Extension of Lindow's Tables for Numerical Differentiation Using Newton-Stirling and Newton-Bessel Differences

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Lindow [1] gives several short tables of coefficients for obtaining the first and second derivative at intermediate points by differentiation of either the Newton-Stirling or Newton-Bessel interpolation formula [2]. The former involves central differences $\mu\delta_0^{2m+1}$ and δ_0^{2m} which are on the line with $f(x_0)$, and the latter involves central differences $\delta_{\frac{1}{2}}^{2m+1}$ and $\mu\delta_{\frac{1}{2}}^{2m}$ which are on the line with $f\left(x_0 + \frac{h}{2}\right)$, h being the tabular interval. Lindow's tables of Newton-Stirling coefficients are intended to interpolate for $f'(x_0 + ph)$ and $f''(x_0 + ph)$ for either $0 \leq p \leq .25$ or for $.75 \leq p \leq 1.00$, in the latter case by choosing a new x_0 = original $x_0 + h$ (and corresponding central differences opposite the next entry) and a new p where now $- .25 \leq$ new $p \leq 0$. His tables of Newton-Bessel coefficients are intended to interpolate for $f'(x_0 + ph)$ and $f''(x_0 + ph)$ for $.25 \leq p \leq .75$, or in the new variable p_1 given by $p = \frac{1}{2} + p_1$, for $- .25 \leq p_1 \leq .25$. The reason for employing both Newton-Stirling and Newton-Bessel formulas (i.e. both tabular and mid-point central differences) is that by proper choice of p or p_1 the argument is never more than one-fourth of a tabular interval away from the central differences employed, which should yield high accuracy.

The formulas for $f'(x)$ and $f''(x)$ are of the form:

$$(1) \quad f'(x_0 \pm ph) = \frac{1}{h}\left\{\mu\delta_0 \pm p\delta_0^2 + \sum_{m=1}^{n} (A_{2m+1}(p)\mu\delta_0^{2m+1} \pm A_{2m+2}(p)\delta_0^{2m+2})\right\} + R_{1,n}$$

obtained by single differentiation of the Newton-Stirling interpolation formula, to be used for $\pm p$ ranging from $-.25$ to $.25$.

$$(2) \quad f'(x_0 + ph) \equiv f'\left(x_0 + \frac{h}{2} \pm p_1 h\right)$$

$$= \frac{1}{h}\left\{\delta_{\frac{1}{2}} \pm p_1\mu\delta_{\frac{1}{2}}^2 + \sum_{m=1}^{n} (B_{2m+1}(p_1)\delta_{\frac{1}{2}}^{2m+1} \pm B_{2m+2}(p_1)\mu\delta_{\frac{1}{2}}^{2m+2})\right\} + R_{2,n},$$

obtained by single differentiation of the Newton-Bessel interpolation formula, to be used for $p = \frac{1}{2} \pm p_1$ ranging from .25 to .75, or $\pm p_1$ ranging from $-.25$ to .25.

$$(3) \quad f''(x_0 \pm ph)$$

$$= \frac{1}{h^2}\left\{\delta_0^2 \pm p\mu\delta_0^3 + \sum_{m=1}^{n} (C_{2m+2}(p)\delta_0^{2m+2} \pm C_{2m+3}(p)\mu\delta_0^{2m+3})\right\} + R_{3,n},$$

obtained by double differentiation of the Newton-Stirling interpolation formula,

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to be used for $\pm p$ ranging from $-.25$ to $.25$.

$$(4)\quad f''(x_0 + ph) \equiv f''\left(x_0 + \frac{h}{2} \pm p_1 h\right)$$
$$= \frac{1}{h^2}\left\{\mu\delta_{\frac12}^2 \pm p_1\delta_{\frac12}^3 + \sum_{m=1}^{n} \left(D_{2m+2}(p_1)\mu\delta_{\frac12}^{2m+2} \pm D_{2m+3}(p_1)\delta_{\frac12}^{2m+3}\right)\right\} + R_{4,n},$$

obtained by double differentiation of the Newton-Bessel interpolation formula, to be used for $p = \frac{1}{2} \pm p_1$ ranging from .25 to .75, or $\pm p_1$ ranging from $-.25$ to .25.

