A Modified Chebyshev-Everett Interpolation Formula

1. Introduction. It is a well-known property of the Chebyshev polynomials that, of all polynomials with leading coefficient unity, they possess the smallest absolute upper bound when the argument is allowed to vary between their limits of orthogonality. This property suggests the use of such polynomials as a means of interpolation. As is explained in Kopal's recent book on numerical analysis [1], a power series rearranged as a series of Chebyshev polynomials is reduced to an economic form, insofar as the "maximum" accuracy is attained with a very small number of terms. We use Kopal's notation [1] here.

The purpose of this note is to rearrange the Everett interpolation formula,

$$f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0^{\text{ii}}\Delta_0^{\text{ii}} + E_1^{\text{ii}}\Delta_1^{\text{ii}} + \cdots + E_0^{(2n)}\Delta_0^{(2n)} + E_1^{(2n)}\Delta_1^{(2n)}$$

as a series of suitable Chebyshev polynomials, thereby reducing considerably the number of terms depending on m, required to give a prescribed accuracy, while retaining the use of even order differences only. Following Miller [2] the polynomials $C_{2j+1}(2-2m)$ and $C_{2j+1}(2m)$ could be used, where $C_j(2m) = 2\cos(j\cos^{-1} m)$. A disadvantage in doing so is the non-vanishing of such polynomials at tabular points. To overcome this difficulty Kopal [1] and Miller [7] have suggested an expansion in terms of the polynomials

$$C_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} \quad \text{and} \quad C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}.$$

Such an expansion Kopal calls the *Modified Chebyshev-Everett Interpolation Formula* and states that "$\cdots$ it represents potentially the most powerful and economic interpolation formula which can as yet be devised" [1]. See also Clenshaw and Olver [9].

Explicitly, following [1], p. 513, let

$$(1)\quad f(m) = f(0) + m\Delta_{\frac{1}{2}} + \sum_{j=1}^{n}\left[\beta_0{}^jC_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} + \beta_1{}^jC_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}\right]$$

$$(2)\quad = f(0) + m\Delta_{\frac{1}{2}} + \sum_{j=1}^{n}\frac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!} \times\left[N_0{}^{(2j)}C_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} + N_1{}^{(2j)}C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}\right]$$

where the coefficients $N_0{}^{(2j)}$ and $N_1{}^{(2j)}$ will be designated *modified differences* of the $2j$-th order. Owing to the symmetry of the Everett coefficients $E^{(2n)}$ and to the fact that $C_{2j+1}(x)$ is an odd function of x, the $N_1^{(2j)}$'s will be the same function of the Δ_1's as the $N_0^{(2j)}$'s are of the Δ_0's.

2. Calculation of the Modified Differences $N_{0,1}^{(2j)}$. The coefficients $N_{0,1}^{(2j)}$ are determined from the equivalence of the two expressions

$$
\begin{aligned}
f(m) &= f(0) + m\Delta_{\frac{1}{2}} + \sum_{j=1}^{n} \left(E_0^{(2j)}\Delta_0^{(2j)} + E_1^{(2j)}\Delta_1^{(2j)}\right) \\
&= f(0) + m\Delta_{\frac{1}{2}} + \sum_{j=1}^{n} \frac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)\,!} \\
&\qquad \times\left[N_0^{(2j)}C_{2j+1}\left\{(2-2m)\cos\frac{\pi}{4j+2}\right\} + N_1^{(2j)}C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}\right]
\end{aligned}
$$

or, in effect, from

$$
\sum_{j=1}^{n} E_1^{(2j)}\Delta_1^{(2j)} = \sum_{j=1}^{n} \frac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)\,!} N_1^{(2j)}C_{2j+1}\left\{2m\cos\frac{\pi}{4j+2}\right\}. \tag{3}
$$

Both sides of (3) are expanded in powers of m and the coefficients of equal powers equated. The procedure is quite straightforward, but the algebra is rather unwieldy, and is not reproduced here.

