NOTES

152.—Asymptotic Formulas for the Hermite Polynomials. Asymptotic formulas for the Hermite polynomials $H_n(x)$ are derived by the method of Liouville-Steklov from the integral equation

$$\exp(-\tfrac{1}{2}x^2)\,H_n(x) = \lambda_n \cos(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi) + N^{-\frac{1}{2}}\int_0^x t^2 \exp(-\tfrac{1}{2}t^2)\,H_n(t)\sin(N^{\frac{1}{2}}[x-t])dt$$

where

$$H_n(x) = (-1)^n \exp(x^2)\left(\frac{d}{dx}\right)^n \exp(-x^2), \quad N = 2n+1$$

and

$$\lambda_n = \begin{cases} |H_n(0)| = n!/(\tfrac{1}{2}n)! & \text{if } n \text{ is even} \\ |N^{-\frac{1}{2}}H_n'(0)| = N^{-\frac{1}{2}}(n+1)!/((n+1)/2)! & \text{if } n \text{ is odd.} \end{cases}$$

The method is an iterative process where one obtains the k-th approximation by placing the $(k-1)$-th approximation under the integral sign to obtain a series in powers of $N^{-\frac{1}{2}}$ which is quite accurate when n is both large and large in comparison with x. Szegö[1] and others do not give the actual coefficients in the asymptotic series, but content themselves with a proof of the existence of the expansions and mention of only the first term or so. Each complete iteration in the process becomes progressively more tedious, and yields only one more term in the expansion. The writer felt that it would be useful to have the explicit expressions for the asymptotic formulas considerably beyond the first term. The expansions which are given below are the result of 9 iterations for both n even and n odd.

These formulas were tested to calculate $H_{19}(x)$ and $H_{20}(x)$ for $x = 1(1)5$ and they permitted a relative error of about 10^{-7} at $x = 1$ which increased steadily with x to about 10^{-2} at $x = 5$.

These formulas can also be used to compute the earlier zeros of $H_n(x)$ by an iterative process.

From the definitions of $A(x)$ and $B(x)$ below, we have

$$0 = A(x) + N^{-\frac{1}{2}}\tan(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)B(x).$$

We then employ the iteration formula

$$x_{i+1} = N^{-\frac{1}{2}}\{(\tfrac{1}{2}n + m)\pi - \arctan(N^{\frac{1}{2}}A(x_i)/B(x_i))\}$$

where m is a suitably chosen integer (positive, negative, or zero). For the r-th zero of $H_n(x)$ we may use as initial approximation

$$x_0 = \begin{cases} N^{-\frac{1}{2}}(r-\tfrac{1}{2})\pi & n \text{ even} \\ N^{-\frac{1}{2}}(r-1)\pi & n \text{ odd.} \end{cases}$$

When applied to H_{19} and H_{20} this process gave a relative error of about 10^{-9} for the first two zeros and about 10^{-3} for the last two.

The asymptotic formulas in question may be given as follows.
$\exp(-\frac{1}{2}x^2)\, H_n(x)/\lambda_n = A(x)\cos(N^{\frac{1}{2}}x - \frac{1}{2}n\pi) + B(x)\, N^{-\frac{1}{2}}\sin(N^{\frac{1}{2}}x - \frac{1}{2}n\pi)$
where

$$A(x) = A_0(x) + N^{-1}A_1(x) + \ldots + N^{-4}A_4(x) + O(n^{-5})$$
$$B(x) = B_0(x) + N^{-1}B_1(x) + \ldots + N^{-4}B_4(x) + O(n^{-5}).$$

The polynomials $A_i(x)$, $B_i(x)$ differ slightly in the two cases:

