5-Point

$r = (hf_x' - \mu\delta_0 + \frac{1}{6}\mu\delta_0^3)/\Delta,$

$s = \frac{1}{2}\mu\delta_0^3/\Delta,$

$t = \frac{1}{6}\delta_0^4/\Delta,\ u = v = 0,$

where $\Delta = \delta_0^2 - \frac{1}{12}\delta_0^4.$

6-Point

$r = (hf_x' - \mu\delta_0 + \frac{1}{6}\mu\delta_0^3 - \frac{1}{30}\mu\delta_0^5)/\Delta,$

$s = (\frac{1}{2}\mu\delta_0^3 - \frac{1}{8}\mu\delta_0^5)/\Delta,$

$t = \frac{1}{6}\delta_0^4/\Delta,$

$u = \frac{1}{24}\mu\delta_0^5/\Delta,\ v = 0,$

where $\Delta = \delta_0^2 - \frac{1}{12}\delta_0^4.$

7-Point

$r = (hf_x' - \mu\delta_0 + \frac{1}{6}\mu\delta_0^3 - \frac{1}{30}\mu\delta_0^5)/\Delta,$

$s = (\frac{1}{2}\mu\delta_0^3 - \frac{1}{8}\mu\delta_0^5)/\Delta,$

$t = (\frac{1}{6}\delta_0^4 - \frac{1}{36}\delta_0^6)/\Delta,$

$u = \frac{1}{24}\mu\delta_0^5/\Delta,$

$v = \frac{1}{120}\delta_0^6/\Delta,$

where $\Delta = \delta_0^2 - \frac{1}{12}\delta_0^4 + \frac{1}{90}\delta_0^6.$

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This work was sponsored in part by the Office of Air Research.

[1] H. E. SALZER, "Table of coefficients for obtaining the first derivative without differences," NBS, *Applied Mathematics Series No. 2*, 1948.

[2] D. GIBB, *A Course in Interpolation and Numerical Integration for the Mathematical Laboratory*. London, Bell, 1915.

[3] H. E. SALZER, "A new formula for inverse interpolation," Amer. Math. Soc., *Bull.*, v. 50, 1944, p. 513–516

RECENT MATHEMATICAL TABLES

915[F].—K. FRÜCHTL, "Statistische Untersuchung über die Verteilung von Primzahl-Zwillingen," Öster. Akad. Wiss., *math.-nat. Kl.*, *Anz.*, 1950, p. 226–232.

Information on the distribution of twin primes $(p, p + 2)$ and quadruplets $(p, p + 2, p + 6, p + 8)$ is given for the first 1020000 natural numbers. The information is based on the old table of CHERNAC.[1] The main table gives the number of prime pairs in each of the 1020 chileads $1000n < p < 1000(n + 1)$ for $n = 0(1)1019$. There is no chilead devoid of prime pairs and only three ($n = 688,851,927$) with but a single prime pair. The rows of this table are added to give the number of prime pairs in each of the 102 myriads $10000n < p < 10000(n + 1)$ for $n = 0(1)101$. These frequencies are, in turn, added by tens to give a 10 entry table for each interval of 100000. The grand total gives 8168 prime pairs $< 10^6$.

As for the quadruplets, there are enumerated in each of the 10 intervals of 100000; the total number is 166. On p. 232 the largest quadruplet, $p + 8$, is given for each of the 172 cases below 1020000.

Previous tables concerning twin primes are mentioned in *MTAC*, v. 4, p. 84. The reviewer has not as yet attempted to reconcile discrepancies between these and the tables under review. No doubt some of these are due to errata in Chernac.

D. H. L.

[1] L. CHERNAC, *Cribrum Arithmeticum* etc., Deventer, 1811.

916[I, O].—R. E. BEARD, "Some notes on approximate product integration," Inst. of Actuaries, *Jn.*, v. 73, part II, 1948, 356–403.

The author studies approximation formulas of the form:

$$\int_a^b f(x)\phi(x)dx = [\sum_{r=1}^{n} a_r f(x_r)]\cdot \int_a^b \phi(x)dx + R_n \int_a^b \phi(x)dx,$$

$a \leq x_1, x_2, \cdots, x_n \leq b$, where the a_r, x_r are independent of $f(x)$ and $\int_a^b \phi(x)dx \neq 0$. $\phi(x)$ is generally a known, tabulated function and $f(x)$ is a complicated or empirically given function. One of the more important applications of such formulas is the evaluation of continuous single premiums in the field of life and disability contingencies where $\phi(x) = v^x$ is the discount factor and $f(x)$ is derived from a mortality or disability table for one or more lives.

The paper is divided into two parts: a theoretical part in which formulas are developed for the determination of a_r, x_r and the remainder term R_n, and a part containing tables for the a_r and x_r based on the formulas of the first part for low orders (i.e. small n).

In the theoretical part the following cases are considered:

(1) Neither the a_r nor the x_r are given; R_n depends on $f^{(2n)}(x)$.

(2) $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$; R_n depends on $f^{(n+1)}(x)$.

(3) $n = 3$, $a_1 = a_3$; R_3 depends on $f^{(5)}(x)$.

(4) The x_r are given, R_n depends on $f^{(n)}(x)$.

