synchronously with the f graph which records the x motion of the plate. The operator moves the plate so that the $y = f(x)$ and $x = \varphi(y)$ graphs have the same ordinate y on the above-mentioned line and hence the desired output $\varphi(f(x))$ is obtained from the recorder. The article contains details of the construction and of the application mentioned in the title.

F. J. M.

NOTES

147.—Stability of Difference Relations in the Solution of Ordinary Differential Equations. In a recent communication, J. Todd[1] demonstrated the danger of replacing a differential equation, for computational purposes, by a difference equation of higher order. H. Rutishauser[2] has since given some general criteria for determining the stability of difference approximations to ordinary differential equations. In the present note, some standard step-by-step methods of integrating ordinary linear differential equations are examined for stability.

Let a linear differential equation be replaced by a finite difference approximation of order p (i.e., one involving $p + 1$ tabular values). Then the nth tabular entry is calculated from

$$y_n + A_1 y_{n-1} + A_2 y_{n-2} + \cdots + A_p y_{n-p} = 0, \tag{1}$$

where $A_1, A_2, \cdots, A_p$ are functions of x and of the interval length h. Now suppose the errors existing in the entries $y_{n-p}, y_{n-p+1}, \cdots, y_{n-1}$ are $\epsilon_{n-p}, \epsilon_{n-p+1}, \cdots, \epsilon_{n-1}$ respectively, then the consequent error in y_n is ϵ_n where

$$\epsilon_n + A_1 \epsilon_{n-1} + A_2 \epsilon_{n-2} + \cdots + A_p \epsilon_{n-p} = 0. \tag{2}$$

Consider also for convenience that the above errors result entirely from errors in the initial values $y_1, y_2, \cdots, y_p$. Then the general error given by equation (2) is

$$\epsilon_n = a_1 \lambda_1^n + a_2 \lambda_2^n + \cdots + a_p \lambda_p^n, \qquad (n > p),$$

where $a_1, a_2, \cdots, a_n$ are constants and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the roots of the auxiliary equation

$$\lambda^p + A_1 \lambda^{p-1} + A_2 \lambda^{p-2} + \cdots + A_p = 0. \tag{3}$$

The condition for stability is that all the roots of equation (3) lie inside or on the unit circle.

Todd[1] considered the differential equation

$$y'' = -y,$$

and its fourth order central difference replacement

$$y_n - 16 y_{n-1} + (30 - 12h^2) y_{n-2} - 16 y_{n-3} + y_{n-4} = 0.$$

As h approaches zero, the roots of the corresponding auxiliary equation tend to 1, 1, $7 - \sqrt{48}$, and $7 + \sqrt{48}$, the last root quoted being responsible for the instability found by Todd. The fourth order backward difference formula, however, is

$$(35 + 12h^2)y_n - 104y_{n-1} + 114y_{n-2} - 56y_{n-3} + 11y_{n-4} = 0,$$

with auxiliary equation roots tending to 1, 1, $\sqrt{11/35}$, and $\sqrt{11/35}$ as h approaches zero. This formula is thus stable at least for sufficiently small h. In general, because of the smaller coefficients employed in calculating y_n from a backward difference formula, the chance of multiplying an error will be correspondingly less. Thus in any step-by-step method, stability is more likely to result from using backward than central differences.

The use of backward difference formulae, however, does not ensure stability, for consider the equation

$$y' = -y, \tag{4}$$

where y is to be computed for increasing x. Write

$$hy_0' = \nabla y_0 + \nabla^2 y_0/2 + \nabla^3 y_0/3 + \cdots + \nabla^n y_0/n, \tag{5}$$

leading to the auxiliary equation,

$$h + (1 - 1/\lambda) + (1 - 1/\lambda)^2/2 + (1 - 1/\lambda)^3/3 + \cdots + (1 - 1/\lambda)^n/n = 0. \tag{6}$$

It can be shown that for $h = 0$ the roots of equation (6) other than $\lambda = 1$ have modulus less than unity for $n \leq 6$. For $n = 7$, there is a pair of conjugate complex roots approximately equal to $\pm i$. For $n \geqq 8$ there will be at least one pair of conjugate roots of modulus greater than unity. (The greatest pair is found easily by GRAEFFE's root-squaring method.) Table I demonstrates the instability of the twelfth order backward difference formula applied to equation (4). The values of e^{-x} at decimal intervals from 0 to 1.1 required to start the computations, were taken from five figure tables. The theoretical and computed values from 1.2 to 2 are given in rows (1) and (2) respectively in table I.