The following general relations can be obtained by equating the coefficients of m^{2n-3} and m:

$$
\begin{aligned}
N^{(2n-4)} &= \Delta^{(2n-4)} - \frac{1}{4}\left(\frac{n}{3} - \frac{1}{2(n-1)}\sec^2\frac{\pi}{4n-2}\right)\Delta^{(2n-2)} \\
&\quad + \left\{\frac{(n+1)(5n+6)}{1440} - \frac{\sec^4\dfrac{\pi}{4n+2}}{64n(2n-1)} - \frac{\sec^2\dfrac{\pi}{4n-2}}{32(n-1)}\right. \\
&\qquad\qquad \left.\times\left(\frac{n+1}{3} - \frac{1}{2n}\sec^2\frac{\pi}{4n+2}\right)\right\}\Delta^{(2n)} + \cdots
\end{aligned}
$$

and

$$
\sum_{j=1}^{n} (-1)^j \frac{(j\,!)^2}{(2j+1)\,!}\Delta_1^{(2j)} = \sum_{j=1}^{n} (-1)^j \frac{\sec^{2j}\dfrac{\pi}{4j+2}}{2^{2j}(2j)\,!} N_1^{(2j)}.
$$

These relations are of assistance in computing the modified differences.

If differences up to the tenth are retained the following closed forms for the coefficients N^{ii}, N^{iv}, $\cdots$, N^{x} are obtained:

$$\begin{aligned} N^{\mathrm{x}} &= \Delta^{\mathrm{x}}, \\ N^{\mathrm{viii}} &= \Delta^{\mathrm{viii}} + a_5\Delta^{\mathrm{x}}, \\ N^{\mathrm{vi}} &= \Delta^{\mathrm{vi}} + a_4\Delta^{\mathrm{viii}} + b_5\Delta^{\mathrm{x}}, \\ N^{\mathrm{iv}} &= \Delta^{\mathrm{iv}} + a_3\Delta^{\mathrm{vi}} + b_4\Delta^{\mathrm{viii}} + c_5\Delta^{\mathrm{x}}, \\ N^{\mathrm{ii}} &= \Delta^{\mathrm{ii}} + a_2\Delta^{\mathrm{iv}} + b_3\Delta^{\mathrm{vi}} + c_4\Delta^{\mathrm{viii}} + d_5\Delta^{\mathrm{x}}, \end{aligned}$$

where

$$a_j = -\frac{1}{4}\left(\frac{j+1}{3} - \frac{1}{2j}\sec^2\frac{\pi}{4j+2}\right),$$

$$b_j = \frac{(j+1)(5j+6)}{1440} - \frac{\sec^4\dfrac{\pi}{4j+2}}{64j(2j-1)} + a_j\frac{\sec^2\dfrac{\pi}{4j-2}}{8(j-1)},$$

$$c_5 = -\frac{139}{6048} + \frac{1}{5\cdot 3^3\cdot 2^{11}}\sec^6\frac{\pi}{22} - \frac{a_5}{7\cdot 2^8}\sec^4\frac{\pi}{18} + \frac{b_5}{24}\sec^2\frac{\pi}{14},$$

$$c_4 = -\frac{41}{3024} + \frac{5}{16\cdot 8!}\sec^6\frac{\pi}{18} - \frac{a_4}{960}\sec^4\frac{\pi}{14} + \frac{b_4}{16}\sec^2\frac{\pi}{10},$$

$$d_5 = \frac{479}{151200} - \frac{15}{2^7\cdot 10!}\sec^8\frac{\pi}{22} + \frac{5a_5}{16\cdot 8!}\sec^6\frac{\pi}{18} - \frac{b_5}{8\cdot 5!}\sec^4\frac{\pi}{14} + \frac{c_5}{16}\sec^2\frac{\pi}{10}.$$

The numerical values of the coefficients a, b, c, d are given in Table I.

If the $N^{(2j)}$ are multiplied by $\dfrac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!}$, the actual coefficients of the Chebyshev polynomials $\beta_{0,1}{}^{(j)}$ in equation (1) are obtained. These are given in Table II ($i = 0,1$).

As will be seen, the coefficients $\beta^{(j)}$ diminish very rapidly and in practice it will only be necessary to use about two terms of the series (1). Should it be found unnecessary to retain the higher differences, they should simply be omitted from the $\beta^{(j)}$ or $N^{(2j)}$.