If $m_i = \int_a^b x^i\phi(x)dx / \int_a^b \phi(x)dx$ are the moments of $\phi(x)$ which are assumed to exist and to be known, the x_r in the cases (1) and (2) are found to be the roots of a polynomial of degree n whose coefficients are determinants the elements of which are multiples of the moments m_i. In the cases (1) and (4) the a_r are expressed as quotients of two determinants whose elements depend in a simple manner on the x_r and the m_i. Simpson's rule, Weddle's and Hardy's formulas are examples of formulas belonging to case (4).

In deriving his formulas for the remainder term R_n the author makes an application of the mean value theorem which is correct only if $\phi(x)$ is non-negative and $0 \leq a < b$ and arrives thereby at an expression for R_n of the form $f^{(k)}(\xi_1)\frac{m_r}{k!} - f^{(k)}(\xi_2)h(x_1, \cdots, x_r)$ where k has the values indicated above, $h(x_1, \cdots, x_r)$ is a function of the x_r only which in many cases can be expressed as a simple function of the m_i, and $0 < \xi_1, \xi_2 < b$. He then simplifies this expression by stating that, for practical purposes, ξ_1 and ξ_2 may be replaced by a common value with $0 < \xi < b$ which is evidently correct only in particular cases. The remainder terms given by the author are therefore useless in many cases for the purpose of measuring the accuracy of the approximation formulas. It is, in any event, doubtful whether remainder terms expressed by means of derivatives of high order are useful for empirically given functions $f(x)$.

The author proceeds to develop from the general formulas the formulas corresponding to low values of n. In many cases he first makes a linear transformation of the variable x to a variable X so as to make $m_1 = 0$ and $m_2 = 1$ and then expresses the remaining moments by means of Pearson's β_i. In those formulas where β_i for $i \geq 3$ appear he makes the further assumption that the β_i for $i \geq 3$ can be expressed as functions of β_1 and β_2 in the same manner as for Pearsonian distribution functions.

In another special application the author considers special functions $\phi(x)$ such as $\phi(x) = 1$, $e^{-\lambda x}$, $\exp(-\{(x-m)/c\}^2)$, $(1-x)^{m_1}(1+x)^{m_2}$.

No attempt has been made to classify the various formulas by some optimum properties or by the magnitude of the remainder terms, the only test being a comparison of the true values of some actuarial functions with the results of various approximation formulas.

In the numerical part of the paper the following tables are given:

Table 1: Solutions to 6D for X_1, X_2, a_1, a_2 for the two point formulas of case (1) corresponding to $\beta_1 = 0(.1)3.0(.1)3.0$.

Table 2: Solutions to 6D for X_1, X_2, X_3, a_1, a_2, a_3 for the three point formula of case (1) corresponding to $\beta_1 = 0(.2)2$ and $\beta_2 = 2(.5)6$.

Table 3: Solutions to 6D for X_1, X_2, X_3 for the three point formula of case (2) corresponding to $\beta_1 = 0(.01).50$.

Table 4: Solutions to 6D for X_1, X_2, X_3, $a_1 = a_3 = \dfrac{1}{2+A}$, $a_2 = \dfrac{A}{2+A}$ for case (3) corresponding to $\beta_1 = 0(.2)2$ and $\beta_2 = 2.0(.5)6.0$.

Table 5: Moments m_1, m_2, m_3, σ to 3D and $\sqrt{\beta_1}$ to 5D for the continuous function $(1+i)^{-x}$ for $i = .02(.01).06$ and $n = 5(5)60$.

Table 6: A similar table for discrete moments.

Table 7: Solutions to 5D for a_1, a_2, a_3 for the three point formula of case (4), where $x_1 = 0$, $x_2 = \dfrac{m}{2}$, $x_m = m$ and $\phi(x) = (1+i)^{-x}$ for $m = 5(5)50$ and $i = .02(.01).06$.

Table 8: The values to 3D of $\bar{e}_x$, $\bar{e}_{xx}$, $\bar{e}_{xxx}$, $\bar{e}_{xxxx}$, $\bar{e}_{xxxxx}$ for the A 1924–29 ultimate mortality tables for $x = 15(1)80$.

Table 9: σ to 3D, $(\beta_1)^{\frac{1}{2}}$ to 5D, β_2 to 4D for the function $(\mu l)_{xx\cdots}$ for 1 to 5 lives of the A 1924–29 ultimate mortality table.

There are also a few auxiliary tables for the purpose of evaluating remainder terms.

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917[I].—LEROY F. MEYERS & ARTHUR SARD, "Best approximate integration formulas," *Jn. Math. Phys.*, v. 29, 1950, p. 118–123.