Case I *n even*

$$A_0(x) = 1$$
$$A_1(x) = -x^6/72 + x^2/4$$
$$A_2(x) = x^{12}/31104 - 11x^8/1440 + 19x^4/96$$
$$A_3(x) = -x^{18}/33592320 + 17x^{14}/622080 - 18889x^{10}/3628800 + 1091x^6/5760 - 19x^2/32$$
$$A_4(x) = x^{24}/67722117120 - 23x^{20}/671846400 + 11153x^{16}/522547200 - 177127x^{12}/43545600 + 14601x^8/71680 - 631x^4/384$$
$$B_0(x) = x^3/6$$
$$B_1(x) = -x^9/1296 + x^5/15 - x/4$$
$$B_2(x) = x^{15}/933120 - 7x^{11}/12960 + 901x^7/20160 - 19x^3/48$$
$$B_3(x) = -x^{21}/1410877440 + x^{17}/933120 - 2131x^{13}/5443200 + 4421x^9/120960 - 241x^5/384 + 19x/32$$
$$B_4(x) = x^{27}/3656994324480 - 13x^{23}/14108774400 + 8503x^{19}/9405849600 - 400187x^{15}/1306368000 + 21500581x^{11}/638668800 - 7159x^7/7680 + 631x^3/192$$

Case II *n odd*

$$A_0(x) = 1$$
$$A_1(x) = -x^6/72 + x^2/4$$
$$A_2(x) = x^{12}/31104 - 11x^8/1440 + 19x^4/96 - 1/4$$
$$A_3(x) = -x^{18}/33592320 + 17x^{14}/622080 - 18889x^{10}/3628800 + 1111x^6/5760 - 21x^2/32$$
$$A_4(x) = x^{24}/67722117120 - 23x^{20}/671846400 + 11153x^{16}/522547200 - 59159x^{12}/14515200 + 132641x^8/645120 - 325x^4/192 + 21/32$$
$$B_0(x) = x^3/6$$
$$B_1(x) = -x^9/1296 + x^5/15 - x/4$$
$$B_2(x) = x^{15}/933120 - 7x^{11}/12960 + 901x^7/20160 - 7x^3/16$$
$$B_3(x) = -x^{21}/1410877440 + x^{17}/933120 - 2131x^{13}/5443200 + 13333x^9/362880 - 1237x^5/1920 + 21x/32$$
$$B_4(x) = x^{27}/3656994324480 - 13x^{23}/14108774400 + 8503x^{19}/9405849600 - 400537x^{15}/1306368000 + 7195607x^{11}/212889600 - 152141x^7/161280 + 671x^3/192.$$

H. E. SALZER

NBSCL

[1] G. SZEGÖ, *Orthogonal Polynomials*. Amer. Math. Soc. *Colloq. Publications*, v. 23, 1939, p. 212–213.

153.—ANALYTICAL APPROXIMATIONS, [See also NOTE 143]. The following twenty-two approximations are all concerned with the so-called offset circle probability function defined by the integral

$$q(R,x) = \int_R^\infty e^{-\frac{1}{2}(\rho^2+x^2)} I_0(\rho x)\rho d\rho,$$

in which $I_0(Z)$ is the usual Bessel function. Among other properties, $q(R,x) + q(x,R) = 1 + e^{-\frac{1}{2}(R^2+x^2)} I_0(Rx)$.

The $q(R,x)$ function has been finely tabulated to 6D by the joint effort of NBSINA and RAND.

(13) To better than .0028 over $-\infty \leqslant x \leqslant \infty$,
$$\lim_{R\to 0} \frac{1 - q(R,x)}{1 - q(R,0)} = e^{-\frac{1}{2}x^2} \doteq \frac{1}{(1 + .123x^2 + .010x^4)^4}.$$

(14) To better than .0014 over $-\infty \leqslant x \leqslant \infty$,
$q(1,x) \doteq 1 - .393(1 + .093x^2 + .007x^4)^{-4}$.

(15) To better than .00014 over $-\infty \leqslant x \leqslant \infty$,
$q(1,x) \doteq 1 - .3935(1 + .0968x^2 + .0047x^4 + .00028x^6)^{-4}$.

(16) To better than .0035 over $-\infty \leqslant x \leqslant \infty$,
$q(2,x) \doteq 1 - .865(1+.038x^2 + .004x^4)^{-4}$.