3. Besselian and Comrie-type Modified Differences. Clearly, the well-known Bessel interpolation formula can also be regrouped in terms of the Chebyshev polynomials, and the modified differences corresponding to it obtained in a similar manner. Modified differences arising in connection with Bessel's formula were first introduced into computational practice by E. W. Brown [3], and subsequently (but independently) by Camp [4] and Comrie [5]. Comrie, in particular, has done so much to propagate their use that they are usually associated with his name. It must be emphasized that the Comrie type of (Besselian) modified differences is not identical with what may be called the Chebyshev-Bessel modified differences, which would be obtained by an analogous procedure to that developed in the preceding sections of this note.

TABLE I

	Δ_i^{ii}	Δ_i^{iv}	Δ_i^{vi}	Δ_i^{viii}	Δ_i^{x}
N_i^{ii}	1	−0.18090 16994 37	0.03717 66239 55	−0.00807 38992 79	0.00180 85304 04
N_i^{iv}		1	−0.28949 60381 83	0.07283 14707 77	−0.01772 12103 93
N_i^{vi}			1	−0.38444 50665 04	0.11351 63325 91
N_i^{viii}				1	−0.47448 31950 54
N_i^{x}					1

TABLE II

	Δ^{ii}	Δ^{iv}	Δ^{vi}	Δ^{viii}	Δ^{x}	j	$\cos\frac{\pi}{4j+2}$
β_i^1	0.03207 50149 55	−0.00580 24247 15	0.00119 24407 69	−0.00025 89704 40	0.00005 80086 40	1	0.86602 54037 84
β_i^2		0.00033 46856 93	−0.00009 68901 81	0.00002 43756 51	−0.00000 59310 36	2	0.95105 65162 95
β_i^3			0.00000 18516 20	−0.00000 07118 46	0.00000 02101 89	3	0.97492 79121 82
β_i^4				0.00000 00061 77	−0.00000 00029 31	4	0.98480 77530 12
β_i^5					0.00000 00000 14	5	0.98982 14418 81

Comrie used modified differences $M^{(i)}$ whose coefficients were determined to minimize the error of the Bessel interpolation formula truncated after a given number of terms. Comrie never published details of the process by which these constants were determined; his argument has been reconstructed in [1], Chapter II. Although these constants were determined by Comrie with specific reference to Bessel's formula, it has become customary to use them in Everett's formula also. In this case an expansion

$$\begin{aligned} f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0{}^{\text{ii}}\Delta_0{}^{\text{ii}} + E_1{}^{\text{ii}}\Delta_1{}^{\text{ii}} + \cdots \\ + E_0{}^{(2j)}(\Delta_0{}^{(2j)} + A_j\Delta_0{}^{(2j+2)} + B_j\Delta_0{}^{(2j+4)} + \cdots) \\ + E_1{}^{(2j)}(\Delta_1{}^{(2j)} + A_j\Delta_1{}^{(2j+2)} + B_j\Delta_1{}^{(2j+4)} + \cdots) \end{aligned}$$

is used, where the coefficients A_j, B_j are those determined by Comrie for the Bessel formula.

For convenience, write

$$r_{2j+1} = \frac{\sec^{2j+1}\left(\dfrac{\pi}{4j+2}\right)}{2^{2j+1}(2j+1)!} C_{2j+1}\left(2m\cos\frac{\pi}{4j+2}\right).$$

Consider a specific example where differences up to the eighth are retained. Then $f(m)$ may be expanded:

$$f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_1{}^{\text{ii}}\Delta_1{}^{\text{ii}} + E_1{}^{\text{iv}}\Delta_1{}^{\text{iv}} + E_1{}^{\text{vi}}\Delta_1{}^{\text{vi}} + E_1{}^{\text{viii}}\Delta_1{}^{\text{viii}} \tag{4}$$

$$+ \text{ similar terms in } E_0\Delta_0,$$

$$\begin{aligned} f(m) = f(0) + m\Delta_{\frac{1}{2}} + r_3(\Delta_1{}^{\text{ii}} + a\Delta^{\text{iv}} + b\Delta_1{}^{\text{vi}} + c\Delta_1{}^{\text{viii}}) \\ + r_5(\Delta_1{}^{\text{iv}} + d\Delta_1{}^{\text{vi}} + e\Delta_1{}^{\text{viii}}) \\ + r_7(\Delta_1{}^{\text{vi}} + f\Delta_1{}^{\text{viii}}) \\ + r_9\Delta_1{}^{\text{viii}} + \cdots. \end{aligned} \tag{5}$$