For each $A = c_0x(0) + c_1x(1) + \cdots + c_mx(m)$ that is an approximation to $\int_0^m x(t)\,dt$ which is exact whenever $x(t)$ is a polynomial in t of degree n, $m \geq 1$, $n \geq 0$, there is a kernel function $k(t)$ such that, when $x(t)$ is of class C^{n+1},

$$R[x] \equiv \int_0^m x(t)\,dt - A = \int_0^m x^{(n+1)}(t)k(t)\,dt.$$

The kernel function is defined explicitly by

$$k(t') = R[\psi_{t'}] = -R[\phi_{t'}]$$

where

$$\psi_{t'} = \psi_{t'}(t) = \begin{cases} 0 \\ (t - t')^n/n!, \end{cases} \qquad \phi_{t'} = \phi_{t'}(t) = \begin{cases} (t - t')^n/n! & \text{if } t \leq t' \\ 0 & \text{if } t > t'. \end{cases}$$

By Schwarz's inequality

$$|R[x]| \leq \left(\int_0^m k^2(t)\,dt \right)^{\frac{1}{2}} \left(\int^m x^{(n+1)}(t)^2 dt \right)^{\frac{1}{2}}.$$

The *best* approximation A, for given m and n, is defined as that which minimizes $J \equiv \int_0^m k^2(t)\,dt$.

In the present paper, the authors give the best integration formulas for $n = 1$, $m = 1(1)20$; $n = 2$, $m = 2(1)12$; $n = 3$, $m = 2(1)9$. For these values of n and m, they tabulate the exact values of $c_0, c_1, \cdots, c_m$ and J. In an earlier paper by SARD[1] the best integration formulas are given for $n = 0$, all m, and $n = 1, 2, 3$, $m \leq 6$.

The authors derive some fundamental algebraic relationships between the c_i's and J. Then for the case $n = 1$, recursive relations are derived which afford a complete characterization of the best integration formulas for any m. Finally, the authors give some conjectures about the convergence of the coefficients, some of which are true for $n = 0$ and 1, but which are open questions for $n \geq 2$.

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NBSCL

[1] A. SARD, "Best approximate integral formulas; best approximation formulas," *Amer. Jn. Math.*, v. 71, 1949, p. 80–91.

918[I].—LEROY F. MEYERS & ARTHUR SARD, "Best interpolation formulas," *Jn. Math. Phys.*, v. 29, 1950, p. 198–206.

The tabular values $x(0)$, $x(1)$, $x(2)$ of a function $x(t)$ being given, the problem discussed here is the approximation of $x(u)$ by an expression of the form $A = a_0x(0) + a_1x(1) + \cdots + a_mx(m)$, where the value of m and the coefficients $a_0 = a_0(u), a_1 = a_1(u), \cdots, a_m = a_m(u)$ are to be determined. If A is an exact approximation whenever $x(t)$ is a polynomial of degree n, there is a kernel function $k(t, u)$, such that when $x(t)$ is of class C^{n+1},

$$R[x] = x(u) - A = \int_K x^{(n+1)}(t)k(t, u)\,dt,$$

where K is the smallest interval containing u and those values of $0, 1, \cdots, m$ for which the corresponding $a_0, a_1, \cdots, a_m$ are not zero. For each u, $k(t, u)$ is a broken polynomial in t consisting of at most $m + 3$ arcs, which is defined explicitly by

$$k(t', u) = R[\psi_{t'}] = -R[\phi_{t'}],$$

where

$$\psi_{t'} = \psi_{t'}(t) = \begin{cases} 0 \\ (t - t')^n/n!, \end{cases} \qquad \phi_{t'} = \phi_{t'}(t) = \begin{cases} (t - t')^n/n! & \text{if } t \leq t', \\ 0 & \text{if } t > t'. \end{cases}$$

By Schwarz's inequality

$$|R[x]| \leq M\left[\int_K x^{(n+1)}(t)^2 dt/|K|\right]^{\frac{1}{2}}$$

where the modulus M is defined by $M = M(u) = [|K| \int_K k(t, u)^2 dt]^{\frac{1}{2}}$. For given m, n, u, that A is called *best* which minimizes the modulus M, and it is denoted by $A_{m,n,u}$. The authors report that for $n = 0$, all m, and $n = 1$, $m = 1(1)4$, $n = 2$, $m = 2$, the conventional polynomial interpolation is best, but not for $n = 2$, $m = 3, 4, 5$.

The determination of $A_{m,n,u}$ involves the prior determination of a number of approximations $B_{m,n,u}$ (for different values of m and u) where $B_{m,n,u}$ is that A which minimizes $J \equiv \int_{min(u,0)}^{max(u,m)} k(t, u)^2 dt$. The modulus of $B_{m,n,u}$ is denoted by $M_{m,n,u}$.

The authors tabulate the auxiliary function $\mu = M^2_{m,2,u}/\theta(u)$ where $\theta(u) = (u - [u])^2(1 - u + [u])^2/120$ for $m = 2(1)5$, $u = 0(.1)2$; $m = 3(1)5$, $u = 2.1(.1)2.5$; 2D. The table of μ enables one (a) to compare moduli, (b) to identify $A_{m+j,2,u}$ with $B_{m,2,u}$, $0 \leq 5 - m$, where each different value of m corresponds to a different range of u, and (c) to find the proper argument by translation in using the main table..

