On the Enumeration of Majority Games

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1. Introduction. This paper presents a combinatorial method for the enumeration of all strong weighted majority games, and an arithmetical method for the enumeration of the games having "homogeneous" weights in the sense of von Neumann and Morgenstern [1]. In contrast with the combinatorial method of von Neumann and Morgenstern [1], both of these methods (a) do not generate extra isomorphic copies of the same game, but (b) do generate some unwanted objects (so that tests are required), and (c) are recursive on the number of players. The known (von Neumann and Morgenstern [1], Gurk and Isbell [2]) complete list of 30 strong simple games on $n \leqq 6$ players (21 of which are majority games) is supplemented by a complete list of the 114 strong majority games with 7 non-dummy players.

A *strong simple game* (for the rest of this paper, a game) on a finite set N of players is (1) intuitively, a scheme for distributing power to coalitions of players, all-or-none (this is the sense of "simple"), and so that if the players are partitioned into two parties, one must win (this is the sense of "strong"). (2) Precisely, it is a set of subsets of N called *winning sets*, such that (a) any set containing a winning set is winning, and (b) for any $S \subset N$, either S or N-S is winning but not both. (3) In this paper, we shall identify two games if there is a one-to-one correspondence between their players identifying their families of winning sets; to name a more definite object one must say, not "the game G", but "the game G with ordered players $P_1, \cdots, P_n$", or a similar phrase. Beyond this, note that an n-player game (e.g. the 435-player game of the House of Representatives) may be converted into an $(n + k)$-player game by adjoining k "voteless" players or *dummies*. The phrase "an n-player game G" does not here imply that all n players are non-dummies. (In the first part of the argument we need games with dummies; afterward we shall exclude them.)

A *majority game* is a game for which one can assign numerical weights $w_1, \cdots, w_n$ to the players so that the winning sets are precisely those sets which have more than half the total weight. Some of the w_i may be negative or zero; it is easy to see that the corresponding players must be dummies. (Since every superset of a winning set wins, a player can have negative weight w_i only if $| w_i |$ is so small that it makes no difference.) Given a game G, the question whether G is a majority game is effectively decidable, since it turns on a finite system of linear inequalities. No better method than the obvious ones is known. It is clear that every majority game has non-negative integral weights—even positive integral weights, for any weights may be assigned to the dummies provided the weights of the other players are made large enough.

The method described below for enumerating all majority games depends on the determination of the game with ordered players with weights $(w_0, w_1, \cdots, w_n)$ by the systems $(w_0 + w_1, w_2, \cdots, w_n)$ and $(w_0 - w_1, w_2, \cdots, w_n)$. The choice

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of weights and orderings gives no trouble. However, the method does not generate weights, but only a combinatorial description of the game. There is an obvious formula for weights for the large game in terms of "suitable" weights for the small games, but we have no version of the method which does not involve both the solution of systems of linear inequalities and the examination of systems which turn out to be inconsistent. There is, however, a fairly simple combinatorial criterion which screens out all the inconsistent systems through $n = 7$.

Weights $(w_1, \cdots, w_n)$ for a game are called *efficient* if every winning set S contains a subset which wins by exactly one vote. (This is equivalent to saying that every minimal winning set wins by one vote. Von Neumann and Morgenstern [1] call these "homogeneous" weights, but this term seems likely to lead to confusion.) It is known [3] that efficient weights w_i for a game with ordered players are unique; they are non-negative integers; for any other non-negative integral weights v_i belonging to the same game with ordered players, $w_i \leqq v_i$ for all i; and when the non-zero weights w_i are arranged in non-decreasing order, they grow no faster than the Fibonacci numbers. We call a non-decreasing sequence of positive integers $(w_1, \cdots, w_k)$ an *efficient sequence* if it is an initial segment of some non-decreasing efficient game weights $(w_1, \cdots, w_k, \cdots, w_n)$. We find a characterization of these and a recursive arithmetical method for generating them, which generates nothing but efficient sequences. It is necessary, of course, to select those which are actually game weights; for $n = 7$ there are 23 games in the 99 efficient sequences. The test for this is simple, and the whole computation is far easier than the other one described above.