A comparison of these formulae shows that

$$\begin{aligned} r_3 &= E_1{}^{\text{ii}} \\ ar_3 + r_5 &= E_1{}^{\text{iv}}. \end{aligned}$$

Now if we regard the last three terms of (5) as error terms then

$$\begin{aligned} f(m) &\sim f(0) + m\Delta_{\frac{1}{2}} + r_3(\Delta_1{}^{\text{ii}} + a\Delta_1{}^{\text{iv}} + b\Delta_1{}^{\text{vi}} + c\Delta_1{}^{\text{viii}}) + \cdots \\ &= f(0) + m\Delta_{\frac{1}{2}} + E_1{}^{\text{ii}}N_1{}^{\text{ii}} + \cdots. \end{aligned}$$

Reference to Comrie's paper [5] shows that $N_1{}^{\text{ii}}$ is in fact identical with the modified second difference used by Comrie.

However, there seems to be no simple relation between $N_1{}^{\text{iv}}$ and higher order modified differences, and the higher Comrie modified differences.

4. Example. As an example, an interpolation has been made in a table of Fermi-Dirac functions at present being prepared by the author.

A portion of this table, together with 1st, 2nd, 4th, 6th, 8th, and 10th differences, is reproduced below. It has been chosen because the differences do not fall off rapidly, as can be seen.

η	$F(\eta)$	$\Delta_{\frac{1}{2}}$	Δ^{ii}	Δ^{iv}	Δ^{vi}	Δ^{viii}	Δ^{x}
−2.0	0.1145 8783						
		6921 403					
−1.5	1838 0186		3748 501	297 363	−135 029	65 578	−20 496
		10669 904					
−1.0	2905 0090		5259 311	130 749	−125 114	110 942	−119 432
		15929 215					
−0.5	*4497 9305*		*6900 870*	*−160 979*	*−4 256*	*36 873*	*−68 774*
		22830 085					
0.0	6780 9389		*8381 450*	*−456 963*	*153 475*	*−105 970*	*113 458*
		31211 535					
0.5	9902 0925		9405 068	−599 471	205 237	−135 355	132 585
		40616 603					
1.0	1.3963 7528		9829 215	−536 743	121 643	−32 156	−17 467
		50445 818					
1.5	1.9008 3346		9716 618	−352 371	5 895	53 577	−82 922
		60162 437					
2.0	2.5024 5783						

From this table we find, e.g., the value of $F(-0.3)$ from the formula: $(m=0.4)$,

$$F(-0.3) = F(-0.5) + 0.4\Delta_{\frac{1}{2}} + E_0^{ii}(0.4)N_0^{ii} + E_1^{ii}(0.4)N_1^{ii} + \beta_0^2 C_5\left(1.2\cos\frac{\pi}{10}\right) + \beta_1^2 C_5\left(0.8\cos\frac{\pi}{10}\right)$$

where

$$N_0^{ii} = 0.06929\ 411 \qquad \beta_0^2 = -\ 0.00000\ 052$$

$$N_1^{ii} = 0.08470\ 882 \qquad \beta_1^2 = -\ 0.00000\ 171.$$

The values of $C_5\left(1.2\cos\frac{\pi}{10}\right)$ and $C_5\left(0.8\cos\frac{\pi}{10}\right)$ are obtained by linear interpolation in the table of Chebyshev polynomials [6].

For comparison, columns (1) and (2) below give respectively the value of $F(-0.3)$, as calculated with the same number of terms in the interpolation formula, by the straight Everett formula,

$$f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0^{ii}\Delta_0^{ii} + E_1^{ii}\Delta_1^{ii} + E_0^{iv}\Delta_0^{iv} + E_1^{iv}\Delta_1^{iv}$$

and the modified Everett formula using Comrie's coefficients in the modified fourth difference,

$$f(m) = f(0) + m\Delta_{\frac{1}{2}} + E_0^{ii}\Delta_0^{ii} + E_1^{ii}\Delta_1^{ii} + E_0^{iv}M_0^{iv} + E_1^{iv}M_1^{iv}$$

where $M_{0,1}^{iv} = \Delta_{0,1}^{iv} - 0.207\Delta_{0,1}^{vi} + 0.045\Delta_{0,1}^{viii}$. Column (3) gives the value computed by the present method. The value obtained by direct calculation is 0.53193 157. It will be seen that the present method gives substantially better results for this case.