The principal table in the article is the collection of formulas for $B_{m,2,u}$, $0 \leq u \leq [(m + 1)/2]$, $m = 2, 3, 4$, and $M^2_{m,2,u}$, $u \leq [(m + 1)/2]$, $m = 2, 3, 4, 5$. In the $B_{m,2,u}$, the coefficients of $x_0, x_1, \cdots, x_m$ are given as exact polynomials in u. The $M^2_{m,2,u}$ are expressed as either exact polynomials in u, or as exact polynomials in u multiplied by $\theta(u)$. The expressions for $B_{m,2,u}$ and $M^2_{m,2,u}$ are different for u lying within different ranges.

The rest of the paper is concerned with the derivation of those formulas and their transformation under a linear transformation of the t-axis, $t^* = bt + c$, $b \neq 0$.

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NBSCL

919[J].—T. M. Cherry. "Summation of slowly convergent series," Camb. Phil. Soc., *Proc.*, v. 46, 1950, p. 436–449.

This paper studies two transformations of the remainder that are helpful in numerical summations.

The power series considered is $\sum_0^\infty C_r t^r$, and, in the remainder $\sum_n^\infty C_r t^r$, C_r is written as a product $c_r f(r)$, where it is supposed that $f(z)$ has an asymptotic expansion

$$f(z) \sim Az^\alpha + A_1 z^{\alpha_1} + A_2 z^{\alpha_2} + \cdots$$

with appropriate conditions. Then, if

$$c_n + c_{n+1}t + c_{n+2}t^2 + \cdots = \phi_n(t)$$

and

$$\vartheta = t\frac{d}{dt}, \quad D_t = \frac{d}{dt}, \quad D_n = \frac{d}{dn}, \quad \Delta f(n) = f(n+1) - f(n),$$

the transformations are

$$\sum_{r=0}^{\infty} c_{n+r}t^{n+r}f(n+r) = t^n \sum_{r=0}^{p-1} \vartheta^r\phi_n(t)\cdot D_n{}^r f(n)/r! + R_{n,p}$$

$$\sum_{r=0}^{\infty} c_{n+r}t^{n+r}f(n+r) = t^n \sum_{r=0}^{p-1} t^r D_t{}^r\phi_n(t)\cdot \Delta^r f(n)/r! + S_{n,p}$$

where $R_{n,p}$, $S_{n,p}$ are "error-terms."

The assumption of an asymptotic expansion for $f(z)$ is needed for the theoretical development, although its coefficients do not occur in the numerical applications. On the other hand, although $\phi_n(t)$ is not, in theory, very restricted, it must be easily accessible numerically which means, virtually, that it must be chosen in such a way that a closed expression for it is known.

A thorough discussion of the error terms $R_{n,p}$, $S_{n,p}$ is given, and upper bounds for them are obtained. In practice these upper bounds exceed the true error considerably, so that a closer rough estimate has been sought which is more useful practically, although it gives the order of the error only; for the numerical examples tested the true error rarely exceeds the estimate, and never by a factor exceeding about 1.2.

Two particular families of functions $\phi_n(t)$ are studied in considerable detail, and are used in the numerical examples. These are binomial remainder functions.

The first family is defined by

$$B_{0,n}(t) = B_0(t) = 1 + t + t^2 + \cdots = (1-t)^{-1}$$

$$B_{1,n}(t) = \frac{1}{n} + \frac{t}{n+1} + \frac{t^2}{n+2} + \cdots$$
$$= t^{-n}\left\{-\ln(1-t) - t - \tfrac{1}{2}t^2 - \cdots - \frac{t^{n-1}}{n-1}\right\}$$

$$B_{2,n}(t) = \frac{1}{n(n+1)} + \frac{t}{(n+1)(n+2)} + \cdots = \frac{1}{t}\left\{\frac{1}{n} - (1-t)B_{1,n}(t)\right\}$$

$$B_{3,n}(t) = \frac{1}{n(n+1)(n+2)} + \frac{t}{(n+1)(n+2)(n+3)} + \cdots$$
$$= \frac{1}{2t}\left\{\frac{1}{n(n+1)} - (1-t)B_{2,n}(t)\right\}$$

and so on.

The second family, with

$$b_n = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} = \frac{2n-1}{2n}b_{n-1}$$

is given by

$$\mathbf{B}_{\frac{1}{2},n}(t) = b_n + b_{n+1}t + b_{n+2}t^2 + \cdots$$
$$= t^{-n}\{(1-t)^{-\frac{1}{2}} - 1 - \tfrac{1}{2}t - \cdots - b_{n-1}t^{n-1}\}$$

$$\mathbf{B}_{\frac{3}{2},n}(t) = \frac{b_n}{n+1} + \frac{b_{n+1}t}{n+2} + \cdots = \frac{2}{t}\{b_n - (1-t)\mathbf{B}_{\frac{1}{2},n}(t)\}$$

$$\mathbf{B}_{\frac{5}{2},n}(t) = \frac{b_n}{(n+1)(n+2)} + \frac{b_{n+1}t}{(n+2)(n+3)} + \cdots$$
$$= \frac{2}{3t}\left\{\frac{b_n}{n+1} - (1-t)\mathbf{B}_{\frac{3}{2},n}(t)\right\}$$

and so on.