It is known [3] that for $n \geqq 4$ there are at least 2^{n-4} efficient weighted majority games with n non-dummy players. We obtain the upper bound $(n - 1)!$—actually an upper bound for the number of efficient sequences. There is even more room between the known bounds for general majority games with n non-dummy players; we find there are more than 2^n for $n \geqq 8$, but no upper bound is known short of 2^{2^n} (the number of sets of sets of players).

A little experimentation with the methods presented here will quickly show the desirability of an arithmetical method for enumerating all majority games. This would presumably involve a unique choice of weights for each game. For $n \leqq 7$ each n-player majority game has *minimum* non-negative integral weights w_i. (That is, for any non-negative integral v_i defining the same game and order, $w_i \leqq v_i$ for each i.) Such weights are clearly unique. However, they do not always exist, as we show with a certain 12-player game.

To put the classes of games in perspective, consider the author's manuscript list of 559 non-majority 7-player games without dummies, thought to be free from duplications. It was computed by the von Neumann-Morgenstern method some years ago and 109 of the 114 majority games were found.

2. The operators H_1 and H_2. We wish to formalize the construction described by the symbol $(w_0 - w_1, w_2, \cdots, w_n)$. A *motion* of a game is a permutation of its players taking winning sets to winning sets. Under the group of motions of the game the players sweep through transitivity sets which we call *roles.* A *court* is a majority game with a distinguished role. A *natural ordering* of the players of a majority game is an ordering $(P_1, \cdots, P_n)$ such that the game has weights w_i such that $w_1 \geqq$

$w_2 \geqq \cdots \geqq w_n$. However, a *natural ordering* of the players of a court is an ordering $(P_1, \cdots, P_n)$ such that P_1 is an element of the distinguished role and for some weights w_i, $w_2 \geqq \cdots \geqq w_n$.

LEMMA 1. *If $(w_i, \cdots, w_n)$ and $(v_1, \cdots, v_n)$ are two systems of weights for a majority game G with ordered players $P_1, \cdots, P_n$, and $w_1 \geqq \cdots \geqq w_n$, and α is a permutation of the integers from 1 to n such that $v_{\alpha(1)} \geqq \cdots \geqq v_{\alpha(n)}$, then $P_i \rightarrow P_{\alpha(i)}$ is a motion of G.*

Proof. If $w_i \geqq w_j$ and S is a winning set containing j but not i then replacing j by i in S yields another winning set, since it does not decrease the weight. If also $v_j \geqq v_i$ then i and j may be exchanged (by a motion leaving all other players fixed). Then the lemma follows by induction.

For any majority game G of $n + 1 \geqq 2$ players, we define an n-player court $H_1(G)$ and an n-player game $H_2(G)$ as follows. It suffices to define games with ordered players, H with players $Q_1, \cdots, Q_n$, H' with players $R_1, \cdots, R_n$, specifying that $H_1(G)$ consists of the game H with the role of Q_1 distinguished and $H_2(G)$ is the game H'. Let $(w_0, \cdots, w_n)$ be any system of weights for G, arranged so that $w_0 \geqq \cdots \geqq w_n$. We define H by its weights $(w_0 - w_1, w_2, \cdots, w_n)$, and H' by its weights $(w_0 + w_1, \cdots, w_n)$.

LEMMA 2. *$H_1(G)$ and $H_2(G)$ are determined by the game G; and together they determine G.*

Proof. The first half is a routine analysis of definitions, and we omit it. For the converse, let G and G' be majority games with $H_1(G) = H_1(G') = H_1$ and $H_2(G) = H_2(G') = H_2$. Naturally order the player $Q_1, \cdots, Q_n$ of the court H_1 and the players $R_1, \cdots, R_n$ of the court H_2. Since $H_1 = H_1(G)$, there exist weights $w_0 \geqq \cdots \geqq w_n$ for G and a one-to-one correspondence φ mapping the last $n - 1$ players $P_2, \cdots, P_n$, of G upon $n - 1$ of the players of H_1 so that the omitted player of H_1 is a member of the distinguished role and the numbers $w_0 - w_1, w_2, \cdots, w_n$, assigned according to φ, are weights for H_1. By Lemma 1 and the definition of a role, there is a motion α of H_1 such that $\alpha(\varphi(P_j)) = Q_j$ for $j = 2, \cdots, n$. By the same device we may regularize the correspondences of G to H_2, G' to H_1 and G' to H_2, so that players with like indices from 2 to n correspond. For any set S of indices in $\{0, 1, \cdots, n\}$, by considering the four possible values of $S \cap \{0, 1\}$, one can give necessary and sufficient conditions for $\{P_i \mid i \in S\}$ to be a winning set in G, in terms of H_1, H_2, and S. The same conditions are necessary and sufficient for $\{P_i' \mid i \in S\}$ to win in G'. Therefore the correspondence $P_i \leftrightarrow P_i'$ is an isomorphism between G and G'.