		1.	2.	3.
$F(-0.3) =$		0.44979305	0.44979305	0.44979305
	+	9132034	9132034	9132034
	−	441656	441656	443482
	−	469361	469361	474369
	−	1875	1845	11
	−	4913	5306	318
		0.53193534	0.53193171	0.53193159

In most applications high order differences will fall off rapidly and probably only the modified second difference will be required. If this is calculated and printed with a table, interpolation becomes a very simple procedure.

5. Use of the Expansion. An interesting discussion about the merits of a Chebyshev-Everett interpolation formula is recorded in Appendix 11 of reference [7] to which the reader is referred. (This report was received after the major part of this note had been written.)

It should be emphasized that the chief advantage of this type of expansion is that the number of terms in the interpolating expansion of a function $f(m)$, dependent on m, is reduced considerably. Clearly, the expansion will have its greatest use when several interpolations within a given interval are required. The complicated modified differences, once calculated, are available for all interpolations within the relevant interval, the number of terms dependent on m being usually only two, *viz.*,

$$C_3\left\{2m\cos\frac{\pi}{6}\right\} \quad \text{and} \quad C_3\left\{(2-2m)\cos\frac{\pi}{6}\right\}.$$

A table of the above functions would be advantageous, especially when the expansion is being used for sub-tabulation. For purposes other than sub-tabulation, values of the Chebyshev functions can be obtained by simple interpolation in the existing tables [6] and [8]. For convenience, values of $\cos\dfrac{\pi}{4j+2}$ are given to twelve decimal places in Table II.

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Note added in proof. Dr. L. Fox of the National Physical Laboratory, Teddington, Middlesex, England, has recently informed the author in a paper as yet unpublished, that in fact the Everett form of the modified Chebyshev-Bessel interpolation formula gives slightly better results (in the sense that the maximum possible residual error is smaller) than the formula advocated here.

Iterative Procedure for Evaluating a Transient ResponseThrough its Power Series

Introduction. We discuss here a particular method of evaluating a time function, such as a transient response, from its Laplace transform. We shall assume the Laplace transform of $F(t)$,

$$\bar{F}(S) = \int_0^\infty e^{-st}F(t)\,dt \tag{1}$$

is given, and is of the form

$$\bar{F}(S) = \frac{A_0{}^0S^{N-1} + A_1{}^0S^{N-2} + \cdots + A^0{}_{N-2}S + A^0{}_{N-1}}{S^N + b_1S^{N-1} + b_2S^{N-2} + \cdots + b_{N-1}S + b_N}. \tag{2}$$

(The superscript 0 of the A's does not indicate a power, but is used as a superscript for reasons which will appear presently.)

We require in (2) that the denominator be of higher degree than the numerator, which is necessary if $F(t)$ is a Laplace transform. We shall always assume also that we have $\bar{F}(S)$ written with the coefficient of S^N in the denominator equal to unity, as in equation (2).

The method described below is an iterative method for obtaining the quantities α_j in MacLaurin series for $F(t)$, which we write in the form

$$F(t) = \sum_{j=0}^{\infty} \alpha_j \frac{t^j}{j!}. \tag{3}$$

The method has the advantage that it is not necessary to know the roots of the numerator or denominator of $\bar{F}(S)$. The series (3) will converge for all positive values of t, although for larger values of t a great many terms may be needed. This, however, is not too great a disadvantage for digital purposes, since the method for computing the α_j's is an iterative one. It is necessary, however, to have some idea of how many terms will be needed, and a criterion for this is also given below and discussed. The theoretical derivations are given in the Appendix.

Method for Computing the α_j's. The method for computing the α_j's, the theory of which is discussed in the Appendix, is as follows: We start with the coefficients

$$A_0{}^0,\ A_1{}^0,\ \cdots,\ A^0{}_{N-1},\ b_1,\ b_2,\ \cdots,\ b_N \quad \text{of (2).}$$