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On a Punched-Card Method of Solving Certain Integral Equations

1. Introduction. In the present paper we are concerned with the numerical solution of the homogeneous Fredholm equation

$$\phi(y) = \int_0^1 p(x, y)\phi(x)\,dx, \tag{1.1}$$

where the given kernel $p(x, y)$ satisfies the conditions

$$p(x, y) > 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1 \tag{1.2}$$

and

$$\int_0^1 p(x, y)\,dy = 1, \qquad 0 \le x \le 1. \tag{1.3}$$

The solution $\phi(y)$ is to be a non-negative function such that

$$\int_0^1 \phi(y)\,dy = 1. \tag{1.4}$$

Equations of the type (1.1) have received considerable attention in the theory of probability,[1,2] and the above problem has been formulated with this in mind. Although all the examples which have been considered by the authors are special cases of (1.1), the numerical method to be explained is immediately applicable to the following more general equation

$$\phi(y) = f(y) + \int_a^b K(x, y, \lambda)\phi(x)\,dx, \tag{1.5}$$

where $f(y)$, $K(x, y, \lambda)$ are given functions, λ is a parameter (supposed known) and a, b are constants. This integral equation includes the general non-homogeneous Fredholm equation

$$\phi(y) = f(y) + \lambda \int_a^b K(x, y)\phi(x)\,dx \tag{1.6}$$

as a special case.

Numerical methods for the solution of integral equations have been considered by a number of writers.[3–10] The methods employed may be classified briefly as follows: (a) Methods which involve the solution of a system of simultaneous linear equations, (b) Iterative procedures, (c) Relaxation methods, (d) Monte Carlo methods. Of these, only (b) has been found suitable for use with punched cards and such a method of solving equation (1.1) is explained below.

2. The Method of Solution. It is clear from (1.4) that the function $\phi(y)$ may be regarded as a probability density function. It is then natural to approximate its graph by means of a histogram. The general form of this histogram will often be suggested by the particular problem or may be ascertained by a study of the kernel $p(x, y)$. Thus, as a first approximation, we replace $\phi(y)$ by the step-function

$$(2.1)\quad \begin{aligned} \phi_0(y) &= c_j^{(0)}, \quad (j-1)/n \le y < j/n, \quad j = 1, 2, \cdots, n-1, \\ &= c_n^{(0)}, \quad (n-1)/n \le y \le 1, \end{aligned}$$

where n is a positive integer—the number of steps in the histogram—and the $c^{(0)}$'s are a set of positive constants. It is not essential at this point that these constants should satisfy (1.4), i.e., that

$$(2.2)\qquad \sum_{j=1}^{n} c_j^{(0)} = n.$$

Substituting from (2.1) in the right member of (1.1), we find an approximation to $\phi(y)$, viz. $\phi^*(y)$, where

$$(2.3)\qquad \phi^*(y) = \sum_{j=1}^{n} c_j^{(0)} \int_{(j-1)/n}^{j/n} p(x, y)\, dx, \qquad 0 \le y \le 1.$$

Since $p(x, y)$ is known, the integral appearing in the above equation may be evaluated for any given value of y. Let the value of this integral be denoted by $A_j(y)$. Then equation (2.3) becomes

$$(2.4)\qquad \phi^*(y) = \sum_{j=1}^{n} c_j^{(0)} A_j(y), \qquad 0 \le y \le 1.$$

In particular, (2.4) provides an approximation to $\phi(y)$ at each value $y = y_k$ $(k = 1, 2, \cdots, n)$, where y_k represents an arbitrary point in the kth interval $((k-1)/n \le y < k/n)$. In practice, it was found convenient to take $y_k = (k-1)/n$. The values $\phi^*(y_k)$ $(k = 1, 2, \cdots, n)$, calculated from (2.4), provide a new set of constants $c_k^{(1)}$ which in turn can be used to define a new step function $\phi_1(y)$ to replace $\phi_0(y)$. Explicitly,

$$(2.5)\qquad c_k^{(1)} = \sum_{j=1}^{n} c_j^{(0)} A_j(y_k), \qquad k = 1, 2, \cdots, n,$$

and

$$(2.6)\qquad \phi_1(y) = \begin{cases} c_k^{(1)}, & (k-1)/n \le y < k/n \\ c_n^{(1)}, & (n-1)/n \le y \le 1. \end{cases}$$

If necessary, the constants $c_k^{(1)}$ may be normalized to satisfy (1.4) by multiplying each $c_k^{(1)}$ by $n \Big/ \left(\sum_{j=1}^{n} c_j^{(1)} \right)$. A further approximation $\phi_2(y)$, defined in terms of constants $c_k^{(2)}$, can be found in precisely the same way, and this procedure may be continued indefinitely.

This iterative procedure is continued until the normalized values of the constants $c_k^{(r)}$ from two successive iterations, $r = m$ and $r = m + 1$, differ by less than some pre-assigned amount. An analytical proof that this procedure converges is given by BERNIER.[4] In practice, the final approximation depends somewhat on the number n of steps in the approximating histogram—the larger n is, the better the approximation.