We call an n-player court H_1 and n-player majority game H_2 *compatible* if there exists a majority game G such that $H_1(G) = H_1$ and $H_2(G) = H_2$. Evidently one can define a combinatorial scheme on the lines of a game, which must be the game G if G exists. The necessary and sufficient condition for compatibility is then the solvability of the system of linear inequalities defining weights for G, together with the axioms for a game. Reformulating, one has

THEOREM 1. *An n-player court H_1 with naturally ordered players Q_i and an n-player majority game H_2 with naturally ordered players R_i are compatible if and only if both*

(a) *whenever $\{Q_i \mid i \in S\}$ is a winning set in H_1 containing the player Q_1, then $\{R_i \mid i \in S \text{ and } i \neq 2\}$ is a winning set in H_2, and*

(b) *there exist weights u_i for H_1, v_i for H_2, such that $u_i = v_i$ for $i = 2, \cdots, n$.*

Proof. The necessity of (a) and (b) is obvious. Conversely, suppose (a) and (b) are valid. We may replace each u_i with max $(u_i, 0)$, and v_j with max $(v_j, 0)$, without affecting either game (since players with negative weight are dummies) or the validity of (a) and (b). We claim the numbers $(v_1 + u_1)/2$, $(v_1 - u_1)/2$, $u_2, \cdots, u_n$ are weights for a game G with players $P_0, P_1, \cdots, P_n$. Call the subsets of $N = \{P_0, \cdots, P_n\}$ having more than half the total weight *winning sets*. For any $S \subset N$, not both S and N-S are winning. One must win unless they have the same weight; but in that case it is clear that two complementary sets in H_1 or in H_2 would have the same weight, which is impossible. To prove that any superset of a winning set wins, it suffices to adjoin players one at a time. Adjoining a player other than P_1 cannot decrease the weight. Next suppose S is a winning set containing P_0 and not P_1. Then the set $\{Q_j \mid j = 1 \text{ or } j \geqq 2 \text{ and } P_j \in S\}$ is a winning set in H_1, by computation; the set $\{R_j \mid j = 1 \text{ or } j \geqq 2 \text{ and } P_j \in S\}$ is a winning set in H_2, by (a) and the fact that supersets of winning sets win in H_2; and $S \cup \{P_1\}$ wins in G, by computation. If S is a winning set containing neither P_0 nor P_1, then N-S contains both but fails to win; hence N-S-$\{P_1\}$ cannot win either, and $S \cup \{P_1\}$ is a winning set. Thus G is a game, and therefore a majority game. To show that $H_1 = H_1(G)$ and $H_2 = H_2(G)$ it remains to prove that the ordering $(P_0, \cdots, P_n)$ is natural. Replacing P_1 with P_0 never turns a winning set into a losing set, since $u_1 \geqq 0$. If S is a winning set containing P_2 and not P_1, and furthermore S contains P_0, then $S \cup \{P_1\} - \{P_2\}$ wins in virtue of (a). Then the same is true if P_0 is not in S, by passage to the complement. For $i = 2, \cdots, n - 1$, the natural ordering of the players of H_1 implies that replacing P_{i+1} by P_i never turns a winning set into a losing set. Therefore we may rearrange the given weights for G into weights $w_0 \geqq \cdots \geqq w_n$ by successive transpositions, and the proof is complete.

The author's experience suggests that (b) of Theorem 1 may as well be ignored in computation; this is discussed in Section 4 below.

COROLLARY 1. *For every n-player court H_1 there exists an n-player majority game compatible with H_1.*

In fact there exists an n-player majority game compatible with all n-player courts. One system of weights for it is $(1, 0, \cdots, 0)$; but as we noted before, any weights may be assigned to the $n - 1$ dummies provided the first weight is large enough.