In the nth pair of terms only, the right hand term within the parentheses does not appear if the interpolation series (1), (2), (3), or (4) is not taken beyond $\mu\delta_0^{2n+1}$, $\delta_{\frac12}^{2n+1}$, δ_0^{2n+2} or $\mu\delta_{\frac12}^{2n+2}$ respectively. The formulas for the remainder terms $R_{i,n}$, $i = 1, 2, 3$, or 4 depend also upon the parity of the order of the last difference retained in (1), (2), (3), or (4). To find $R_{i,n}$ explicitly one may differentiate the formulas for the remainder terms in the Newton-Stirling or Newton-Bessel formula ([3], p. 100, 102), making use of the properties of divided differences under repeated differentiation ([3], p. 66–67). For (1)–(4) there will be altogether eight different formulas for $R_{i,n}$, having either two, three, four or six terms, the use of which is quite troublesome and time-consuming, since each term involves the evaluation of a polynomial of higher degree and the estimation of a higher order derivative. For most practical problems one can almost always bypass the work in using the theoretically exact expression for $R_{i,n}$, by simply observing the last term retained in any of (1)–(4) in conjunction with the rate at which the terms are falling off in magnitude.

Following are the explicit expressions for the coefficients $A_r(p)$, $B_r(p_1)$, $C_s(p)$ and $D_s(p_1)$, as (even or odd) polynomials in p or p_1, for $r = 3(1)10$ and $s = 4(1)10$:

$$A_3(p) = \frac{3p^2 - 1}{6},\quad A_4(p) = \frac{2p^3 - p}{12},\quad A_5(p) = \frac{5p^4 - 15p^2 + 4}{120},$$

$$A_6(p) = \frac{3p^5 - 10p^3 + 4p}{360},\quad A_7(p) = \frac{7p^6 - 70p^4 + 147p^2 - 36}{5040},$$

$$A_8(p) = \frac{2p^7 - 21p^5 + 49p^3 - 18p}{10080},$$

$$A_9(p) = \frac{9p^8 - 210p^6 + 1365p^4 - 2460p^2 + 576}{3\ 62880},$$

$$A_{10}(p) = \frac{5p^9 - 120p^7 + 819p^5 - 1640p^3 + 576p}{18\ 14400}.$$

$$B_3(p_1) = \frac{12p_1^2 - 1}{24},\qquad B_4(p_1) = \frac{4p_1^3 - 5p_1}{24},$$

$$B_5(p_1) = \frac{80p_1^4 - 120p_1^2 + 9}{1920},\quad B_6(p_1) = \frac{48p_1^5 - 280p_1^3 + 259p_1}{5760},$$

$$B_7(p_1) = \frac{448p_1^6 - 2800p_1^4 + 3108p_1^2 - 225}{3\ 22560},$$

$$B_8(p_1) = \frac{64p_1^7 - 1008p_1^5 + 3948p_1^3 - 3229p_1}{3\ 22560},$$

$$B_9(p_1) = \frac{2304p_1^8 - 37632p_1^6 + 1\ 57920p_1^4 - 1\ 54992p_1^2 + 11025}{928\ 97280},$$

$$B_{10}(p_1) = \frac{1280p_1^9 - 42240p_1^7 + 4\ 21344p_1^5 - 13\ 82480p_1^3 + 10\ 57221p_1}{4644\ 86400}.$$

$$C_4(p) = \frac{6p^2 - 1}{12}, \qquad C_5(p) = \frac{2p^3 - 3p}{12},$$

$$C_6(p) = \frac{15p^4 - 30p^2 + 4}{360}, \quad C_7(p) = \frac{3p^5 - 20p^3 + 21p}{360},$$

$$C_8(p) = \frac{14p^6 - 105p^4 + 147p^2 - 18}{10080},$$

$$C_9(p) = \frac{6p^7 - 105p^5 + 455p^3 - 410p}{30240},$$

$$C_{10}(p) = \frac{15p^8 - 280p^6 + 1365p^4 - 1640p^2 + 192}{6\ 04800}.$$

$$D_4(p_1) = \frac{12p_1^2 - 5}{24}, \qquad D_5(p_1) = \frac{4p_1^3 - 3p_1}{24},$$

$$D_6(p_1) = \frac{240p_1^4 - 840p_1^2 + 259}{5760}, \quad D_7(p_1) = \frac{336p_1^5 - 1400p_1^3 + 777p_1}{40320},$$

$$D_8(p_1) = \frac{448p_1^6 - 5040p_1^4 + 11844p_1^2 - 3229}{3\ 22560},$$

$$D_9(p_1) = \frac{576p_1^7 - 7056p_1^5 + 19740p_1^3 - 9687p_1}{29\ 03040},$$

$$D_{10}(p_1) = \frac{11520p_1^8 - 2\ 95680p_1^6 + 21\ 06720p_1^4 - 41\ 47440p_1^2 + 10\ 57221}{4644\ 86400}.$$