Tables are given of $\mathbf{B}_{\frac{1}{2},10}(t) = \sum_{s=0}^{\infty} b_{10+s}t^s = R(r,\theta) + iI(r,\theta)$ to 4 decimals for $r = 0.7(0.05)1$, $\theta = 0°(5°)90°$, and have been used for the numerical examples, which concern the Kapteyn series

$$\sum_{r=1}^{\infty} x^r J_r(ry)$$

and are very fully considered.

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920[K, L].—B. V. GNEDENKO, *Kurs Teorii Veroiatnostei* [A Course in the Theory of Probabilities]. Moscow and Leningrad, 1950, 388 p.

On p. 372–385 there are tables of

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x = [0(.01)3.99; 4D]$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int^x e^{-z^2/2} dz, \quad x = 0(.01)2(.02)3(.2)4(.5)5;\ \text{4D up to 2.98, 5D to 8D thereafter}$$

$$P_k(a) = \frac{a^k e^{-a}}{k!}, \quad a = .1(.1)1(1)9, \quad k = [0(1)27; 6D]$$

$$\sum_{m=0}^{k} \frac{a^m(e^{-a})}{m!}, \quad a = .1(.1)1(1)3, \ k = [0(1)15; 6D]$$

$$P(x) = \frac{1}{2^{(k-2)/2}\Gamma\left(\frac{k}{2}\right)} \int_x^{\infty} z^{k-1} e^{-z^2/2} dz, \quad x = 1(1)30, \ k = [1(1)29; 4D]$$

$$S(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{((n-1)\pi)^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{x} \left(1 + \frac{z^2}{n-1}\right)^{-n/2} dz$$

$$n = 2(1)20, \infty, \ x = [0(.1)6; 3D], \ x = \infty; 5D$$

$$K(x) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2x^2}, \quad x = .28(.01)2.50(.05)3; \text{ mostly 6D}$$

$$\ln \frac{1-\beta}{\alpha} \quad \alpha = .001, .01(.01).05, .1, .15$$
$$\beta = [.001, .01(.01).05, .1, .15; 3D]$$

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921[L].—AIRCRAFT RADIATION SYSTEMS LABORATORY, "Tables of modified cosine-integral," Stanford Research Institute, 1951, viii + 56 p.

These tables contain 6D values of

$$\overline{\mathrm{Ci}}\,(x) = \int_0^x \frac{1-\cos t}{t}\,dt$$

for $x = 0(.001)10(.01)50$. The computation was carried out, largely on IBM machines, by the Telecomputing Corporation of Burbank, California.

The introduction, by C. T. TAI, discusses the connection of $\overline{\mathrm{Ci}}$ with the cosine-integral function and the application of the tables, and describes the computation and preparation of the tables. A bibliography is appended.

The work was sponsored by the U. S. Air Force.

A. E.

922[L, M, Q, S].—M. P. BARNETT & C. A. COULSON, "The evaluation of integrals occurring in the theory of molecular structure. Parts I and II," Roy. Soc. London, *Philos. Trans.*, v. 243A, 1951, p. 221–249.

Table 1, p. 233. Modified Bessel functions of the first kind, $I_{n+\frac{1}{2}}(x)$, to 7S for $n = -1(1)4$ and $x = .5(.5)10$.

Table 2, p. 233. Modified Bessel functions of the third kind, $K_{n+\frac{1}{2}}(x)$, to 7S for $n = -1(1)4$ and $x = .5(.5)10(1)25$.

These tables are used for the numerical computation of the functions ζ defined by the expansion

$$r^{m-1}e^{-r} = (t\tau)^{-\frac{1}{2}} \sum_{n=0}^{\infty} (2n+1)P_n(\cos\theta)\zeta_{m,n}(1, t; \tau)$$

where $r^2 = t^2 + \tau^2 - 2t\tau\cos\theta$. With these functions the authors form

$$\mathbf{Z}_{m,n,l+\frac{1}{2}}(\kappa, \tau) = \int_0^\infty e^{-\kappa t}\zeta_{m,n}(1, t; \tau)t^{l+\frac{1}{2}}dt$$

and discuss the computation of **Z** by both numerical integration and analytical methods.

In the memoir it is shown that a large number of integrals occurring both in nuclear physics and astrophysics can be reduced to known integrals and to **Z** integrals. Formulas are listed for more than 180 integrals.

A. E.

923[L].—C. L. BARTBERGER, "The magnetic field of a plane circular loop," *Jn. Appl. Phys.*, v. 21, 1950, p. 1108–1114.

The integrals

$$I_1 = \pi^{-1} \int_0^\pi (1 - b \cos \theta)^{-\frac{3}{2}} d\theta$$

$$I_2 = \pi^{-1} \int_0^\pi (1 - b \cos \theta)^{-\frac{3}{2}} \cos \theta d\theta$$

are expressed in terms of complete elliptic integrals. Series expansions are also given in ascending powers of b and $1 - b$.

Table I (p. 1110–1111) gives 6D values of I_1, and table II (p. 1111–1112) 6D values of I_2, for $b = 0(.001).809$.

Table III (p. 1113–1114) gives 6D values of I_1, I_2, $I_1 - I_2$, $(1 - b)I_1$, $(1 - b)I_2$ for $b = .8(.001)1$, and table IV (p. 1114) 6D values of the same functions as table III for $b = .995(.0001)1$.

