spaced values of $C_1$ and $C_2$. $\varphi$ is the velocity potential and $\psi$ is the streamline function and thus the two sets of curves intersect at right angles. The plot divides the given area into curvilinear rectangles, which in the case of incompressible flows are equisided. For the compressible case, Bernoulli's equation gives a relation between the sides of these curvilinear rectangles. The authors give a historical discussion of graphical methods in this problem and describe, with examples, a process of starting with the incompressible case and arriving at the desired result. For this purpose a number of mechanical aids are described. A four legged caliper is described which will give the proper size for the other side of the rectangle, when one side is given and a cam device for the same purpose is described. To permit a ready modification of the plot, the latter is realized by wires and pegs which have two holes bored at right angles through their sides. Lord Rayleigh has pointed out that the incompressible case can be plotted very effectively by using a double mesh of wires. For this purpose each peg has an additional pair of perpendicular holes making an angle of 45° with the first pair and the double mesh involves both the sides and the diagonals of the smaller squares. The authors also discuss the representation of the equations of fluid flow by means of a resistance network. Here the resistances are to be adjusted until the proper relation between potential and current flow is obtained.

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**22.** W. W. SOROKA, "Experimental aids in engineering design analysis," *Mech. Engineering*, v. 71, 1949, p. 831–837.

This is a survey article with a bibliography, covering certain analogue techniques for the solutions of partial and ordinary differential equations and algebraic equations.

**23.** M. A. WOODBURY, "Inverting modified matrices," Princeton Univ., Statistical Research Group, *Memorandum Report* 42, Princeton, New Jersey. 4 hectographed leaves.

This is a resumé, including the work of SHERMAN and MORRISON, of formulas for inverting matrices which have been obtained from a matrix with a known inverse by bordering or by modifying either an individual element or the elements in one row or column or both or in a number of rows and columns. "None of the proofs are difficult but considerable exploration was necessary to discover useful forms."

F. J. M.

## NOTES

**128.** GAUSS TO GERLING ON RELAXATION.—In his recent book[7] on matrices ZURMÜHL traces the relaxation method[2] of solving linear equations back to DEDEKIND'S report of GAUSS'S lectures.[7] It is believed by some computers, however, that Gauss's method is a different one—namely, the related method given by SEIDEL.[3] In the interest of giving Gauss his proper credit as a proponent of relaxation, the following translation of a letter by Gauss[4] is offered. Moreover, in this same letter Gauss introduces a useful trick,[7] now generally forgotten, which seems to improve the convergence of the relaxation process in a wide class of problems. The letter to C. L. GERLING dated at Göttingen, 26 Dec. 1823 is as follows.[1]

My letter got to the post office too late and was brought back to me. I am therefore opening it to add the practical directions on elimination. However, this method has many small special advantages which can be learned only by use.

As an example I take your measurements for Orber-Reisig.[5]

[The angle measurements communicated by Gerling (taken from a sheet found with Gauss's papers) were as follows, where 1 denotes Berger Warte, 2 Johannisberg, 3 Taufstein, and 4 Milseburg:

| Repetitions | Angle |
|---|---|
| 13 | 1.2 = 26°44′ 7″.423 |
| 28 | 1.3 = 77 57 53.107 |
| 26 | 1.4 = 136 21 13.481 |
| 50 | 2.3 = 51 13 46.600 |
| 6 | 2.4 = 109 37 1.833 |
| 78 | 3.4 = 58 23 18.161.] |

I first make

$$[\text{the angle toward}]\ 1 = 0,$$

whence from 1.3

$$3 = 77^\circ 57' 53''.107$$

(I prefer this, because 1.3 has more weight than 1.2); then from

$$\left.\begin{array}{r|l|l} 13 & 1.2 & 2 = 26^\circ 44'\ 7''.423 \\ 50 & 2.3 & 2 = \qquad\quad 6.507 \end{array}\right\} 2 = 26^\circ 44'\ 6''.696;$$

finally from

$$\left.\begin{array}{r|l|l} 26 & 1.4 & 4 = 136^\circ 21' 13''.481 \\ 6 & 2.4 & 4 = \qquad\quad 8.529 \\ 78 & 3.4 & 4 = \qquad\quad 11.268 \end{array}\right\} 4 = 136^\circ 21' 11''.641.$$