The purpose of these present tables is to extend Lindow's tables which go only as far as the coefficients of the 6th differences in formulas (1)–(4) above, and give only 5D up to the 4th difference coefficients and 4D for the 5th and 6th difference coefficients. These present tables give the coefficients $A_r(p)$, $B_r(p_1)$, $C_s(p)$ and $D_s(p_1)$ at the same interval of .01 and the same range of p or p_1 from 0 to .25, as occur in Lindow, but for every difference as far as the 10th difference inclusive, and to 10 significant figures. This represents a considerable extension of Lindow's original tables and should be useful in many calculations of the first and second derivatives at intermediate points, which arise in numerical differentiation work and in the solution of first or second order differential equations, where Lindow's tables are entirely inadequate. Anyone who has performed numerical differentiation, especially for the second derivative, is aware of the great loss in significant figures due to the power of h in the denominator, as well as the subtraction of nearly equal terms in the numerator. For functions that are not determined by either measurement, observation, experiment or approximation, but which are mathematically defined so as to be computable to any degree of precision, the only limitation to the accuracy in numerical differentiation is due to the truncating error in using a finite number of terms of the formulas (1)–(4) and the computing error due to the carrying of a fixed number of decimal places or significant figures in the computation. Thus Lindow's original tables severely limit the accuracy attainable for mathematically defined functions because of those two mentioned reasons. These present 10S tables, as far as the 10th difference, are intended primarily to reduce considerably both truncation and computational errors.

The only other conveniently available tables for numerical differentiation at intermediate points, employing central-type differences, appear to be those of Davis [4] who gives, at intervals of .01, the Newton-Stirling and Newton-Bessel coefficients, but only as far as the 5th difference and only to 5D, and the Everett [2] coefficients as far as the 6th central difference and to 10D. However, all of Davis' coefficients are for the first derivative only. Davis' 10D Everett coefficients, giving 8-point accuracy, might suffice for most problems requiring just the first derivative. But Lindow must be extended anyhow to take care of the equally important second derivative, and thus we might as well be uniform in procedure and accuracy and use Lindow's arrangement for finding also the first derivative.

The present calculation was done originally using only a desk calculator, by exact computation of the numerators in the coefficients $A_r(p)$, $B_r(p_1)$, $C_s(p)$ and $D_s(p_1)$ in (1)–(4), and rounding only after division by the denominators. Thus all tabular entries should be correct to within a half-unit in the last (tenth) significant figure. All entries on the preliminary manuscript were rechecked by Norman Levine, employing the IBM 704, using double-precision (floating point) arithmetic. An additional differencing check was performed by hand upon the entries on the preliminary manuscript. Also a functional check was performed upon every entry in the typewritten final manuscript by computing the first and second derivatives of $(1 + x)^{10}$ for $x = 0(.01).25$ and $x = .50(.01).75$.

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2. E. T. Whittaker & G. Robinson, *Calculus of Observations*, 4th ed., Blackie & Son, London, 1944, p. 38–41.
3. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956, p. 66–67, 99–103.
4. H. T. Davis, *Tables of the Higher Mathematical Functions*, v. I, Principia Press, Bloomington, Ind., 1933, p. 140–147.

Newton-Stirling Coefficients for First Derivative

$$f'(x_0 \pm ph) = \frac{1}{h}\{\mu\delta_0 \pm p\delta_0^2 + A_3(p)\mu\delta_0^3 \pm A_4(p)\delta_0^4 + A_5(p)\mu\delta_0^5 \pm A_6(p)\delta_0^6 + A_7(p)\mu\delta_0^7 \pm A_8(p)\delta_0^8 + A_9(p)\mu\delta_0^9 \pm A_{10}(p)\delta_0^{10} + \cdots\}$$