COROLLARY 2. *Let $f(n)$ be the number of different n-player majority games, $g(n)$ the number of these which have no dummies. Then $f(n + 1) \geqq 2f(n) - 1$ and $g(n + 1) \geqq 2g(n) - 1$. For $n > 6$, $f(n) > 2^n$, and for $n > 7$, $g(n) > 2^n$.*

Proof. Every majority game has at least two roles, with the exception of the games having weights $(1, 1, \cdots, 1)$, of which there is one for each odd number of players. In any case the number of n-player courts is at least $2f(n) - 1$ and the number without dummies is at least $2g(n) - 1$. Corollary 1 and Lemma 2 show that there are at least as many corresponding $n + 1$-player majority games; and clearly G cannot have any dummies when $H_1(G)$ has none. Then if $f(n) > 2^n$, it follows that $f(n + 1) > 2^{n+1}$; and the list at the end of the paper shows that $f(7) = 135 > 2^7$. The same induction applies for g; and though $g(7)$ is only 114, the list shows there are over 256 corresponding courts.

The true order of f and g is not known. No respectable upper bound is known, though one can make slight improvements on 2^{2^n}, which is the number of sets of sets of players. (Of course, the lower bound 2^n may be as wide of the mark.)

3. Efficient sequences. In this section we shall consider only games without dummies; it is also convenient to reverse our convention as to ordering. Let a *sequence* be a finite non-decreasing sequence of positive integers. For any sequence $w = (w_1, \cdots, w_n)$, let $w(N)$ be the total $\Sigma_{i=1}^{i=n} w_i$; for any subset S of $N = \{1, \cdots, n\}$, let $w(S)$ be $\Sigma_{i \in S} w_i$. Then $w(S)$ is called the *weight* of S.

Weights (w_i) for a majority game are *efficient* provided $w(N)$ is an odd number $2p - 1$ and every subset of weight greater than p contains a subset of weight exactly p. These conditions automatically make (w_i) weights for a game, but the absence of zeros does not imply the absence of dummies. (Example: 1, 2, 2, 2.) We are assuming there are no dummies; that is, every index is in some set of weight p. Then it is known [2, 3] that the efficient weights are uniquely determined by the game, namely as minimum integral weights, and that their growth is restricted by the inequality

$$(*) \qquad w_i \leqq 1 + \Sigma_{j=1}^{i-2} w_j .$$

For any sequence w, let us call the non-negative integer p *acceptable* for w provided every set of weight $>p$ contains a subset of weight exactly p. Clearly 0 is acceptable for each w, no integer from 1 to $w_n - 1$ can be acceptable, a sum of acceptable numbers is acceptable, and all sufficiently large numbers are acceptable.

LEMMA 3. *$p > 0$ is acceptable for $(w_1, \cdots, w_{n+1})$ if and only if both p and $p - w_{n+1}$ are acceptable for $(w_1, \cdots, w_n)$.*

Proof. If p is acceptable for the long sequence it is clearly acceptable for a part of the sequence. If $p - w_{n+1}$ is not acceptable for $(w_1, \cdots, w_n)$, one has a set S such that $w(S) > p - w_{n+1}$ and S contains no subset of weight exactly $p - w_{n+1}$. Since the w_i are increasing, one can assure that $w(S) < p$. Then $S \cup \{n + 1\}$ can contain no subset of weight p, a contradiction. The converse is obvious.

Now let us call p *admissible* for w if both p and $w(N) + 1 - p$ are acceptable; p is *proper* if $1 \leqq p \leqq w(N)$. Let the sequence w be *efficient* provided it satisfies $(*)$ and some proper p is admissible for w.

THEOREM 2. *A sequence w is an efficient sequence if and only if w is an initial segment of the (increasing) weights for some efficient weighted majority game.*

Proof. If w is an efficient sequence with proper admissible

$$p \leqq p' = w(N) + 1 - p,$$

then $p + p'$ is acceptable, hence admissible, for $(w_1, \cdots, w_n, p, p')$, by four applications of the lemma. To show that every index is in a set of weight $p + p'$ it suffices, by efficiency, to show that this is true of the least index. Suppose i is the least index actually used; if $i > 1$ then by $(*)$, w_i is $\leqq$ the sum of the preceding weights, and i can be replaced by a set of smaller indices without diminishing the weight, which reduces to a contradiction.

