

Here we have  $p = 0.25$ ,  $p' = 0.25$ . Using (2'), together with a table of  $E_2(x)$ , as in [2], we get  $I(4.025, 7.05) \approx .878, 5410$ , which agrees with result obtained by Pearson.

In principle, throwback of other kinds in the bivariate Everett formula is possible. There is, also, no a priori reason why it could not be accomplished in the trivariate Everett formula, which is stated in [4], page x.

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1. BRIT. ASSOC. FOR ADV. OF SCIENCE, *Mathematical Tables*, v. 1, Cambridge University Press, 1931.
2. NBSCL, *Tables of Bessel Functions of Fractional Order*, v. 1, Columbia Univ. Press, New York, 1948.
3. KARL PEARSON, Editor, *On the Construction of Tables and on Interpolation*, Part II; *Bivariate Tables*, Tracts for Computers, No. III, Cambridge University Press, 1920.
4. KARL PEARSON, Editor, *Tables of the Incomplete Beta-Function*, Cambridge University Press, 1934.
5. KARL PEARSON, Editor, *Tables of the Incomplete  $\Gamma$ -Function*, Cambridge University Press, 1946.
6. BRIT. NAUT. ALMANAC OFFICE, *Interpolation and Allied Tables*, 1936 (1947 reprint), p. 928-929.
7. L. J. COMRIE, *Chambers's Six-Figure Mathematical Tables*, v. II, D. van Nostrand Co., New York, 1949, p. 533.
8. D. R. HARTREE, *Numerical Analysis*, Clarendon Press, Oxford, 1952, p. 70-72.
9. ZDENĚK KOPAL, *Numerical Analysis*, John Wiley & Sons, Inc., New York, 1955, p. 54-64.

## TECHNICAL NOTES AND SHORT PAPERS

### On the Treatment of Monte Carlo Methods in Text Books

Though Monte Carlo methods have not yet reached the text book stage, they are gradually being afforded a brief mention in new books on numerical analysis. This is probably stimulated by a desire for completeness but the brevity of treatment often results in a very narrow picture of Monte Carlo. For any given problem there may be several methods of solution by Monte Carlo and there are general principles which should guide one in choosing between these. A description of a single method, leaving the reader with the impression that this is the sole method, is therefore a mistake. Care must also be taken to see that, though a method may be illustrated on a simple example for which Monte Carlo would not normally be used, it is easily extensible to problems for which Monte Carlo would be needed and provides a reasonably practical method of solving these.

In two otherwise excellent books by Householder [2] and Kopal [4] a technique is given for evaluating integrals by Monte Carlo. It is the only example of Monte Carlo in either book and is rather an unfortunate one; for it is not easily applicable in practice to multiple integrals though it is largely for these that Monte Carlo can be useful; in addition, it is always less efficient, and often much less so, than another well-known simple technique (defined as crude Monte Carlo by Hammersley and Morton [1]). Their comparative efficiency is also an admirable illustration of a general precept to be used when choosing even the simplest Monte Carlo methods.

Consider the estimation of the integral

$$I = \int_0^1 f(x) dx$$

where  $f(x)$  is a prescribed function whose minimum is made zero by a change of origin and whose maximum is a positive constant  $M$ . In the method described in [2] and [4], we draw a rectangle of height  $M$  and unit breadth about the curve  $y = f(x)$  and choose  $n$  points  $(\xi_i, M\eta_i)$ ,  $i = 1, 2, \dots, n$ , at random in this rectangle;  $\xi_i$  and  $\eta_i$  are independent random numbers uniformly distributed in the interval  $(0, 1)$ . We count a score  $t_{1i} \equiv t_1(\xi_i, \eta_i)$  for each point,  $t_{1i}$  being  $M$  if the point lies between  $y = f(x)$  and the  $x$ -axis and zero otherwise. Then the mean score  $\bar{t}_1 = (1/n) \sum_{i=1}^n t_{1i}$  is an unbiased estimator of  $I$  and  $\bar{t}_1/M$  is binomially distributed with parameter  $I/M$ , i.e.,

$$\epsilon(\bar{t}_1) = I \quad \text{and} \quad \text{var } \bar{t}_1 = I(M - I)/n.$$

The crude Monte Carlo technique, first described by Kahn [3] in 1949, consists of using  $t_2 = f(\xi)$  as an estimator of  $I$ . Thus, for the  $n$  random numbers  $\xi_i$ , we have a mean score  $\bar{t}_2 = (1/n) \sum_{i=1}^n f(\xi_i)$  for which clearly

$$\epsilon(\bar{t}_2) = I \quad \text{and} \quad \text{var } \bar{t}_2 = (1/n) \left\{ \int_0^1 [f(x)]^2 dx - I^2 \right\} = (S^2 - I^2)/n.$$

It is clear that, when dealing with multiple integrals, the bounds of the integrand which it is necessary to know for the first method may be very difficult to obtain even when finite; and that any overestimate of them can lead to a considerable increase in  $\text{var } t_1$ . Moreover, since  $f(x) \geq 0$ ,  $S^2 \leq IM$  so that  $\text{var } t_2 \leq \text{var } t_1$ , the equality occurring only when  $f(x)$  is a constant; the importance of this difference is enhanced by the fact that it is the ratio of the variances which measures the relative efficiency of the two methods. For example, when  $f(x) \equiv x$ ,  $IM = 1/2$  and  $S^2 = 1/3$  but  $\text{var } t_1 = 1/4 = 3 \text{ var } t_2$ , so that the second method is three times as efficient as the first. Finally, it should be noted that  $t_2 = \int_0^1 t_1(\xi, \eta) d\eta$ , i.e.,  $M$  times the probability that, for given  $\xi$ , the point  $(\xi, M\eta)$  will lie below  $y = f(x)$ : thus the second estimator is simply related to the first, being its average over  $\eta$ , and its higher efficiency is due to following the principle that random processes should be replaced by analysis wherever possible, as pointed out in [1].

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1. J. M. HAMMERSLEY & K. W. MORTON, "A new Monte Carlo technique: antithetic variates," Cambridge Phil. Soc., *Proc.*, v. 52, 1956, p. 449-475.
2. A. S. HOUSEHOLDER, *Principles of Numerical Analysis*, McGraw-Hill Book Co., Inc., New York, 1953, p. 243.
3. H. KAHN, "A modification of the Monte Carlo method," Seminar on Scientific Computation, *Proc.*, I.B.M., New York, 1949, p. 20-27.
4. Z. KOPAL, *Numerical Analysis*, Chapman and Hall Ltd., London, 1955, p. 428-429.