synchronously with the f graph which records the x motion of the plate. The operator moves the plate so that the $y = f(x)$ and $x = \varphi(y)$ graphs have the same ordinate y on the above-mentioned line and hence the desired output $\varphi(f(x))$ is obtained from the recorder. The article contains details of the construction and of the application mentioned in the title.

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NOTES

147.—STABILITY OF DIFFERENCE RELATIONS IN THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS. In a recent communication, J. TODD[1] demonstrated the danger of replacing a differential equation, for computational purposes, by a difference equation of higher order. H. RUTISHAUSER[2] has since given some general criteria for determining the stability of difference approximations to ordinary differential equations. In the present note, some standard step-by-step methods of integrating ordinary linear differential equations are examined for stability.

Let a linear differential equation be replaced by a finite difference approximation of order p (i.e., one involving $p + 1$ tabular values). Then the nth tabular entry is calculated from

$$y_n + A_1 y_{n-1} + A_2 y_{n-2} + \cdots + A_p y_{n-p} = 0, \tag{1}$$

where $A_1, A_2, \cdots, A_p$ are functions of x and of the interval length h. Now suppose the errors existing in the entries $y_{n-p}, y_{n-p+1}, \cdots, y_{n-1}$ are $\epsilon_{n-p}, \epsilon_{n-p+1}, \cdots, \epsilon_{n-1}$ respectively, then the consequent error in y_n is ϵ_n where

$$\epsilon_n + A_1 \epsilon_{n-1} + A_2 \epsilon_{n-2} + \cdots + A_p \epsilon_{n-p} = 0. \tag{2}$$

Consider also for convenience that the above errors result entirely from errors in the initial values $y_1, y_2, \cdots, y_p$. Then the general error given by equation (2) is

$$\epsilon_n = a_1\lambda_1{}^n + a_2\lambda_2{}^n + \cdots + a_p\lambda_p{}^n, \qquad (n > p),$$

where $a_1, a_2, \cdots, a_n$ are constants and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the roots of the auxiliary equation

$$\lambda^p + A_1\lambda^{p-1} + A_2\lambda^{p-2} + \cdots + A_p = 0. \tag{3}$$

The condition for stability is that all the roots of equation (3) lie inside or on the unit circle.

TODD[1] considered the differential equation

$$y'' = -y,$$

and its fourth order central difference replacement

$$y_n - 16y_{n-1} + (30 - 12h^2)y_{n-2} - 16y_{n-3} + y_{n-4} = 0.$$

As h approaches zero, the roots of the corresponding auxiliary equation tend to 1, 1, $7 - \sqrt{48}$, and $7 + \sqrt{48}$, the last root quoted being responsible for the instability found by Todd. The fourth order backward difference formula, however, is

$$(35 + 12h^2)y_n - 104y_{n-1} + 114y_{n-2} - 56y_{n-3} + 11y_{n-4} = 0,$$

with auxiliary equation roots tending to 1, 1, $\sqrt{11/35}$, and $\sqrt{11/35}$ as h approaches zero. This formula is thus stable at least for sufficiently small h. In general, because of the smaller coefficients employed in calculating y_n from a backward difference formula, the chance of multiplying an error will be correspondingly less. Thus in any step-by-step method, stability is more likely to result from using backward than central differences.

The use of backward difference formulae, however, does not ensure stability, for consider the equation

$$y' = -y, \tag{4}$$

where y is to be computed for increasing x. Write

$$hy_0' = \nabla y_0 + \nabla^2 y_0/2 + \nabla^3 y_0/3 + \cdots + \nabla^n y_0/n, \tag{5}$$

leading to the auxiliary equation,

$$h + (1 - 1/\lambda) + (1 - 1/\lambda)^2/2 + (1 - 1/\lambda)^3/3 + \cdots + (1 - 1/\lambda)^n/n = 0. \tag{6}$$