p	$A_3(p)$	$A_4(p)$	$A_5(p)$	$A_6(p)$	$A_7(p)$	$A_8(p)$	$A_9(p)$	$A_{10}(p)$
.00	−0.16666 66667	0	.(1)33333 33333	0	−.(2)71428 57143	0	.(2)15873 01587	0
.01	− .16661 66667	−.(3)83316 66667*	.(1)33320 83375	.(3)11108 33342	−.(2)71399 40615	−.(4)17852 28195	.(2)15866 23715	.(5)31736 99340
.02	− .16646 66667	−.(2)16653 33333	.(1)33283 34000	.(3)22200 00267	−.(2)71311 92698	−.(4)35675 40349	.(2)15845 90549	.(5)63419 76753
.03	− .16621 66667	−.(2)24955 00000	.(1)33220 86708	.(3)33258 35358	−.(2)71166 18392	−.(4)53440 22919	.(2)15812 03443	.(5)94994 15729
.04	− .16586 66667	−.(2)33226 66667	.(1)33133 44000	.(3)44266 75200	−.(2)70962 26026	−.(4)71117 67362	.(2)15764 64654	.(4)12640 61059
.05	− .16541 66667	−.(2)41458 33333	.(1)33021 09375	.(3)55208 59375	−.(2)70700 27260	−.(4)88678 72628	.(2)15703 77337	.(4)15760 17187
.06	− .16486 66667	−.(2)49640 00000	.(1)32883 87333	.(3)66067 31467	−.(2)70380 37078	−.(3)10609 44766	.(2)15629 45548	.(4)18852 73177
.07	− .16421 66667	−.(2)57761 66667	.(1)32721 83375	.(3)76826 40058	−.(2)70002 73785	−.(3)12333 61387	.(2)15541 74242	.(4)21912 94946
.08	− .16346 66667	−.(2)65813 33333	.(1)32535 04000	.(3)87469 39733	−.(2)69567 59001	−.(3)14037 50765	.(2)15440 69266	.(4)24935 51653
.09	− .16261 66667	−.(2)73785 00000	.(1)32323 56708	.(3)97979 92075	−.(2)69075 17655	−.(3)15718 28281	.(2)15326 37362	.(4)27915 16224
.10	− .16166 66667	−.(2)81666 66667	.(1)32087 50000	.(2)10834 16667	−.(2)68525 77976	−.(3)17373 11310	.(2)15198 86161	.(4)30846 65895
.11	− .16061 66667	−.(2)89448 33333	.(1)31826 93375	.(2)11853 84209	−.(2)67919 71488	−.(3)18999 19462	.(2)15058 24178	.(4)33724 82733
.12	− .15946 66667	−.(2)97120 00000	.(1)31541 97333	.(2)12855 40693	−.(2)67257 32996	−.(3)20593 74832	.(2)14904 60813	.(4)36544 54165
.13	− .15821 66667	−.(1)10467 16667	.(1)31232 73375	.(2)13837 26078	−.(2)66539 00578	−.(3)22154 02242	.(2)14738 06340	.(4)39300 73504
.14	− .15686 66667	−.(1)11209 33333	.(1)30899 34000	.(2)14797 81520	−.(2)65765 15574	−.(3)23677 29486	.(2)14558 71910	.(4)41988 40465
.15	− .15541 66667	−.(1)11937 50000	.(1)30541 92708	.(2)15735 49479	−.(2)64936 22573	−.(3)25160 87570	.(2)14366 69537	.(4)44602 61683
.16	− .15386 66667	−.(1)12650 66667	.(1)30160 64000	.(2)16648 73813	−.(2)64052 69397	−.(3)26602 10953	.(2)14162 12101	.(4)47138 51223
.17	− .15221 66667	−.(1)13347 83333	.(1)29755 63375	.(2)17535 99881	−.(2)63115 07091	−.(3)27998 37791	.(2)13945 13335	.(4)49591 31089
.18	− .15046 66667	−.(1)14028 00000	.(1)29327 07333	.(2)18395 74640	−.(2)62123 89904	−.(3)29347 10167	.(2)13715 87824	.(4)51956 31721
.19	− .14861 66667	−.(1)14690 16667	.(1)28875 13375	.(2)19226 46749	−.(2)61079 75274	−.(3)30645 74336	.(2)13474 50996	.(4)54228 92498
.20	− .14666 66667	−.(1)15333 33333	.(1)28400 00000	.(2)20026 66667	−.(2)59983 23810	−.(3)31891 80952	.(2)13221 19111	.(4)56404 62222
.21	− .14461 66667	−.(1)15956 50000	.(1)27901 86708	.(2)20794 86751	−.(2)58834 99273	−.(3)33082 85308	.(2)12956 09261	.(4)58478 99611
.22	− .14246 66667	−.(1)16558 66667	.(1)27380 94000	.(2)21529 61360	−.(2)57635 68560	−.(3)34216 47560	.(2)12679 39357	.(4)60447 73769
.23	− .14021 66667	−.(1)17138 83333	.(1)26837 43375	.(2)22229 46953	−.(2)56386 01676	−.(3)35290 32960	.(2)12391 28121	.(4)62306 64666
.24	− .13786 66667	−.(1)17696 00000	.(1)26271 57333	.(2)22893 02187	−.(2)55086 71722	−.(3)36302 12084	.(2)12091 95078	.(4)64051 63599
.25	− .13541 66667	−.(1)18229 16667	.(1)25683 59375	.(2)23518 88021	−.(2)53738 54864	−.(3)37249 61054	.(2)11781 60546	.(4)65678 73652

* Throughout these tables the numbers in parenthesis denotes the number of 0's between the decimal point and the first significant digit.