In order to improve the approximation still further, I find, from

$$\left.\begin{array}{r|l|l} 13 & 1.2 & 1 = -\,0''.727 \\ 28 & 1.3 & 1 = \quad 0 \\ 26 & 1.4 & 1 = -\,1.840 \end{array}\right\} 1 = -\,0''.855.$$

Since any common alteration of all directions is allowed when dealing only with relative position, I alter all four by $+0''.855$ and set

$$\begin{array}{l} 1 = \quad 0^\circ\ 0'\ 0''.000 + a \\ 2 = \ \ 26\ 44\ \ 7.551 + b \\ 3 = \ \ 77\ 57\ 53.962 + c \\ 4 = 136\ 21\ 12.496 + d. \end{array}$$

In the indirect[6] process it is very advantageous to ascribe a variation to *each* direction. You can easily convince yourself of this, if you compute the same example without this trick,[7] so that you lose the great convenience of always having as a control the sum of the absolute terms = 0. Now I form the four normal equations by the following scheme (in actual application, and if the terms are more numerous, I separate the positive and negative terms): [The constants are expressed in units of the third decimal place.]

| | | | |
|---|---|---|---|
| $ab - 1664$ | $ba + 1664$ | $ca + 23940$ | $da - 25610$ |
| $ac - 23940$ | $bc + 9450$ | $cb - 9450$ | $db + 18672$ |
| $ad + 25610$ | $bd - 18672$ | $cd - 29094$ | $dc + 29094$. |

The normal equations are therefore:

$$\begin{aligned} 0 &= +\quad 6 + 67a - 13b - 28c - 26d \\ 0 &= -\quad 7558 - 13a + 69b - 50c - 6d \\ 0 &= -\ 14604 - 28a - 50b + 156c - 78d \\ 0 &= +\ 22156 - 26a - 6b - 78c + 110d; \\ &\text{sum} = 0. \end{aligned}$$

In order to eliminate indirectly,[6] I note that, if 3 of the quantities $a, b, c, d$ are set equal to 0, the fourth gets the largest value when $d$ is chosen as the fourth.[8] Naturally every quantity must be determined from its own equation, and hence $d$ from the fourth. I therefore set $d = -201$ and substitute this value. The absolute terms then become: +5232, −6352, +1074, +46; the other terms remain the same.

Now I let $b$ take its turn, find $b = +92$, substitute, and find the absolute terms: +4036, −4, −3526, −506. And thus I continue until there is nothing more to correct. Of this whole calculation I actually write only the following table:[9]

| | $d=-201$ | $b=+92$ | $a=-60$ | $c=+12$ | $a=+5$ | $b=-2$ | $a=-1$ |
|---|---|---|---|---|---|---|---|
| + 6 | +5232 | +4036 | + 16 | −320 | + 15 | +41 | −26 |
| − 7558 | −6352 | − 4 | + 776 | +176 | +111 | −27 | −14 |
| −14604 | +1074 | −3526 | −1846 | + 26 | −114 | −14 | +14 |
| +22156 | + 46 | − 506 | +1054 | +118 | − 12 | 0 | +26. |

Insofar as I carry the calculation only to the nearest 2000-th of a second, I see that now there is nothing more to correct. I therefore collect

| $a=-60$ | $b=+92$ | $c=+12$ | $d=-201$ |
|---|---|---|---|
| + 5 | − 2 | | |
| − 1 | | | |
| −56 | +90 | +12 | −201 |

and add the common correction +56, whence

$$a = 0 \qquad b = +146 \qquad c = +68 \qquad d = -145;$$

therefore the values [of the directions] are

| | |
|---|---|
| 1 | 0° 0′ 0″.000 |
| 2 | 26 44 7.697 |
| 3 | 77 57 54.030 |
| 4 | 136 21 12.351. |

Almost every evening I make a new edition of the tableau, wherever there is easy improvement.[10] Against the monotony of the surveying business, this is always a pleasant entertainment; one can also see immediately whether anything doubtful has crept in, what still remains to be desired, etc. I recommend this method to you for imitation. You will hardly ever again eliminate directly, at least not when you have more than 2 unknowns. The indirect procedure can be done while half asleep, or while thinking about other things.[11]

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[1] The preparation of this translation was sponsored (in part) by the Office of Naval Research. The author wishes to acknowledge the assistance of Drs. FRITZ JOHN and OLGA TAUSSKY-TODD.