3. The Plan of Computation for Punched Cards. The numerical calculations involved in the iterative procedure described above may be performed using punched card equipment. The following IBM machines, or their equivalent, are required:

1. The Alphabetical Accounting Machine, Type 416,
2. The Calculating Punch, Type 602-A,
3. The Reproducing Punch, Type 513,
4. The Sorter, Type 070.

An outline of the plan for carrying out these calculations is given below.

I. Prepare a deck of n^2 cards, to be called the A deck. On the $[(k-1)n+j]$th card, $(k, j = 1, 2, \cdots, n)$, punch the following information:

(i) the card number $(k - 1)n + j$ in card columns 78–80,
(ii) the numerical value of $A_j(y_k)$ in card columns 1–5,
(iii) the group number j in card columns 75, 76.

The number of card columns required for each of the fields (i) to (iii) will depend on the particular problem; the positions of these fields on the card are, of course, arbitrary.

II. Prepare a second deck of n cards, to be called the C deck. On the jth card $(j = 1, 2, \cdots, n)$, punch the following information:

(i) the card number j in card columns 75, 76,
(ii) the number jn in card columns 78–80,
(iii) the number $c_j^{(0)}$ in card columns 65–70,
(iv) an X in card column 7.

For succeeding iterations the numbers $c_j^{(0)}$ in field (iii) will be replaced by $c_j^{(r)}$ $(r = 1, 2, \cdots)$. Field (iv) serves a dual purpose; it provides a ready means of distinguishing the cards of the C deck from those of the A deck and it also provides the means of controlling multiplication (cf. III below).

III. Carry out the following steps in order:

(1) Place the C deck in front of the A deck and sort on card columns 75, 76. The jth card of the C deck, bearing the number $c_j^{(0)}$, is now followed by all those A cards in which are punched the numbers $A_j(y_k)$ to be multiplied by $c_j^{(0)}$.

(2) Using group multiplication, form the products $c_j^{(0)}A_j(y_k)$ and punch each answer in card columns 8–15 of the corresponding A card.

(3) Sort out the C cards, place them behind the A deck and sort on card columns 78–80. The A cards are now arranged in n groups, the kth group being followed by the kth C card. The sum of the products $c_j^{(0)}A_j(y_k)$ in each group will give the corresponding unnormalized value of $c_k^{(1)}$. The

cards of the C deck are here used merely to provide a break in the control and so allow a total to be taken.

(4) Add the products $c_j^{(0)} A_j(y_k)$ and list the resulting sums $c_k^{(1)}$. This step is performed on the tabulator and the next deck of C cards containing $c_k^{(1)}$ in place of $c_k^{(0)}$ may be prepared simultaneously by summary punching.

(5) Sort out the old C deck.

The first iteration is now completed.

It is not necessary to calculate normalized constants $c_k^{(r)}$ at each stage, i.e., a set of constants satisfying

$$\sum_{k=1}^{n} c_k^{(r)} = 1. \tag{3.1}$$

However, since the criterion for terminating the procedure depends on a comparison of normalized constants, we must eventually find such normalized values. When they are required, these are easily found.

4. Illustrative Example. Consider a point moving at random on a segment of the x-axis lying between reflecting barriers at $x = 0$ and $x = 1$. Let $p(x, y)\, dx$ denote the probability density of the point moving in one step from a position in the interval $(x, x + dx)$ to a position with abscissa y. Then $\phi(y)\, dy$ is the probability of the point being in the interval $(y, y + dy)$ after an infinite number of steps.

With regard to the actual motion of the point, we make the following assumptions:

(a) The point moves in steps whose lengths are normally distributed with mean .1 and standard deviation .02. It follows that single steps involving more than one reflection at a barrier have such small probabilities that they may be ignored.

(b) Steps from the central reference point $y_0 = .5$ take place in either direction with equal probabilities.

(c) Starting from any point other than y_0, steps toward y_0 have probability .7 whereas steps away from y_0 have probability .3.

It follows that

$$p(x, y) = \begin{cases} .7\, g(y - x) + 0.3\, g(y + x), & 0 \le x \le y < y_0, \\ .3\, g(x - y) + 0.3\, g(x + y), & 0 \le y \le x < y_0, \\ .7\, g(x - y), \quad y < y_0 \quad x > y_0 \text{ or } y > y_0 & \\ \qquad\qquad x < y_0 \text{ or } y = y_0 \quad x \ne y_0, & \\ .3\, g(y - x) + 0.3\, g(2 - x - y), & y_0 < x \le y \le 1, \\ .7\, g(x - y) + 0.3\, g(2 - x - y), & y_0 < y \le x \le 1, \\ .5\, g(x - y), \quad x = y_0 \quad 0 \le y \le 1, & \end{cases} \tag{4.1}$$

where

$$g(t) = (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp \{-\tfrac{1}{2}(t - .1)^2/\sigma^2\} \quad (\sigma = .02). \tag{4.2}$$