Table I

x	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
(1)	.30199	.27253	.24660	.22313	.20190	.18268	.16530	.14957	.13534
(2)	.30148	.27341	.24708	.22244	.20337	.18665	.16017	.14168	.16065

Next consider the stability of the integration formulae due to ADAMS and MOULTON. Adams' method is based on the formula

$$(y_1 - y_0)/h = y_0' + 1/2\nabla y_0' + 5/12\nabla^2 y_0' + 3/8\nabla^3 y_0' + 251/720\nabla^4 y_0' + \cdots \tag{7}$$

where a sufficient number of starting values for y, y' is supposed computed by an independent method (e.g., by Taylor series). Consider again the first order equation (4). Adams' formula leads to the auxiliary equation

$$F(\lambda) = \lambda - 1 + h\{1 + 1/2(1 - 1/\lambda) + 5/12(1 - 1/\lambda)^2 + 3/8(1 - 1/\lambda)^3 + 251/720(1 - 1/\lambda)^4\} = 0, \tag{8}$$

where only fourth differences are retained and it is clear that there is stability as h tends to zero. Now from equation (8), $F(-\infty) < 0$, and $F(-1) = -2 + 551h/45$. There is therefore a root of modulus greater than unity when h exceeds 90/551, and the method is stable only for sufficiently small tabular interval. Moreover if higher order differences are retained, the maximum value of h for which the method is stable is decreased.

Similar arguments show that Moulton's method based on the formula

$$(9)\quad (y_0 - y_{-1})/h = y_0' - 1/2\nabla y_0' - 1/12\nabla^2 y_0' - 1/24\nabla^3 y_0' - 19/720\nabla^4 y_0' - \cdots$$

is also unstable for large values of the tabular interval when differences higher than the first are retained. The upper limit on h for stability for a given number of differences is very much higher than in Adams' method.

It has been remarked by RUTISHAUSER[2] that the error equation, corresponding to a non-linear differential equation of the form

$$y^{(n)} = f(x, y, y^{(1)}, \cdots, y^{(n-1)}),$$

is linear. The above arguments with certain modifications can therefore be applied to the stability associated with equations of this form.

A. R. MITCHELL
J. W. CRAGGS

Department of Mathematics
St. Andrews University
Scotland

[1] J. TODD, "Solution of differential equations by recurrence relations," *MTAC*, v. 4, 1950, p. 39–44.
[2] H. RUTISHAUSER, "Uber die Instabilität von Methoden zur Integration gewöhnlicher Differentialgleichungen," *Zeit. angew. Math. Phys.*, v. 3, 1952, p. 65–74.

148.—TWO NON-ELEMENTARY DEFINITE INTEGRALS. The two integrals in question are

$$F(x) = \int_0^x t^t dt, \qquad G(x) = \int_0^x t^{-t} dt$$

and are of interest because of the peculiar branching properties of the integrands and because they lead to series with unusually rapid convergence. Integrals of these types have been encountered in some recent studies of transients in networks. They can be evaluated numerically as follows.

As usual, we interpret t^t as

$$e^{t \log t} = \sum_{n=0}^{\infty} (t \log t)^n/n!$$

The integral of the general term of this series

$$I_n = \frac{1}{n!}\int_0^x (t \log t)^n dt$$

can be expressed in terms of the complete and incomplete Gamma function by means of the transformation

$$u = -(n+1)\log t.$$

In fact

$$(-1)^n(n+1)^{n+1}I_n = 1 - \frac{\Gamma_y(n+1)}{\Gamma(n+1)}$$

where

$$\Gamma_y(n+1) = \int_0^y e^{-u}u^n du$$

and

$$y = -(n+1)\log x.$$

However, since available tables[1] of the incomplete Gamma function are not convenient for this application, it was decided to use a direct evaluation of the series.

