$, and τ is a dimensionless speed variable related to the Mach number by $\tau = M^2(2\beta + M^2)^{-1}$ and to the reduced velocity w by $\tau = w^2$.

Particular solutions tabulated are $Z_m = F(a_m, b_m; c_m, \tau)$, where $a_m + b_m = m + \beta$, $a_mb_m = -\frac{1}{2}\beta m(m - 1)$, $c_m = m + 1$; and $\bar{Z}_{-m}$, where $\bar{Z}_{-m}$ denotes a second independent solution; *not negative values of the parameter m*. This second solution $\bar{Z}_{-m}$ contains a logarithmic term and has been normalized to approach unity as w tends to zero.

The other functions were computed from the relations

$$L_m = w^mZ_m \qquad \bar{L}_{-m} = w^m\bar{Z}_{-m},$$
$$\psi_m = \rho(mL_m - m^{-1}wL'_m), \qquad \bar{\psi}_{-m} = \rho(m\bar{L}_{-m} - m^{-1}w\bar{L}'_{-m})$$

where $\rho = (1 - w^2)^{5/2}$, and a prime denotes d/dw.

Values of the functions for $w = 6^{-\frac{1}{2}}$, sonic speed, were found by interpolation using NEWTON's backward formula.

The tables give 6S values of the functions

$$Z_m, Z'_m, \bar{Z}_{-m}, \bar{Z}'_{-m}, L_m, L'_m, \bar{L}_{-m}, \bar{L}'_{-m}, \psi_m, \bar{\psi}_{-m}$$

for $m = 2(1)10$ and $w = .01(.01).5$ and for $w = 6^{-\frac{1}{2}}$ to which latter value of w a separate table is devoted.

The tables were produced by the computational facilities of the Naval Ordnance Laboratory.

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912[L].—NBSCL, *Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments*. Columbia Univ. Press, 1950. xl, 427 p. 20 × 26.5 cm. \$7.50.

This volume maintains the high standards of the earlier tables, prepared by the computation group under A. N. LOWAN. It presents values of the real and imaginary parts of the second solutions Y_0 and Y_1 of the Bessel equation, according to the magnitude ρ of the argument $\rho e^{i\varphi}$ over the range [0(0.01)10; 10D] and according to the phase angle φ over the range [0°(5°)90°]. Auxiliary tables of $Y_0 - (2/\pi)J_0 \log \rho$ and $Y_1 - (2/\pi)J_1 \log \rho + (2/\pi\rho)e^{-i\varphi}$, over the range [$\rho = 0(0.01)0.50$; $\varphi = 0°$-(5°)90°; 10D], are provided, to facilitate interpolation for small values of ρ. In addition, the position of some of the zeros of $Y_0 = Y_1'$ and of Y_1 and Y_0' off the real axis are given, together with values of Y_0 and Y_1 at some of these points. A table of 5-point Lagrange interpolation coefficients is also provided, for use with the tables. In the introduction are given contour plots for the general behavior of Y_0 and Y_1 in the complex plane; and a discussion of the properties of the functions and the method of calculation of the tables.

Together with the earlier volume,[1] giving corresponding values of J_0 and J_1 for complex values of the argument, these tables will allow many acoustical and microwave problems to be computed for complex boundary conditions on cylindrical surfaces. It is particularly satisfactory to have both functions J of the first kind and functions Y of the second kind, for the same values of the order and argument. Many applications of Bessel functions require knowledge of both functions and so many tables of these functions, heretofore published, have lost a great part of their value because only the J's were tabulated, not the Y's. In the present case the computing group is to be congratulated in resisting the temptation to compute J_2, J_3 etc., and in turning to the more difficult but more useful Y_0 and Y_1.

Perhaps some of our larger digital machines will eventually take the place of tables, making up in speed of output what they inevitably lack in judgment; they show little sign as yet of their ability to produce further tables, as carefully planned, edited, and produced as the tables under review. Perhaps, in the distant future, when we have a large digital computer in each laboratory, we will not need tables of the sort reviewed here (though the present reviewer thinks we always will). At any rate, for a long time to come, these tables and the many others produced by Lowan's able group will be standard and much used tools of physicists and engineers.

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[1] NBSCL, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments.* 2nd ed., New York, 1947 [*MTAC*, v. 3, p. 25].

913[L].—A. T. Price, "Electromagnetic induction in a semi-infinite conductor with a plane boundary," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 385–410.

Table I, p. 398, gives 3 and 4D values of Y_1, Y_2, Z_1, Z_2 for $\xi, \eta = 0, .5, 1(1)5, 10$ where

$$Y_n = \int_0^\infty f_n e^{-u\xi} \cos u\eta \, du, \qquad Z_n = \int_0^\infty f_n e^{-u\xi} \sin u\eta \, du,$$

$$f_1 = 1 - u\sqrt{2}\{(1 + u^4)^{\frac{1}{2}} - u^2\}^{\frac{1}{2}}, \qquad f_2 = 2u^2 - u\sqrt{2}\{(1 + u^4)^{\frac{1}{2}} + u^2\}^{\frac{1}{2}}.$$

A. E.

914[L].—P. Rhodes, "Fermi-Dirac functions of integral order," R. Soc. London, *Proc.*, v. 204A, 1950, p. 396–405.

The functions

$$F_n(\eta) = \int_0^\infty x^n(e^{x-\eta} + 1)^{-1}dx$$

arise in the theoretical treatment of assemblies of particles subject to Fermi-Dirac statistics. For the cases $n = \frac{1}{2}, \frac{3}{2}$ numerical tables are known.[1]

In the present paper n is a positive integer. A series convergent for negative η, a relation between $F_n(\eta)$ and $F_n(-\eta)$, and a polynomial approximation for large (positive) η are derived.

Table 1 (p. 404) gives the approximating polynomial for $n = 1(1)4$.

Table 2 (p. 404) gives 7D values of $F_n(\eta)/n!$ for $n = 1(1)4$, $\eta = -4(.1)0$.

A. E.

[1] J. McDougall & E. C. Stoner, "The computation of Fermi-Dirac functions," R. Soc. London, *Phil. Trans.*, v. 237A, 1938, p. 67–104.

C. Truesdell, "On a function which occurs in the theory of the structure of polymers," *Annals Math.*, s. 2, v. 46, 1945, p. 150 [*MTAC*, v. 1, p. 445].

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **883** (van der Pol & Speziali), **889** (DeLury), **894** (Hartley & Pearson), **895** (Howell), **902** (Akademiâ Nauk SSSR).

190.—A. J. C. Cunningham, *Binomial Factorisations.* V. 2, London, 1924.

Page 189, line 6
for 39249421 *read* 30249421 = 1291·23431

D. H. L.

191.—A. J. C. Cunningham & H. J. Woodall, *Factorisation of* $(y^n \mp 1)$. London, 1925.

Page 17, $n = 66$, *delete* the factor 3
$n = 77$, *insert* the factor 463.

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192.—J. P. Stanley & M. V. Wilkes, *Table of the Reciprocal of the Gamma Function for Complex Argument.* Computation Centre, Univ. of Toronto, 1950 [*MTAC*, v. 5, p. 25–26].

The following errors occurred in the preparation for press. A number of illegible entries are also noted. Thanks are due to Miss C. M. Munford of the University Mathematical Laboratory, Cambridge, and to Dr. van Wijngaarden of the Mathematical Centre of Amsterdam, who helped in the discovery of these errors.