For the converse, $(*)$ is established in a previous paper [3], and the rest is clear from the lemma. Note that $(*)$ assures that for $n > 1$, one of p and $w(N) + 1 - p$ is $>w_n$, so that subtracting w_n leaves a proper integer.

THEOREM 3. *Let* $(w_1, \cdots, w_n)$ *be efficient. Then* $(w_1, \cdots, w_n, w_{n+1})$ *is efficient is and only if* (1) $w_{n+1} \geqq w_n$, (2) $w_{n+1} \leqq 1 + \Sigma_{i=1}^{n-1} w_i$, *and* (3) $w_{n+1} = h - k$ *for some h and k which are admissible for* $(w_1, \cdots, w_n)$. *A proper p is admissible for*

$$(w_1, \cdots, w_{n+1})$$

if and only if both p and $p - w_{n+1}$ *are admissible for* $(w_1, \cdots w_n)$.

Proof. The last statement follows from the lemma and the rest follows from that.

Note that since 0 is always admissible, the possible values of w_{n+1} always include the proper admissible numbers. There are not always at least two of these. However, it is known [3] that there are at least 2^{n-4} n-player games with efficient weights, for $n \geqq 4$. From Theorem 3 and the proof of Theorem 2 we see that the number of efficient sequences of length n is an increasing function of n and lies between the numbers of efficient game weights of length n and of length $n + 2$. In the other direction we have

THEOREM 4. *For an efficient sequence* $(w_1, \cdots, w_n)$ *there are at most n values of* w_{n+1} *which make* $(w_1, \cdots w_{n+1})$ *efficient. Hence the number of efficient sequences of length n is between* 2^{n-4} *and* $(n - 1)!$.

Proof. There are at most n acceptable integers s_k between w_n and $w(N) + 1 - w_n$, namely, w_n, $w_n + w_{n-1}, \cdots, w(N)$; for clearly if this sequence skips over p, $s_k < p < s_{k+1}$, then at step $k + 1$ we have a set of weight $>p$ which drops below p if its smallest element is removed. Now w_{n+1} itself need not be acceptable, but $w(N) + 1 - w_{n+1}$ is. For there are p and p' acceptable for $(w_1, \cdots, w_n)$ such that $p + p' = w(N) + w_{n+1} + 1$. By the lemma, both $p - w_{n+1}$ and $p' - w_{n+1}$ are acceptable for $(w_1, \cdots, w_p)$; hence so is their sum, which is $w(N) + 1 - w_{n+1}$. This establishes the inductive step, and a glance at the case $n = 1$ establishes the upper bound $(n - 1)!$. The lower bound 2^{n-4} is established in [3] for $n \geqq 4$, and the cases $n = 1, 2, 3$ are easily verified.

4. The enumeration. A method for generating all majority games of $n + 1$ players from a list of all n-player majority games is as follows. List all the courst (H, r), H an n-player majority game and r a role of H. For each (H, r) there is at least one compatible game H_2, as pointed out in Corollary 1 to Theorem 1. All compatible games must be determined. In some cases (e.g. when H_2 has weights $(1, 0, \cdots, 0)$) one can write down weights for the corresponding $n + 1$-player game G by rote. In general, however, one must screen the possible values of H_2 and solve the systems of linear inequalities defining weights for G. For hand computation it seems best to apply the necessary condition (a) of Theorem 1, and for each H_2 satisfying that condition, to attempt to solve the inequalities for weights for G with the side conditiont $w_i \geqq 0$. In a run such as the enumeration of the 7-player majority games one develops subroutines; for example, there are different courts H_1, H_1', such that H_2 is compatible with H_1 if and only if H_2 is compatible with H_1', and one may make mental lists of the corresponding classes of games H_2.

It appears to be feasible though far from routine to code this method for automatic computation. No attempt has been made. For automatic computation it might be preferable simply to ignore the combinatorial conditions (a) and deal entirely in inequalities. However, (a) is a highly effective screen; for $n \leqq 6$ every pair (H_1, H_2) satisfying (a) is compatible.

The method expounded by von Neumann and Morgenstern [1] for enumerating all strong simple games involves, necessarily, a listing of winning sets (no other way is known for describing these games in general); and it also involves obtaining and then eliminating multiple isomorphic copies of each game. No other method is known for this problem. However, for generating a list of majority games, the present method is far superior. Note that if one were to employ the method of von Neumann and Morgenstern for this special purpose it would still be necessary to examine the inequalities determining whether each game has weights.

