On Irregular Negative Determinants of Exponent $9n$

It may first be stated that this article is concerned only with the irregular negative determinants which occur in the study of quadratic forms. (The technical terms used below, *e.g.*, "determinant," "irregular," "exponent," "duplicating," "opposite," and "genera II, IV, VIII, etc.," are fully explained in Mathews, "Theory of Numbers" (Part I., 1892) on pages 59, 178, 182, 149, 68, and 132 respectively [1].)

Only three examples of the above determinants have as yet been given in mathematical literature, *viz.*, $-D = 3299$ (Scholz and Taussky) [2], 6075 (Pepin) [3], and 11907 (Gauss) [4]. The writer, in his Thesis for the London Ph.D. degree written in 1953, gives 55 more examples, the complete table of 58 determinants up to the limit 150,000 being:

$-D =$	3299	56403	94284	117612
	6075	58563	97200	120987
	11907	60075	99387	122715
	17739	64395	101675	128547
	23571	70227	103627	132300
	24300	70956	105219	134379
	27675	75411	107163	135675
	29403	76059	107419	140211
	33075	77571	108675	140940
	35235	81675	109907	141075
	41067	81891	110700	146043
	46899	85995	111051	148419
	47628	87075	114075	149499
	52731	87723	115425	
	54675	93555	116883	

A characteristic property of the determinants of exponent 9 given in the above list (with the exception of -3299) is that they possess 13 classes each of which when duplicated produces its own opposite. These may be termed "critical classes."

For the determinant $-D = 6075$ they are:

4	1	1519
9	3	676
36	3	169
36	15	175
76	9	81
79	18	81
81	36	91
19	9	324
25	5	244
31	1	196
49	1	124
61	5	100
76	29	91

In the exceptional case, $-D = 3299$, there are only 4 critical classes, the single genus it possesses having 81 classes, which tabulate into 9 periods of 9 classes each. All the other members of the list of 58 determinants given above have genera II, IV, VIII, etc.

In an attempt to find a relation between some of these determinants it was found that 25 of them were of the form $-D = 6075 + 5832p$, *viz.*, 6075, 11097, 17739, 23571, etc. Another series was found to be of the form $27675 + 27000p$, giving 27675, 54675, 81675, etc., [5].

In these series, when the critical classes (a, b, c) are arranged so that a is a perfect square, it is found that each member gives the same values of a and b; *e.g.*, in the above example, 6075, the critical classes can be shown thus:

$-D = 6075$				$-D = 11907$	$-D = 17739$
a	b	c		c	c
4	1	1519		2977	4435
9	3	676		1324	1972
36	3	169		331	493
36	15	175		337	499
81	9	76		148	220
81	18	79		151	223
81	36	91	for the next two	163	235
324	9	19	members 11907,	37	55
324	45	25	17739 the	43	61
324	63	31	c-values are:	49	67
324	99	49		67	85
324	117	61		79	97
324	153	91		109	127

and for all the others the c-values increase respectively in arithmetical progression.

It is found that the determinants (with the exception of $-D = 3299$) can be arranged in 20 series as follows:

Series No.	Form	Members
1	$6075 + 5832p$	6075, 11907, 17739, 23571, 29403, 35235, 41067, 46899, 52731, 58563, 64395, 70227, 76059, 81891, 87723, 93555, 99387, 105219, 111051, 116883, 122715, 128547, 134379, 140211, 146043.
2	$27675 + 27000p$	27675, 54675, 81675, 108675, 135675.
3	$33075 + 27000p$	33075, 60075, 87075, 114075, 141075.
4	$24300 + 46656p$	24300, 70956, 117612.
5	$47628 + 46656p$	47628, 94284, 140940.
6	$75411 + 74088p$	75411, 149499.
7	$11907 + 74088p$	11907, 85995. (11907 also appears in Ser. 1)
8	$33075 + 74088p$	33075, 107163. (33075 also appears in Ser. 3)

Series No.	Form	Members
9	$132300 + 74088000p$	132300
10	$97200 + 1728000p$	97200
11	$115425 + 91125p$	115425
12	$110700 + 216000p$	110700
13	$101675 + 343000p$	101675
14	$77571 + 474552p$	77571
15	$56403 + 580093704p$	56403
16	$103627 + 170584511336p$	103627
17	$107419 + 7066834559000p$	107419
18	$109907 + 982107784p$	109907
19	$120987 + 64964808p$	120987
20	$148419 + 35937000p$	148419.