A. E.

924[L].—A. Fletcher, "Tables of two integrals and of Spielrein's inductance function," *Quart. Jn. Mech. Appl. Math.*, v. 4, 1951, p. 223–235.

The tables (p. 229–232) are of

$$I = \int_\alpha^1 (K - E) dk, \qquad J = \int_\alpha^1 (K - E) k^{-3} dk$$

and $-16\pi(I - \alpha^3 J)/[3(1 - \alpha)^2]$.

Values are given to 10D, 10D and 6D respectively. The range of α is 0(.01)1. Half a dozen small auxiliary tables are also given.

925[L].—Harvard University, Computation Laboratory, *Annals*, v. 14: *Tables of the Bessel Functions of the First Kind of Orders Seventy-Nine through One Hundred Thirty-Five.* Cambridge, Mass., Harvard University Press, 1951, viii, 614 p. 19.5 × 26.7 cm. $8.00.

This is the twelfth and final volume of the monumental set of Tables of Bessel Functions of the First Order, published by Harvard during the past five years—six volumes in 1947, three in 1948, two in 1949 and one in 1951. The tabular parts of the volumes fill 7652 pages. The previous 11 volumes have been reviewed in *MTAC:* v. 2, p. 261–262, 344; v. 3, p. 102, 185–186, 367, 474–475; v. 4, p. 22, 92. Roughly speaking we now have here 10D tables of all $J_n(x)$, for $x = 0(.01)100$, when $n = 0(1)111$; and for $x = 0(.1)$-100, when n has any positive integral value > 111; for $n > 135$ the values of $J_n(x)$ are always less than 10^{-10}. In addition to what is thus stated, for $n = 0(1)3$, $x = [0(.001)25(.01)100; 18D]$; for $n = 4(1)15$, $x = [0(.001)$-$25(.01)100; 10D]$. Detailed information concerning interpolation in the whole range is given in *Annals*, v. 3 and 5; for 10D interpolation the work is not excessive.

Further, in the present volume we have 10D tables of $J_n(n)$ for $n = 0(1)$-100. Zero values are not given, since the values found by the Computation Laboratory were presented to the Royal Society Committee for use in connection with their second volume of Bessel function tables. For $n \geq 92$, $x \leq 100$, $J_n(x) \neq 0$. For $J_n(n)$, in the Harvard range, we had earlier: Meissel (1891), $n = 20$ to 20D; Meissel (1895), $n = 1(1)24$ to 18D; Airey (1916),

$n = 1(1)50(5)100$ to 6D; WATSON (1922), $n = 1(1)50$ to 7D; and HAYASHI (1930), $n = 2$ to 101D, $n = 10$ to 61D, $n = 20$ to 41D, $n = 30$ to 35D, $n = 40$ to 35D, $n = 50$ to 30D, $n = 100$ to 18D. Hence most of the values in this special Harvard table are new.

In the recent Russian table of FADDEEVA and GAVURIN, RMT **852**, the argument extends to 124.9, at interval .1, so that some 6D of $J_n(x)$ for all orders, $n = 0(1)120$ supplement values given in the Harvard tables. So also for 5D zeros < 125, of $J_n(x)$; the last zero is for $J_{115}(x)$.

The remarkable Automatic Sequence-Controlled Calculator on which these tables were computed carried the values of $J_n(x)$, for $n = 0(1)3$ to 23D, and for $n > 3$ to not less than 13D and most of the time much more than this; its electromagnetic typewriters also wrote out, for checking purposes, 10 differences in every case, and also finally produced the 18D or 10D copy which could be sent directly to the printer for offset reproduction. The computation of these tables was only a tiny fraction of the work achieved in the ASCC since its activities began in 1945, and have continued to the present, 24 hours a day, 7 days a week.

Only one tabular slip has ever been found in the published volumes, $J_3(72.10)$ [*MTAC*, v. 3, p. 41], but this slip was due to some reproduction difficulty, and not to any error in computation or in automatic mechanical checking. In the last line of the second page of the "Preface," of the volume under review, *for* $J(x)$, $n = 0(1)120$, $x = [0(.01)14.99; 8D]$, *read* $J_n(x)$, $n = 0(1)120$, $x = [0(.1)124.9; 6D]$; $n = 0(1)13$, $x = [0(.01)14.99; 8D]$.

In the first page of the Preface, line -2, *for* Claire, *read* Clare.

These Harvard Bessel Function tables, with most of the values new, constitute an outstanding contribution to scientific research.

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926[L, S].—HELMAR KRUPP, "Bestimmung der allgemeinen Lösung der Schrödinger-Gleichung für Coulomb-Potential," Akad. Wiss. Leipzig, *mat.-phys. Kl., Berichte*, v. 97, 1950, no. 8, p. 1–28.

The Schrödinger equation for the Coulomb potential is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + 2\left[-\frac{1}{2n^2} + \frac{1}{r} - \frac{l(l+1)}{2r^2}\right]R = 0, \qquad l = 0, 1, 2, \cdots.$$

Two solutions are written in the form

$$_jR(n, l, r) = \frac{(2r)^l}{(2l+1)!}\, e^{-r/n}\, {}_jM(a, b, x), \qquad j = 1, 2,$$

where $a = l + 1 - n$, $b = 2l + 2$, $x = 2r/n$.