Newton-Bessel Coefficients for First Derivative

$$f'(x_0 + ph) \equiv f'\left(x_0 + \frac{h}{2} \pm p_1 h\right) = \frac{1}{h}\{\delta_{\frac{1}{2}} \pm p_1\mu\delta_{\frac{1}{2}}^2 + B_3(p_1)\delta_{\frac{1}{2}}^3 \pm B_4(p_1)\mu\delta_{\frac{1}{2}}^4 + B_5(p_1)\delta_{\frac{1}{2}}^5 \pm B_6(p_1)\mu\delta_{\frac{1}{2}}^6 + B_7(p_1)\delta_{\frac{1}{2}}^7 \pm B_8(p_1)\mu\delta_{\frac{1}{2}}^8 + B_9(p_1)\delta_{\frac{1}{2}}^9 \pm B_{10}(p_1)\mu\delta_{\frac{1}{2}}^{10} + \cdots\}$$

p_1	$B_3(p_1)$	$B_4(p_1)$	$B_5(p_1)$	$B_6(p_1)$	$B_7(p_1)$	$B_8(p_1)$	$B_9(p_1)$	$B_{10}(p_1)$
.00	−.(1)41666 66667	0	.(2)46875 00000	0	−.(3)69754 46429	0	.(3)11867 94705	0
.01	−.(1)41616 66667	−.(2)20831 66667	.(2)46812 50417	.(3)44960 41675	−.(3)69658 11880	−.(3)10009 31675	.(3)11851 26451	.(4)22758 09919
.02	−.(1)41466 66667	−.(2)41653 33333	.(2)46625 06667	.(3)89891 66933	−.(3)69369 18650	−.(3)20011 29068	.(3)11801 23731	.(4)45498 34292
.03	−.(1)41216 66667	−.(2)62455 00000	.(2)46312 83750	.(2)13476 46036	−.(3)68887 97981	−.(3)29998 58274	.(3)11717 92660	.(4)68202 88662
.04	−.(1)40866 66667	−.(2)83226 66667	.(2)45876 06667	.(2)17955 00853	−.(3)68215 01927	−.(3)39963 86136	.(3)11601 43432	.(4)90853 90750
.05	−.(1)40416 66667	−.(1)10395 83333	.(2)45315 10417	.(2)22421 90104	−.(3)67351 03330	−.(3)49899 80622	.(3)11451 90302	.(3)11343 36154
.06	−.(1)39866 66667	−.(1)12464 00000	.(2)44630 40000	.(2)26874 23147	−.(3)66296 95781	−.(3)59799 11199	.(3)11269 51584	.(3)13592 42635
.07	−.(1)39216 66667	−.(1)14526 16667	.(2)43822 50417	.(2)31309 09839	−.(3)65053 93579	−.(3)69654 49207	.(3)11054 49636	.(3)15830 81596
.08	−.(1)38466 66667	−.(1)16581 33333	.(2)42892 06667	.(2)35723 60640	−.(3)63623 31677	−.(3)79458 68231	.(3)10807 10839	.(3)18056 76765
.09	−.(1)37616 66667	−.(1)18628 50000	.(2)41839 83750	.(2)40114 86707	−.(3)62006 65610	−.(3)89204 44478	.(3)10527 65586	.(3)20268 52628
.10	−.(1)36666 66667	−.(1)20666 66667	.(2)40666 66667	.(2)44480 00000	−.(3)60205 71429	−.(3)98884 57143	.(3)10216 48254	.(3)22464 34540