The integral for I_n can be expressed as a sum of $n + 1$ terms as follows:[2]

$$I_n = x^{n+1}\sum_{r=0}^{n}\frac{(-1)^r(\log x)^{n-r}}{(n+1)^{r+1}(n-r)!}.$$

Summing over n and collecting the coefficient of each power of $\log x$ we obtain the double series

$$F(x) = \sum_{n=0}^{\infty} I_n = \sum_{r=0}^{\infty}(\log x)^r/r!\sum_{s=1}^{\infty}(-1)^{s+1}x^{s+r}(s+r)^{-s}.$$

Similarly

$$G(x) = \sum_{n=0}^{\infty}(-1)^n I_n = \sum_{r=0}^{\infty}(-1)^r(\log x)^r/r!\sum_{s=1}^{\infty}x^{s+r}(s+r)^{-s}.$$

When $x = 1$ we have the unusually rapidly converging series

$$F(1) = 1 - 2^{-2} + 3^{-3} - 4^{-4} + \cdots,$$
$$G(1) = 1 + 2^{-2} + 3^{-3} + 4^{-4} + \cdots.$$

The following values of $F(x)$ and $G(x)$ were computed to 10D by the above double series and checked by numerical integration using tables of fractional powers.[3] Values have been rounded off to 8D. Most of the calculation was made by Mrs. Joan M. Clay.

x	$F(x)$	$G(x)$
0	0.0000 0000	0.0000 0000
.1	0.0870 9546	0.1152 6443
.2	0.1625 6731	0.2478 5617
.3	0.2334 0100	0.3890 5018
.4	0.3027 3442	0.5332 8125
.5	0.3726 1486	0.6763 8768
.6	0.4446 5226	0.8152 2316
.7	0.5202 8866	0.9474 7031
.8	0.6009 4326	1.0715 0820
.9	0.6881 0902	1.1862 9918
1.0	0.7834 3051	1.2912 8600

M. S. Corrington

Radio Corporation of America
Camden, N. J.

[1] KARL PEARSON, *Tables of the Incomplete Γ-Function.* London, H. M. Stationery Office. (Reissue 1934; London, Biometrika Office, University College.)
[2] W. GRÖBNER & N. HOFREITER, *Integraltafel.* Erster Teil, Unbestimmte Integrale. Vienna, 1949, p. 111, formula 2a.
[3] NBSMTP, *Tables of Fractional Powers.* New York, Columbia University Press, 1946.

149.—The CANON DOCTRINAE TRIANGVLORVM (1551) of RHETICUS (1514–1576). Some facts with reference to this excessively rare publication have been given in material about PITISCUS and RHETICUS in *MTAC*, v. 3, p. 394, 396, 553–554, 559–560. It is here noted that the only copies known to have been preserved were in the Bibliothèque Nationale and British Museum. DEMORGAN had a copy in 1845 when he published[1] a description of the work, but this was doubtless in his Library at the University of London, destroyed during the recent World War.

In Catalogue 19, 1952, of the London bookseller E. WEIL, a copy was offered for 27 £ 10 s. Mr. WILLIAM D. MORGAN, of 1764 St. Anthony Ave., St. Paul 4, Minnesota, was so fortunate as to secure this item for adding to his already valuable collection (see *MTAC*, v. 3, p. 562–563). Since Mr. Morgan graciously loaned this precious work to me that a microfilm copy might be made for the Brown University Library, I take the opportunity to add a little to the information already published in *MTAC.* The complete title is as follows: *Canon Doctrinae Triangvlorvm. Nvnc primvm a Georgio Ioachimo Rhetico, in lvcem editvs, cvm privilegio imperiali, Ne qvis haec intra decennivm, quacunque; forma ac compositione, edere, neve sibi vendicere aut operibvs suis inserere ausit. Lipsiae ex officina Wolphgangi Gvnteri. Anno M.D. LI.* In the title page is an obelisk with a man drawing a diagram on the base.