(17) To better than .001 over $-\infty \leqslant x \leqslant \infty$,
$q(2,x) \doteq 1 - .865(1 + .0401x^2 + .00309x^4 + .000075x^6)^{-4}$.

(18) To better than .006 over $0 \leqslant y \leqslant \infty$,
$$\lim_{R\to 0} \frac{1 - q(R,R + y)}{1 - q(R,R)} = e^{-\frac{1}{2}y^2} \doteq \frac{1}{(1 + .015y + .076y^2 + .040y^3)^4}.$$

(19) To better than .00037 over $0 \leqslant y \leqslant \infty$,
$q(.5,.5 + y) \doteq 1 - .1045(1 + .129y + .079y^2 + .056y^3)^{-4}$.

(20) To better than .0007 over $0 \leqslant y \leqslant \infty$,
$q(1,1 + y) \doteq 1 - .267(1 + .203y + .079y^2 + .062y^3)^{-4}$.

(21) To better than .0009 over $0 \leqslant y \leqslant \infty$,
$q(2,2 + y) \doteq 1 - .397(1 + .236y + .066y^2 + .066y^3)^{-4}$.

(22) To better than .0011 over $0 \leqslant y \leqslant \infty$,
$q(4,4 + y) \doteq 1 - .45(1 + .227y + .064y^2 + .065y^3)^{-4}$.

(23) To better than .0013 over $0 \leqslant y \leqslant \infty$,
$$\lim_{R\to\infty} q(R,R+y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \doteq 1 - \frac{.5}{(1+.209y+.061y^2+.062y^3)^4}.$$

(24) To better than .00011 over $0 \leqslant x \leqslant 1$,
$q(1,x) \doteq .6066 + .1500x^2 - .0238x^4$.

(25) To better than .0017 over $0 \leqslant x \leqslant 2$,
$q(2,x) \doteq .135 + .566(x/2)^2 - .096(x/2)^4$.

(26) To better than .0008 over $0 \leqslant x \leqslant 3$,
$q(3,x) \doteq .011 + .231(x/3)^2 + .654(x/3)^4 - .329(x/3)^6$.

(27) To better than .0019 over $0 \leqslant x \leqslant 3$,
$q(3,x) \doteq [.105 + .930(x/3)^2 - .282(x/3)^4]^2$.

(28) To better than .0011 over $0 \leqslant x \leqslant 3$,
$q(3,x) \doteq [.105 + .954(x/3)^2 - .349(x/3)^4 + .043(x/3)^6]^2$.

(29) To better than .002 over $0 \leqslant x \leqslant 4$,
$q(4,x) \doteq [.018 + .581(x/4)^2 + .515(x/4)^4 - .372(x/4)^6]^2$.

(30) To better than .0035 over $0 \leqslant y \leqslant 3$,
$q(3,3 - y) \doteq .568(1 + .157y + .107y^2 + .017y^3)^{-4}$.

(31) To better than .0013 over $0 \leqslant y \leqslant 4$,
$q(4,4-y) \doteq .551(1+.187y+.055y^2+.051y^3)^{-4}$.

(32) To better than .0013 over $0 \leqslant y \leqslant \infty$,
$$\lim_{R\to\infty} q(R,R-y) = \int_{-\infty}^{-y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \doteq \frac{.5}{(1+.209y+.061y^2+.062y^3)^4}.$$

(33) To better than .0004 over $0 \leqslant R \leqslant 1$,
$q(R,R) \doteq 1 - .4921R^2 + .3212R^4 - .0966R^6$.

(34) To better than .0011 over $1 \leqslant R \leqslant \infty$,
$q(R,R) \doteq (R+.183)/(2R-.388)$.

Cecil Hastings, Jr.
James P. Wong, Jr.

The Rand Corporation
1700 Main Street
Santa Monica, California

CORRIGENDA

V. 7, p. 87, l. 11 *for* 6202089 *read* 62020897
for 1858477404602617 *read* 18584774046020617