.11	−.(1)35616 66667	−.(1)22694 83333	.(2)39373 50417	.(2)48816 13376	−.(3)58222 45608	−.(2)10849 18879	.(4)98739 71820	.(3)24642 48828
.12	−.(1)34466 66667	−.(1)24712 00000	.(2)37961 40000	.(2)53120 40693	−.(3)56059 04957	−.(2)11801 92570	.(4)95005 46433	.(3)26801 22902
.13	−.(1)33216 66667	−.(1)26717 16667	.(2)36431 50417	.(2)57389 96911	−.(3)53717 86508	−.(2)12745 95828	.(4)90966 68144	.(3)28938 85357
.14	−.(1)31866 66667	−.(1)28709 33333	.(2)34785 06667	.(2)61621 98187	−.(3)51201 47407	−.(2)13680 58139	.(4)86628 37422	.(3)31053 66083
.15	−.(1)30416 66667	−.(1)30687 50000	.(2)33023 43750	.(2)65813 61979	−.(3)48512 64788	−.(2)14605 09473	.(4)81995 93076	.(3)33143 96367
.16	−.(1)28866 66667	−.(1)32650 66667	.(2)31148 06667	.(2)69962 07147	−.(3)45654 35634	−.(2)15518 80322	.(4)77075 11867	.(3)35208 09000
.17	−.(1)27216 66667	−.(1)34597 83333	.(2)29160 50417	.(2)74064 54047	−.(3)42629 76636	−.(2)16421 01733	.(4)71872 08092	.(3)37244 38378
.18	−.(1)25466 66667	−.(1)36528 00000	.(2)27062 40000	.(2)78118 24640	−.(3)39442 24037	−.(2)17311 05347	.(4)66393 33132	.(3)39251 20609
.19	−.(1)23616 66667	−.(1)38440 16667	.(2)24855 50417	.(2)82120 42582	−.(3)36095 33465	−.(2)18188 23433	.(4)60645 74986	.(3)41226 93613
.20	−.(1)21666 66667	−.(1)40333 33333	.(2)22541 66667	.(2)86068 33333	−.(3)32592 79762	−.(2)19051 88929	.(4)54636 57763	.(3)43169 97227
.21	−.(1)19616 66667	−.(1)42206 50000	.(2)20122 83750	.(2)89959 24251	−.(3)28938 56795	−.(2)19901 35468	.(4)48373 41157	.(3)45078 73301
.22	−.(1)17466 66667	−.(1)44058 66667	.(2)17601 06667	.(2)93790 44693	−.(3)25136 77263	−.(2)20735 97426	.(4)41864 19889	.(3)46951 65802
.23	−.(1)15216 66667	−.(1)45888 83333	.(2)14978 50417	.(2)97559 26119	−.(3)21191 72493	−.(2)21555 09946	.(4)35117 23124	.(3)48787 20914
.24	−.(1)12866 66667	−.(1)47696 00000	.(2)12257 40000	.(1)10126 30219	−.(3)17107 92221	−.(2)22358 08982	.(4)28141 13861	.(3)50583 87133
.25	−.(1)10416 66667	−.(1)49479 16667	.(3)94401 04167	.(1)10489 90885	−.(3)12890 04371	−.(2)23144 31327	.(4)20944 88295	.(3)52340 15366