The back of the title page is blank; then follows a page of Latin verses. On the back of this page is the first of 14 pages of 7D tables of the six trigonometric functions, at interval 10′, arranged for the first time in semiquadrantal display. The degrees are in black, and the minutes and differences are in red. This is the first table in which all trigonometric functions are brought together. Rheticus was the first to define trigonometric functions by means of a right-angled triangle without any reference to a circle.

Immediately following the tables are 6 pages of dialogue between Philomathes, a supposed friend of Rheticus, and Hospes, his pupil. The pupil asks what the intention of the book is, and is answered at length. He suggests that, perhaps, the intention may be to complete the system of Copernicus, by publishing tables from it resembling those then in use. But he is answered that Rheticus would rather that Copernicus himself had not done so much in this line, as he thereby diminished the geometrical practice of the learner, and so on.

An undated 1580 reprint of the *Canon* is in the British Museum.

The copy of the *Canon* before me has evidently had its pages trimmed; but the present size of its pages is 15.8 × 22.5 cm.

R. C. ARCHIBALD

Brown University
Providence, R. I.

[1] DEMORGAN, "On the almost total disappearance of the earliest trigonometrical canon," RAS, *Mo. Not.*, v. 6, 1845, p. 221–228; reprinted with an addition in *Phil. Mag.*, s. 3, v. 26, 1845, p. 517–526.

150.—Square Root on the 602A. The square root setup in the IBM 602A manual [8th ed., p. 81–84] extracts six digit roots of eight digit base numbers at the rate of approximately two roots per minute, using the Newton-Raphson scheme with a fixed (six) number of iterations, with starting values of 3, 30, 300, or 3000, depending on the size of the base number. The 602A must be equipped with division circuits.

About 75 percent of the 602A's in use do not have division circuits. Square roots can be calculated by many iterative schemes, however, of which one of the simplest to program is

$$x_{i+1} = x_i + \tfrac{1}{2}(N - x_i^2) \tag{1}$$

where N is the number whose square root is required, and is taken such that $0 \leq N \leq 1$. The starting value, x_0, is taken as N. The iterations are terminated when

$$|x_{i+1} - x_i| < \epsilon \tag{2}$$

where ϵ is a predetermined small number (say 5 in the 4th decimal place). The error is then less than or equal to $\epsilon N^{-\frac{1}{2}}$.

The rate of convergence is given by the following inequality where $N^{\frac{1}{2}} - x_k$ is the error after the kth iteration:

$$|N^{\frac{1}{2}} - x_k| < (N^{\frac{1}{2}} - N)(1 + N - N^{\frac{1}{2}})^k.$$

One notices that convergence is slow for small values of N.

For N, a 6 digit number, and $\epsilon = 10^{-4}$, square roots to 4D were obtained in the following times.

N	time in seconds
0.9	25
0.8	29
0.7	38
0.6	50
0.5	56
0.4	63
0.3	75
0.2	103
0.1	157

Starting values, x_0, could, of course, be gang punched from a master deck, which would make the setup considerably faster. However, the order of the deck is then disturbed. If the order of the deck is not critical, the remarks below are pertinent; moreover, the entire job (for three digit roots) can be done readily from a master deck of 1000 cards, by collating and gang punching.

If division is available and it is possible to order the cards on the numbers N, then the formula

$$x_{i+1} = \tfrac{1}{2}(x_i + N/x_i) \tag{3}$$

gains speed if the starting value for each card is the result from the previous card. Approximately eight cards per minute can be calculated, with eight digit N's and square roots to 4D.

The answer from the previous card is taken as x_0. The chief problem in the wiring is to obtain a starting value for the first card, since the 602A performs all the programming wired, in dummy form, before the first card is read. If the cards are in descending order on N, the first starting value can be taken as N, provided that the dummy programming can be skipped. This latter can be accomplished by wiring to "read" from an early program through the normal side of a selector which is then latched for the remainder of the run.

Diagrams of the setups used at the Numerical Analysis Laboratory will be published shortly.