With the abbreviations

$$(a)_0 = 1,\ (a)_m = a(a+1)\cdots(a+m-1) \text{ for } m = 1, 2, \cdots$$
$$A = \Psi(-a) \text{ if } a = 0, -1, -2, \cdots$$
$$A = \Psi(a-1) \text{ if } a \neq b-1, b-2, \cdots$$

the definitions of the M are

$$ {}_1M(a, b, x) = \sum_{m=0}^{\infty} \frac{(a)_m x^m}{(b)_m m!} $$

$$ \pi\, {}_2M(a, b, x) = [\ln x + C - \Psi(b - 1) + A]\, {}_1M(a, b, x) + \sum_{m=1}^{\infty} \frac{(a)_m x^m}{(b)_m m!} \sum_{k=0}^{m-1} \left(\frac{1}{a + k} - \frac{1}{b + k} - \frac{1}{1 + k} \right) + \sum_{m=0}^{b-2} \frac{(-)^{m-b}}{m!\Gamma(a)} \Gamma(a - b + m + 1) b!(b - m - 2)!x^{m+1-b}. $$

Tables are given for ${}_1R$, d_1R/dr, ${}_2R$, d_2R/dr for $0 \leq x \leq 15$ (the interval in most cases is .5), for $l = 0$ and $n = \frac{1}{2}\left(\frac{1}{2}\right)\frac{7}{2}$, $l = 1$ and $n = \frac{3}{2}\left(\frac{1}{2}\right)\frac{7}{2}$, $l = 2$ and $n = \frac{5}{2}\left(\frac{1}{2}\right)\frac{7}{2}$. Graphs of these functions are added. A misprint occurs on page 14, line 3 from the bottom and on page 15, line 6; in both places $\binom{-\kappa + n + l + 1}{\kappa}$ should be replaced by $\binom{n + l}{\kappa}$. However the formulas to which the author refers as the source of computation do not contain this misprint.

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927[L].—U. E. KRUSE & N. F. RAMSEY, "The integral $\int_0^\infty y^3 \exp(-y^2 + ix/y)dy$," *Jn. Math. Phys.*, v. 30, 1951, p. 40–43.

In several problems of theoretical physics there arise integrals which can be reduced to the real part, $I(x)$, or the imaginary part, $K(x)$, of the integral mentioned in the title. Table 2 of this paper gives 5D values of both these functions for $x = 0(.1).4(.2)8(.5)20$.

Convergent expansions, useful for $x < 3$, have been given by ZAHN,[1] and Table 1 of the present paper gives numerical values, to 7–9S, of the coefficients up to that of x^{14}. For larger x it is more convenient to use asymptotic series developed by LAPORTE,[2] and 6D values of the first six coefficients in these series are also given in the present paper. [The author remarks that in TORREY's paper[3] the recurrence formula for the coefficients contains a misprint, but Torrey's numerical values are in agreement with the author's.]

Table 2 was computed from these expansions, and V. E. CULLER assisted in the computation.

A. E.

[1] C. T. ZAHN, "Absorption coefficients for thermal neutrons," *Phys. Rev.*, v. 52, 1937, p. 67–71.

[2] O. LAPORTE, "Absorption coefficients for thermal neutrons," *Phys. Rev.*, v. 52, 1937, p. 72–74.

[3] H. C. TORREY, "Notes on intensities of radio frequency spectra," *Phys. Rev.*, v. 59, 1941, p. 293–299.

928[L, S].—N. METROPOLIS & J. R. REITZ, "Solutions of the Fermi-Thomas-Dirac equation," *Jn. Chem. Phys.*, v. 19, 1951, p. 555–573.

Solutions ψ are given of the equation

$$\frac{d^2\psi}{dx^2} = x(\epsilon + \psi^{\frac{1}{2}}x^{-\frac{1}{2}})^3 \qquad (\epsilon^3 = 3\cdot 2^{-5}\cdot \pi^{-2}Z^{-2})$$

in terms of the variable $w = (2x)^{\frac{1}{2}}$. The tables are at intervals of .08 and extend until ψ becomes negative. The parameter Z takes on the 24 values

$$Z = 6\ (4)\ 14,\ 16,\ 18\ (4)\ 26,\ 29\ (4)\ 81,\ 84\ (4)\ 92$$

and there are 8 different initial slopes ψ'. Actually 2ψ is tabulated to 5D. The calculations were done on the ENIAC.

D. H. L.

929[L].—L. PRANDTL, with the assistance of F. VANDREY, "Flieszgesetze normalzäher Stoffe im Rohr," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 169–174.

Table 1. 4S table of

$$\varphi(a) = \tfrac{1}{2}\cosh a - \frac{1}{2a}\sinh a + \frac{1}{a^2}(\cosh a - 1)$$

for $a = 0(.1)5(.2)10$.

Table 2. 3D table of $\varphi(a\xi)/\varphi(a)$ for $\xi = 0(.2).6(.1)1$ and $a = 1(1)10(2)14$. Some of the values were obtained by interpolation: these are put in parentheses.