NEWTON-STIRLING COEFFICIENTS FOR SECOND DERIVATIVE

$$f''(x_0 \pm ph) = \frac{1}{h^2}\{\delta_0^2 \pm p\mu\delta_0^3 + C_4(p)\delta_0^4 \pm C_5(p)\mu\delta_0^5 + C_6(p)\delta_0^6 \pm C_7(p)\mu\delta_0^7 + C_8(p)\delta_0^8 \pm C_9(p)\mu\delta_0^9 + C_{10}(p)\delta_0^{10} \pm \cdots\}$$

p	$C_4(p)$	$C_5(p)$	$C_6(p)$	$C_7(p)$	$C_8(p)$	$C_9(p)$	$C_{10}(p)$
.00	−.(1)83333 33333	0	.(1)11111 11111	0	−.(2)17857 14286	0	.(3)31746 03175
.01	−.(1)83283 33333	−.(2)24998 33333	.(1)11102 77819	.(3)58327 77786	−.(2)17842 56057	−.(3)13556 69646	.(3)31718 91760
.02	−.(1)83133 33333	−.(2)49986 66667	.(1)11077 78444	.(2)11662 22249	−.(2)17798 82619	−.(3)27104 36619	.(3)31637 60225
.03	−.(1)82883 33333	−.(2)74955 00000	.(1)11036 14486	.(2)17485 00202	−.(2)17725 97722	−.(3)40633 98661	.(3)31502 16691
.04	−.(1)82533 33333	−.(2)99893 33333	.(1)10977 88444	.(2)23297 78631	−.(2)17624 07613	−.(3)54136 54349	.(3)31312 74690
.05	−.(1)82083 33333	−.(1)12479 16667	.(1)10903 03819	.(2)29097 24826	−.(2)17493 21035	−.(3)67603 03508	.(3)31069 53156
.06	−.(1)81533 33333	−.(1)14964 00000	.(1)10811 65111	.(2)34880 06480	−.(2)17333 49221	−.(3)81024 47629	.(3)30772 76411
.07	−.(1)80883 33333	−.(1)17442 83333	.(1)10703 77819	.(2)40642 91784	−.(2)17145 05893	−.(3)94391 90286	.(3)30422 74152
.08	−.(1)80133 33333	−.(1)19914 66667	.(1)10579 48444	.(2)46382 49529	−.(2)16928 07255	−.(2)10769 63755	.(3)30019 81432
.09	−.(1)79283 33333	−.(1)22378 50000	.(1)10438 84486	.(2)52095 49207	−.(2)16682 71985	−.(2)12092 89839	.(3)29564 38639
.10	−.(1)78333 33333	−.(1)24833 33333	.(1)10281 94444	.(2)57778 61111	−.(2)16409 21230	−.(2)13408 08512	.(3)29056 91471
.11	−.(1)77283 33333	−.(1)27278 16667	.(1)10108 87819	.(2)63428 56431	−.(2)16107 78596	−.(2)14714 31378	.(3)28497 90905
.12	−.(1)76133 33333	−.(1)29712 00000	.(2)99197 51111	.(2)69042 07360	−.(2)15778 70139	−.(2)16010 70456	.(3)27887 93171
.13	−.(1)74883 33333	−.(1)32133 83333	.(2)97146 78194	.(2)74615 87189	−.(2)15422 24353	−.(2)17296 38221	.(3)27227 59712
.14	−.(1)73533 33333	−.(1)34542 66667	.(2)94937 84444	.(2)80146 70409	−.(2)15038 72161	−.(2)18570 47646	.(3)26517 57149
.15	−.(1)72083 33333	−.(1)36937 50000	.(2)92572 04861	.(2)85631 32812	−.(2)14628 46903	−.(2)19832 12242	.(3)25758 57237
.16	−.(1)70533 33333	−.(1)39317 33333	.(2)90050 84444	.(2)91066 51591	−.(2)14191 84317	−.(2)21080 46096	.(3)24951 36826
.17	−.(1)68883 33333	−.(1)41681 16667	.(2)87375 78194	.(2)96449 05436	−.(2)13729 22532	−.(2)22314 63918	.(3)24096 77804
.18	−.(1)67133 33333	−.(1)44028 00000	.(2)84548 51111	.(1)10177 57464	−.(2)13241 02047	−.(2)23533 81076	.(3)23195 67055
.19	−.(1)65283 33333	−.(1)46356 83333	.(2)81570 78194	.(1)10704 34119	−.(2)12727 65715	−.(2)24737 13638	.(3)22248 96400
.20	−.(1)63333 33333	−.(1)48666 66667	.(2)78444 44444	.(1)11224 88889	−.(2)12189 58730	−.(2)25923 78413	.(3)21257 62540
.21	−.(1)61283 33333	−.(1)50956 50000	.(2)75171 44861	.(1)11738 90342	−.(2)11627 28604	−.(2)27092 92989	.(3)20222 66995
.22	−.(1)59133 33333	−.(1)53225 33333	.(2)71753 84444	.(1)12246 07247	−.(2)11041 25147	−.(2)28243 75776	.(3)19145 16044
.23	−.(1)56883 33333	−.(1)55472 16667	.(2)68193 78194	.(1)12746 08584	−.(2)10432 00451	−.(2)29375 46042	.(3)18026 20653
.24	−.(1)54533 33333	−.(1)57696 00000	.(2)64493 51111	.(1)13238 63552	−.(3)98000 88649	−.(2)30487 23954	.(3)16866 96407
.25	−.(1)52083 33333	−.(1)59895 83333	.(2)60655 38194	.(1)13723 41580	−.(3)91460 69723	−.(2)31578 30617	.(3)15668 63438

Newton-Bessel Coefficients for Second Derivative

$$f''(x_0 + ph) \equiv f''\left(x_0 + \frac{h}{2} \pm p_1 h\right) = \frac{1}{h^2}\{\mu\delta_{\frac{1}{2}}^2 \pm p_1\delta_{\frac{1}{2}}^3 + D_4(p_1)\mu\delta_{\frac{1}{2}}^4 \pm D_5(p_1)\delta_{\frac{1}{2}}^5 + D_6(p_1)\mu\delta_{\frac{1}{2}}^6 \pm D_7(p_1)\delta_{\frac{1}{2}}^7 + D_8(p_1)\mu\delta_{\frac{1}{2}}^8 \pm D_9(p_1)\delta_{\frac{1}{2}}^9 + D_{10}(p_1)\mu\delta_{\frac{1}{2}}^{10} \pm \cdots\}$$