It can be shown that for $h = 0$ the roots of equation (6) other than $\lambda = 1$ have modulus less than unity for $n \leq 6$. For $n = 7$, there is a pair of conjugate complex roots approximately equal to $\pm i$. For $n \geqq 8$ there will be at least one pair of conjugate roots of modulus greater than unity. (The greatest pair is found easily by GRAEFFE's root-squaring method.) Table I demonstrates the instability of the twelfth order backward difference formula applied to equation (4). The values of e^{-x} at decimal intervals from 0 to 1.1 required to start the computations, were taken from five figure tables. The theoretical and computed values from 1.2 to 2 are given in rows (1) and (2) respectively in table I.

Table I

x	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
(1)	.30199	.27253	.24660	.22313	.20190	.18268	.16530	.14957	.13534
(2)	.30148	.27341	.24708	.22244	.20337	.18665	.16017	.14168	.16065

Next consider the stability of the integration formulae due to ADAMS and MOULTON. Adams' method is based on the formula

$$(y_1 - y_0)/h = y_0' + 1/2\nabla y_0' + 5/12\nabla^2 y_0' + 3/8\nabla^3 y_0' + 251/720\nabla^4 y_0' + \cdots \tag{7}$$

where a sufficient number of starting values for y, y' is supposed computed by an independent method (e.g., by Taylor series). Consider again the first order equation (4). Adams' formula leads to the auxiliary equation

$$F(\lambda) = \lambda - 1 + h\{1 + 1/2(1 - 1/\lambda) + 5/12(1 - 1/\lambda)^2 + 3/8(1 - 1/\lambda)^3 + 251/720(1 - 1/\lambda)^4\} = 0, \tag{8}$$

where only fourth differences are retained and it is clear that there is stability as h tends to zero. Now from equation (8), $F(-\infty) < 0$, and $F(-1) = -2 + 551h/45$. There is therefore a root of modulus greater than unity when h exceeds 90/551, and the method is stable only for sufficiently small tabular interval. Moreover if higher order differences are retained, the maximum value of h for which the method is stable is decreased.

Similar arguments show that Moulton's method based on the formula

$$(9) \quad (y_0 - y_{-1})/h = y_0' - 1/2\nabla y_0' - 1/12\nabla^2 y_0' - 1/24\nabla^3 y_0' - 19/720\nabla^4 y_0' - \cdots$$

is also unstable for large values of the tabular interval when differences higher than the first are retained. The upper limit on h for stability for a given number of differences is very much higher than in Adams' method.

It has been remarked by RUTISHAUSER[2] that the error equation, corresponding to a non-linear differential equation of the form

$$y^{(n)} = f(x, y, y^{(1)}, \cdots, y^{(n-1)}),$$

is linear. The above arguments with certain modifications can therefore be applied to the stability associated with equations of this form.

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[1] J. TODD, "Solution of differential equations by recurrence relations," *MTAC*, v. 4, 1950, p. 39–44.

[2] H. RUTISHAUSER, "Uber die Instabilität von Methoden zur Integration gewöhnlicher Differentialgleichungen," *Zeit. angew. Math. Phys.*, v. 3, 1952, p. 65–74.

148.—TWO NON-ELEMENTARY DEFINITE INTEGRALS. The two integrals in question are

$$F(x) = \int_0^x t^t dt, \qquad G(x) = \int_0^x t^{-t} dt$$

and are of interest because of the peculiar branching properties of the integrands and because they lead to series with unusually rapid convergence. Integrals of these types have been encountered in some recent studies of transients in networks. They can be evaluated numerically as follows.

As usual, we interpret t^t as

$$e^{t \log t} = \sum_{n=0}^{\infty} (t \log t)^n / n!$$

The integral of the general term of this series

$$I_n = \frac{1}{n!} \int_0^x (t \log t)^n dt$$

can be expressed in terms of the complete and incomplete Gamma function by means of the transformation

$$u = -(n + 1) \log t.$$