A. E.

930[L].—S. SILVER & W. K. SAUNDERS, "The radiation from a transverse rectangular slot in a circular cylinder," *Jn. Appl. Phys.*, v. 21, 1950, 745–749.

Table I gives values of ka, n, and ϕ_0 for which $H_0^{(2)\prime}(ka)/H_n^{(2)\prime}(ka) \leq .0001$ and $(n\phi_0)^{-1}\sin n\phi_0 \geq .9$.

Table II gives 4D values of

$$\left| \sum_{n=0}^{\infty} \frac{\epsilon_n i^n \cos n\phi}{\sin\theta H_n^{(2)\prime}(ka\sin\theta)} \Big/ \sum_{n=0}^{\infty} \frac{\epsilon_n i^n}{H_n^{(2)\prime}(ka)} \right|$$

for $ka = .8$, $\theta = 10°(10°)90°$, $\phi = 0°(10°)180°$. Here $\epsilon_0 = 1$, $\epsilon_n = 2$ if $n > 0$.

Table III is similar to table II except that $ka = 2.5$.

A. E.

931[L].—E. WOLF, "Light distribution near focus in an error-free diffraction image," Roy. Soc. London, *Proc.*, v. 204A, 1951, 533–548.

In the course of the work the function

$$Q_{2m}(v) = \sum_{s=0}^{2m} (-1)^s [J_s(v)J_{2m-s}(v) + J_{s+1}(v)J_{2m+1-s}(v)]$$

is introduced, where the J are Bessel functions of the first kind.

Table 1 (p. 541) gives 5D values of $Q_{2m}(v)$ for $v = 0(1)15$ and $m = 0(1)$ $M(v)$ where

$$M(v) = v + 1 \text{ for } v \leq 5,\ M(6) = 6,\ M(7) = M(8) = 7,\ M(9) = 8,\ M(10) = M(11) = 9,\ M(12) = 10,\ M(13) = M(14) = 11, \text{ and } M(15) = 12.$$

The results of some numerical computations involving these functions are given in form of diagrams.

A. E.

932[V].—E. Gruschwitz, *Calcul approché de la couche limite laminaire en écoulement compressible sur une paroi conductrice de la chaleur.* Office National d'Études et de Recherches Aeronautiques (O.N.E.R.A.). Publication No. 47, Chatillon-sous-Bagneux, Seine. 1950, 39 p.

As a generalization of the Kármán-Pohlhausen[1] method to steady compressible flow, the velocity component u parallel to a wall $y = 0$ is approximated within the boundary layer by a quartic in $\eta = \text{const} \int_0^y \rho dy$ for a fixed x; moreover, the density ρ is expressed as a rational function of η. These assumptions lead to the system:

$$(1) \qquad (\theta u_e/\nu_e) d\theta/dx = F_1(K) - (K/b_0)[2 - M_e^2 F_2(k)],$$

$$(2) \qquad b_0 = (1 + .2025\, M_e^2)[1 + M_e^2 F_3(K)]/[1 + M_e^2 F_4(k)],$$

$$(3) \qquad K = (\theta^2 b_0/\nu_e) du_e/dx,$$

with known initial conditions at $u_e = 0$ for the determination of K (which is a quintic in Pohlhausen's parameter λ), $b_0 = [\rho_e/\rho]_{\eta=0}$, and the momentum loss $\theta = \int_0^{y_e} (\rho u/\rho_e u_e)(1 - u/u_e) dy$. The velocity u_e, kinematic viscosity ν_e, Mach number M_e and density ρ_e are supposed known in the main stream as well as $y_e = [y]_{u=.99u_e}$, while the functions $F_i(K)$, which depend on the Prandtl number Pr, are tabulated to 3D ($i = 2, 3, 4$) or 4D ($i = 1$) for $K = .094(-.001) - .156$ [Pr $= .725$ ($i = 1, 2, 3, 4$) and Pr $= 1$ ($i = 2$)]. In addition the displacement $\delta^* = \int_0^{y_e} (1 - \rho u/\rho_e u_e) dy$ of the main stream from the wall is given by $\delta^* = \theta[b_0 f_1(K) + M_e^2 f_2(K)]$, the $f_i(K)$ being tabulated to 3D for the above range in K [Pr $= .725$].

In the appendix E. A. Eichelbrenner describes an exact method and presents graphical results indicating that certain aerodynamical quantities (excluding the temperature) are equally well approximated by using Pr $= .725$ or the simpler value Pr $= 1$ [$F_3(K) = F_4(K)$] in Gruschwitz's method.

The tables on pages 13–16 have columns of values of K (as above), λ (generally to 4S), F_3, F_4, f_1, f_2, F_1, F_2, F_2 [Pr $= 1$]. There are also rows of values for $\lambda = \pm 12$, 7.0523, $K = -.157$ (λ, F_1, F_2, F_2 [Pr $= 1$]).

The identities $F_2 = .595 + F_3$ and $f_2 = .405 - F_3$ should hold throughout the table.

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[1] K. Pohlhausen, "Zur näherungsweisen Integration der Differentialgleichungen der laminaren Grenzschicht." *Zeit. angew. Math. Mech.*, v. 1, 1921, p. 252–268.