In every case the factor of p (*i.e.*, the constant difference) is the cube of the product of the square roots of the a-values of the 1st, 2nd, and 5th classes.

All the critical classes of these determinants have been tested by duplication, and their opposites were obtained in every case.

As a further test with a determinant beyond our assumed limit, we may take $-D = 187596$, the 4th member of Series 5, and we have:

$$-D = 187596$$

a	b	c
9	3	20845
16	2	11725
144	30	1309
144	66	1333
81	9	2317
81	18	2320
81	36	2332
1296	414	277
1296	558	385
1296	18	145
1296	306	217
1296	126	157
1296	450	301

the last six of which reduce to:

277	137	745
385	173	565
145	18	1296
217	89	901
157	31	1201
301	149	697.

Choosing one of these classes, say (301, 149, 697), we can duplicate it by compounding (301, 149, 697) with (697, −149, 301).

This is done most simply by a method devised by the writer:

Let the classes be $(a, b, \cdots)$ and $(a', b', \cdots)$ where a and a' are prime to each other, and take p and q such that either ap and $a'q$, or $a'p$ and aq differ by unity. This can nearly always be done mentally, but when a and a' are not small, the values of p and q are more quickly found as the constituents of the penultimate convergent of the continued fraction representing a/a' or a'/a. The required compound class is then given by $(aa', a'bp + ab'q, \cdots)$ or by $(aa', a'bq + ab'p, \cdots)$, care being taken that the signs of p and q are so chosen that the smaller of the two products, *e.g.*, aq and $a'p$, say, shall be negative. Applying this to the case in hand, we get:

$$(697.301, 697.149(-19) + 301(-149)44, \cdots),$$

or

$$(209797, -1973207 -1973356, \cdots),$$
$$(209797, -3946563, \cdots),$$
$$(209797, +39580, 7468),$$
$$(7468, 2240, 697),$$
$$(697, 149, 301),$$
$$(301, -149, 697),$$

which shows that (301, 149, 697) is a critical class, and each of the twelve other classes when similarly tested is found to be a critical class.

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1. G. B. Mathews, *Theory of Numbers*, Part I, 1892.
2. A. Scholz & O. Taussky, "Die Hauptideale der kubischen Klassenkörper imaginär-quadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluss auf den Klassenkörperturm," *Jahrbuch über die Fortschritte der Mathematik*, 60, 1934, p. 126, *J. reine angew. Math.* 171, 1934, p. 19–41.
3. T. Pepin, *Atti. Acad. Pont. Nuovi Lincei*, 33, 1881, p. 354–391.
4. C. F. Gauss, *Disquisitiones Arithmeticae*, Art. 306, VIII, 1801; in *Werke* I, 1863, p. 371; German transl. by H. Maser, 1889, p. 653–654.
5. The 11th member of the second series, *i.e.* −297675, has exponent 27, with 40 critical classes.

TECHNICAL NOTES AND SHORT PAPERS

Selected References on Use of High-Speed Computers for Scientific Computation

The author is often asked to recommend reading to orient mathematicians in the impact of high-speed computers on numerical analysis. The following list was prepared in answer to one such request, but does not pretend to be definitive. The author is indebted to C. B. Tompkins for several suggestions.

For a list of books not necessarily influenced by high-speed computers, but highly pertinent to their use, see G. E. Forsythe, "A numerical analyst's fifteen-foot shelf," *MTAC*, v. 7, 1953, p. 221–228.