From a consideration of the assumptions (a) to (c) or from a study of a graph of $p(x, y)$, we easily see that $\phi(y)$ must be symmetrical about the point $y = 0.5$. Thus, if the number n of histogram steps is even, we must have

$$c_j^{(r)} = c_{n-j+1}^{(r)}, \qquad r = 0, 1, 2, \cdots. \tag{4.3}$$

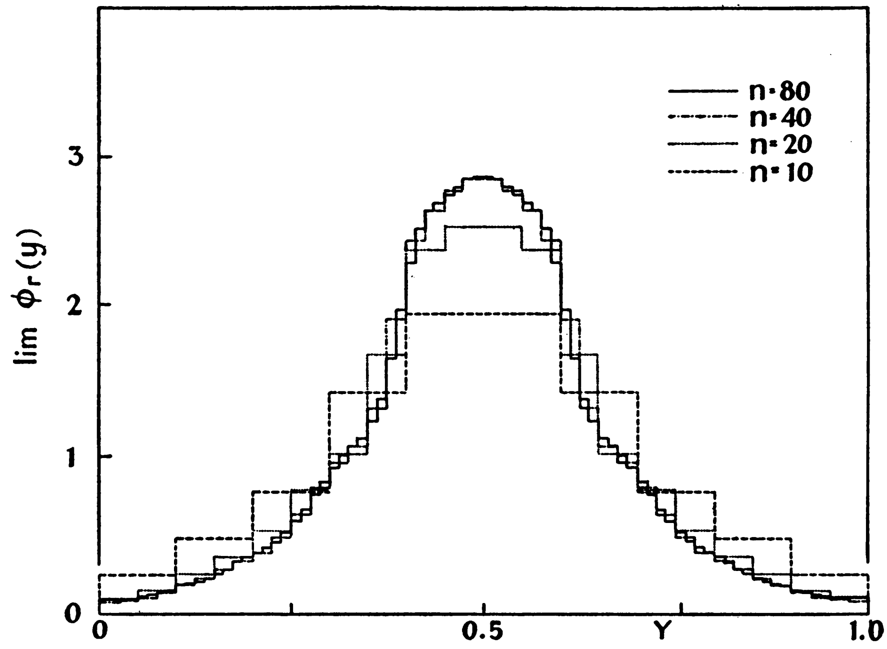

FIG. 1. Limiting n-step histograms approximating $\phi(y)$, $n = 10, 20, 40, 80$.

Here, as elsewhere, r represents the number of the iteration. This means that we need only compute the ordinates of $\phi_{(r)}(y)$ over one-half the range (0, 1).

Approximate results for $\lim_{r \to \infty} \phi_{(r)}(y)$ were obtained for $n = 10, 20, 40$ and 80. In order to obtain agreement between successive iterations to three significant figures, the numbers of iterations required were 12, 24, 33 and 32, respectively. Figure 1 shows a graph of the four limiting histograms, and it is clear that there is quite good agreement among them—they appear to be approximations to the same function. The numerical convergence is exemplified by the successive normalized values of $c_j^{(r)}$ shown in Table I for the case $n = 10$.

TABLE 1. Normalized values of $c_j^{(r)}$ for $n = 10$
r = the number of the iteration; c_j^r is the value of $\phi_{(r)}(y)$ in the interval $((j-1)/n \leq y < j/n)$.

j \ r	1	2	3	4	5	6
5, 6	2.18	1.84	2.05	2.05	2.01	2.00
4, 7	1.57	1.33	1.50	1.24	1.50	1.47
3, 8	0.73	0.67	0.80	0.74	0.80	0.79
2, 9	0.38	0.44	0.45	0.47	0.47	0.49
1, 10	0.19	0.07	0.19	0.23	0.26	0.26

j \ r	7	8	9	10	11	12
5, 6	1.93	1.96	1.97	1.97	1.97	1.97
4, 7	1.48	1.45	1.46	1.46	1.46	1.46
3, 8	0.80	0.80	0.80	0.80	0.80	.0.80
2, 9	0.51	0.51	0.51	0.50	0.50	0.50
1, 10	0.28	0.28	0.28	0.28	0.27	0.27

In this example, the time required per iteration was about 15 minutes for the case $n = 20$. For the larger values of n, the time required was only slightly longer. Since convergence was fairly rapid, a good approximation to the solution $\phi(y)$ could be obtained in a few hours, once the initial A and C decks have been punched.

The method has been applied to about eighteen different cases of the homogeneous equation (1.1). The solutions obtained agree well with those found by Monte Carlo methods and are supported by a limited body of experimental evidence resulting from a study of the one-dimensional movements of fish.

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The Use of Exponential Sums in Step by Step Integration—II

[Continued from *MTAC*, v. 6, p. 63–78]

10. The error analysis for the exponential method is predicated on the convergence of the series representation of the function

$$y/\{(1+y)\ln(1+y)\}, \qquad y = e^{-\lambda h} - 1$$

about $y = 0$ (see equation (13)). The radius of convergence of this series is 1. For small h, $y = e^{-\lambda h} - 1$ is small and there will be convergence for all positive h less than the least $h = h_0$ such that

$$|e^{-\lambda h_0} - 1| = 1. \tag{32}$$

Such an h_0 exists except when λ is real and non-negative, in which case $|y| < 1$ for every positive h.