p_1	$D_4(p_1)$	$D_5(p_1)$	$D_6(p_1)$	$D_7(p_1)$	$D_8(p_1)$	$D_9(p_1)$	$D_{10}(p_1)$
.00	−0.20833 33333	0	.(1)44965 27778	0	−.(1)10010 54067	0	.(2)22761 07546
.01	− .20828 33333	−.(2)12498 33333	.(1)44950 69486	.(3)19267 36119	−.(1)10006 86896	−.(4)33361 66939	.(2)22752 14682
.02	− .20813 33333	−.(2)24986 66667	.(1)44906 95111	.(3)38513 89156	−.(2)99958 55675	−.(4)66682 54746	.(2)22725 36636
.03	− .20788 33333	−.(2)37455 00000	.(1)44834 06153	.(3)57718 77025	−.(2)99775 06455	−.(4)99921 87205	.(2)22680 75040
.04	− .20753 33333	−.(2)49893 33333	.(1)44732 05111	.(3)76861 19644	−.(2)99518 30669	−.(3)13303 89393	.(2)22618 32613
.05	− .20708 33333	−.(2)62291 66667	.(1)44600 95486	.(3)95920 39931	−.(2)99188 41434	−.(3)16599 31329	.(2)22538 13163
.06	− .20653 33333	−.(2)74640 00000	.(1)44440 81778	.(2)11487 56480	−.(2)98785 55610	−.(3)19874 39529	.(2)22440 21580
.07	− .20588 33333	−.(2)86928 33333	.(1)44251 69486	.(2)13370 62617	−.(2)98309 93792	−.(3)23125 10452	.(2)22324 63839
.08	− .20513 33333	−.(2)99146 66667	.(1)44033 65111	.(2)15239 16196	−.(2)97761 80311	−.(3)26347 42301	.(2)22191 46993
.09	− .20428 33333	−.(1)11128 50000	.(1)43786 76153	.(2)17091 11707	−.(2)97141 43218	−.(3)29537 35317	.(2)22040 79174
.10	− .20333 33333	−.(1)12333 33333	.(1)43511 11111	.(2)18924 44444	−.(2)96449 14286	−.(3)32690 92063	.(2)21872 69587
.11	− .20228 33333	−.(1)13528 16667	.(1)43206 79486	.(2)20737 10598	−.(2)95685 28995	−.(3)35804 17718	.(2)21687 28508
.12	− .20113 33333	−.(1)14712 00000	.(1)42873 91778	.(2)22527 07360	−.(2)94850 26527	−.(3)38873 20359	.(2)21484 67277
.13	− .19988 33333	−.(1)15883 83333	.(1)42512 59486	.(2)24292 33022	−.(2)93944 49752	−.(3)41894 11248	.(2)21264 98296
.14	− .19853 33333	−.(1)17042 66667	.(1)42122 95111	.(2)26030 87076	−.(2)92968 45217	−.(3)44863 05120	.(2)21028 35021
.15	− .19708 33333	−.(1)18187 50000	.(1)41705 12153	.(2)27740 70312	−.(2)91922 63136	−.(3)47776 20463	.(2)20774 91961
.16	− .19553 33333	−.(1)19317 33333	.(1)41259 25111	.(2)29419 84924	−.(2)90807 57373	−.(3)50629 79804	.(2)20504 84665
.17	− .19388 33333	−.(1)20431 16667	.(1)40785 49486	.(2)31066 34603	−.(2)89623 85431	−.(3)53420 09988	.(2)20218 29721
.18	− .19213 33333	−.(1)21528 00000	.(1)40284 01778	.(2)32678 24640	−.(2)88372 08435	−.(3)56143 42458	.(2)19915 44748
.19	− .19028 33333	−.(1)22606 83333	.(1)39754 99486	.(2)34253 62027	−.(2)87052 91114	−.(3)58796 13532	.(2)19596 48385
.20	− .18833 33333	−.(1)23666 66667	.(1)39198 61111	.(2)35790 55556	−.(2)85667 01786	−.(3)61374 64683	.(2)19261 60288
.21	− .18628 33333	−.(1)24706 50000	.(1)38615 06153	.(2)37287 15917	−.(2)84215 12336	−.(3)63875 42808	.(2)18911 01118
.22	− .18413 33333	−.(1)25725 33333	.(1)38004 55111	.(2)38741 55804	−.(2)82697 98203	−.(3)66295 00507	.(2)18544 92534
.23	− .18188 33333	−.(1)26722 16667	.(1)37367 29486	.(2)40151 90008	−.(2)81116 38350	−.(3)68629 96348	.(2)18163 57183
.24	− .17953 33333	−.(1)27696 00000	.(1)36703 51778	.(2)41516 35520	−.(2)79471 15254	−.(3)70876 95138	.(2)17767 18691
.25	− .17708 33333	−.(1)28645 83333	.(1)36013 45486	.(2)42833 11632	−.(2)77763 14872	−.(3)73032 68190	.(2)17356 01652