J. HOLLINGSWORTH
F. GRUENBERGER

Numerical Analysis Laboratory
University of Wisconsin
Madison, Wisconsin

151. A NUMERICAL STUDY OF A CONJECTURE OF KUMMER.—The generalization to the cubic case of the well-known (quadratic) Gauss sum was first investigated by KUMMER.[1] He showed that the expression

$$x_p = 1 + 2 \sum_{\nu=1}^{(p-1)/2} \cos(2\pi\nu^3/p) \tag{1}$$

for all $p \equiv 1 \pmod 3$ satisfies the cubic equation

$$f(x) = x^3 - 3px - pA = 0 \tag{2}$$

where A is uniquely determined by the requirements

$$4p = A^2 + 27B^2, \qquad A \equiv 1 \pmod 3. \tag{3}$$

Equation (2) clearly has three real roots for each p. Kummer classified the primes $p \equiv 1 \pmod 3$ according to whether the Kummer sum is the largest, middle or smallest root of equation (2). He conjectured that the asymptotic frequencies for these classes of p are (in the order above) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$. To check this surmise he calculated the first 45 of the x_p and found the densities to be .5333, .3111, .1556.

This problem was brought to the attention of the authors by E. ARTIN who suggested the desirability of further testing the conjecture since its truth would have important consequences in algebraic number theory.

Accordingly the primes $p \equiv 1 \pmod 3$ from 7 through 9,973 were tested. We give below a summary of the resulting densities. In this tabulation we have arbitrarily divided the primes into six groups of 100 each, designated by I, $\cdots$, VI and a final group of eleven primes, VII.

Number of primes $p \equiv 1 \pmod 3$ such that

Group	x_p is the largest root	x_p is the middle root	x_p is the smallest root
I	54	28	18
II	41	38	21
III	46	33	21
IV	39	32	29
V	43	29	28
VI	44	38	18
VII	5	3	3
Total	272	201	138
Density	.4452	.3290	.2258

These results would seem to indicate a significant departure from the conjectured densities and a trend toward randomness.

The method of calculation was this: Each root of (2) lies in one of the intervals $(-2p^{\frac{1}{2}}, -p^{\frac{1}{2}})$, $(-p^{\frac{1}{2}}, +p^{\frac{1}{2}})$, $(p^{\frac{1}{2}}, 2p^{\frac{1}{2}})$ as may be seen directly from the form of (2) with the help of (3). For each relevant p the expression (1) for x_p was evaluated, its sign was determined and its square compared to p. This determined in which of the three intervals just described the x_p lies. To check that x_p was indeed a solution of (2), (3) and to determine the precision of the evaluation the expression

$$f(x_p)/f'(x_p) \tag{4}$$

was then calculated. This latter check was performed by first transforming the x_p into decimal form for tabulation and then retransforming these results back into binary form before evaluating the expression (4). In this manner both the calculation proper and the conversion to decimal form of the results were checked.

The trigonometric expressions appearing in (1) were evaluated by power series. Each angle was reduced mod 2π and then mod π until it lay between $-\pi/2$ and $+\pi/2$. Then the cosine of $\frac{1}{4}$ of this angle was calculated keeping five terms in the series expansion. The "double-angle" formula for cosines was then used twice to obtain the desired cosine.

The calculation involved about 15 million multiplications counting the checking mentioned above. The values of p were introduced in blocks of 200. The entire calculation was carried out twice to ensure reliability. The authors are indebted to Mrs. ATLE SELBERG who programmed and coded the calculation.

J. VON NEUMANN
H. H. GOLDSTINE

Institute for Advanced Study
Princeton, New Jersey

[1] E. E. Kummer, "De residuis cubicis disquisitiones nonnullae analyticae," *Jn. f. d. reine u. angew. Math.*, v. 32, 1846, p. 341–365.

CORRIGENDA

v. 6, p. 262, *insert* Emch, G. F. 247.
v. 6, p. 265, under Myers *insert* 54.
v. 6, p. 268, under Yowell *insert* 254.
v. 7, p. 31, l. 11 of MTE 218; *for* $-\log p$ *read* $\log p$.