The recursive enumeration of all efficient sequences involves additional information which is most easily generated from the beginning, the efficient sequence (w_1), where $w_1 = 1$. One may lay out three columns on a sheet of paper, the first giving the sequence $(w_1, \cdots, w_n)$, the second the list of all integers from 0 to $w(N)$ which are admissible for w, and the third the list of all differences of integers in the second column which satisfy the inequalities (1) and (2) of Theorem 3. The numbers in the third column are precisely the values of w_{n+1} which can be used for an extension $(w_1, \cdots, w_n, w_{n+1})$; Theorem 3 tells how to compute the second column for $(w_1, \cdots, w_{n+1})$, using the entries in the first column of this row and in the second column of the earlier row; then the new third column is computed from the new first and second columns. Coding this computation should be routine; the only complication is the irregularity of the size of the clumps of numbers.

It would certainly be desirable to develop a method for the enumeration of all majority games which, like the method for efficient sequences, should deal directly with arithmetic properties of the weights. A prerequisite would seem to be a scheme for assigning unique weights (not necessarily integers) to each game. The obvious idea of taking minimal integral weights is not enough, because these are not unique. To see this, consider the two sets of weights (1, 3, 5, 6, 8, 11, 12, 23, 28, 31, 31, 38) and (1, 3, 5, 7, 8, 11, 12, 23, 28, 31, 31, 37). One may verify that these weights determine the same game. One then proves twelve lemmas which show that any weights for this game are at least as large as (1, 3, 5, 6, 8, 11, 12, 23, 28, 31, 31, 37). First lemma: the first player is not a dummy; hence his weight in integers must be at least 1. The proof that the second weight must be at least 3 involves finding combinations of 1, 3, 11, and 12. Having all this, one observes that the twelve minimum values do not form weights for any game, since they permit a tie; there are no integers between them and the two sets of weights, and therefore both are minimal.

It can be shown by similar arguments that the 135 sets of weights listed below are all minimum. (Criteria from a previous paper [3] aid in most of the verifications.) In particular, no two of the 135 games are isomorphic. Comparable care has been taken to assure that no game is omitted. The list for $n \leqq 5$ is taken from von Neumann and Morgenstern [1], and the list for $n = 6$ from Gurk and Isbell [2].

The 15 efficient weighted majority games of less than seven players

1	11113	111114	111334
111	11122	111123	112225
1112	11223	111224	112335
11111	111112	111233	

The 6 nonefficient weighted majority games of less than seven players

111222	112234	122334	122345
112223	122233		

The 23 seven-player efficient weighted majority games

1111111	1111223	1112226	1122227
1111113	1111234	1112244	1122355
1111115	1111335	1112336	1122557
1111122	1111344	1112446	1123338
1111124	1111445	1113337	1123558
1111133	1112222	1113447	

The 91 seven-player nonefficient weighted majority games

1111223	1123345	1223344	1234557
1111333	1123356	1223346	1234568
1112224	1123446	1223348	1234579
1112233	1123457	1223357	1244567
1112235	1133344	1223445	1244679
1112334	1133355	1223447	2222333
1112345	1133445	1223449	2223334
1113335	1133456	1223456	2223345
1113346	1133467	1223458	2223367
1122223	1133557	1223467	2233344
1122225	1133568	1223559	2233445
1122234	1222233	1224457	2233456
1122236	1222235	1224558	2233478
1122245	1222334	1224569	2234455
1122333	1222336	1233345	2234556
1122335	1222345	1233446	2234567
1122337	1222347	1233455	2234589
1122344	1222356	1233457	2334456
1122346	1222455	1233468	2344567
1122445	1222556	1233556	2345678
1122447	1222567	1233567	3334455
1122456	1223335	1233578	3345567
1123334	1223337	1234456	

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1. J. Von Neumann & O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press, New Jersey, 1944.

2. H. M. Gurk & J. R. Isbell, "Simple solutions," to appear in *Annals of Mathematics Studies, Contributions to the Theory of Games*, V. 4, Princeton Univ. Press, New Jersey.

3. J. R. Isbell, "A class of majority games," *Quart. Jn. of Math.*, Ser. 2, V. 7, 1956, p. 183–187.