[2] See R. V. SOUTHWELL, *Relaxation Methods in Engineering Science*. Oxford, 1940, and L. FOX, "A short account of relaxation methods," *Quart. Jour. Mech. Appl. Math.*, v. 1, 1948, p. 253–280.

[3] Ludwig Seidel, "Über ein Verfahren, die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, sowie lineare Gleichungen überhaupt, durch successive Annäherung aufzulösen," *Akad. Wiss., Munich, mat.-nat. Abt. Abhandlungen*, v. 11, 1874, p. 81–108. Although Seidel's process is frequently called the Gauss-Seidel process, I know of no place where Gauss mentions it.

[4] The letter is in Gauss's *Werke*, v. 9, 1903, p. 278–281. It is probably the letter referred to in the footnote, p. 257, of E. T. WHITTAKER and G. ROBINSON, *The Calculus of Observations*, 1st edit., London and Glasgow, 1924, as a source of their example of relaxation (p. 257–8).

[5] Words within brackets are translations of inserts by L. KRÜGER, who prepared volume 9 of Gauss's *Werke*.

[6] 'Indirect elimination' was Gauss's term for his iterative process of solving the normal equations. It later came to denote any iterative process for solving linear equations.

[7] The symmetric treatment of the unknowns is an essential idea in this and other letters. Here Gauss mentions only its advantage as a device which sets up a column-sum check to detect errors. In later letters (cf., e.g., Gauss to Gerling, 19 January 1840, *Werke*, vol. 9, p. 250–3) Gauss is convinced that the symmetric treatment of all unknowns yields normal equations whose iterative solution converges significantly faster.

The trick is later described by: Christian Ludwig Gerling (recipient of the letter), *Die Ausgleichungsrechnung der practischen Geometrie*. Hamburg and Gotha, 1843 (p. 157–8, p. 163, p. 386, p. 390); by R. Dedekind, "Gauss in seiner Vorlesung über die Methode der kleinsten Quadrate," *Festschrift zur Feier des 150-jährigen Bestehen der königlichen Gesellschaft der Wissenschaften zu Göttingen*. Berlin, 1901 (pp. 45–59) and *Gesammelte Mathematische Werke*. V. 2, 1931, p. 293–306; and by R. Zurmühl, *Matrizen*, Berlin, 1950, p. 280–282. (Zurmühl is wrong, however, in stating that the trick will improve the convergence for all badly conditioned systems of equations.)

For some discussion of when and why the trick may be expected to improve the convergence of iterative processes for solving linear equations, see George E. Forsythe and THEODORE S. MOTZKIN, "An extension of Gauss's transformation for improving the condition of systems of linear equations," multilithed typescript at the National Bureau of Standards, Los Angeles.

[8] Gauss is here using a method which relaxers recommend: liquidating "that residual . . . which requires the largest 'displacement' " (Fox, *op. cit.*, p. 256).

[9] Study of the table shows the algorithm to be precisely the relaxation method described by Fox (*op. cit.*, p. 255–6). In comparing Gauss's and Fox's presentations, we note that Gauss uses the method mentioned in note 8, and that he "liquidates the residuals" only approximately at each stage, as recommended by Fox (*op. cit.*, p. 257–8) to save unnecessary arithmetic. Gauss does not, however, propose "under-relaxation" or "over-relaxation," as Fox does. On the other hand, Gauss's trick mentioned in note 7 is not mentioned by Fox, although it is extremely helpful in many common problems.

[10] Gauss wrote, "Fast jeden Abend mache ich eine neue Auflage des Tableaus, wo immer leicht nachzuhelfen ist." It is not clear what he meant.

[11] Here one must bear in mind Gauss's gift for mental arithmetic!

## QUERIES

**49.** FIRST USE OF THE TERM HAVERSINE AND THE FIRST TABLE OF HAVERSINES.—The editors of the great Oxford *New English Dictionary* (*NED*) endeavoured to bring together quotations exhibiting the first use