# Explicit Formulae for 25 of the Associated Legendre Functions of the Second Kind

**By Dietrich Suschowk**

Since the present writer knows of no place where to find explicit expressions for some more of Legendres associated functions of the second kind than are given for instance in [1] or [2], he hopes the small collection of formulae at the end of this note might be of some use to others.

The associated Legendre functions $Q_n^k(x)$ of the second kind for $|x| > 1$ and nonnegative integer values of $n$, $k$ can be defined by [1]

$$Q_n^k(x) = \frac{(x^2-1)^{k/2}}{2^n n!} \frac{d^{n+k}}{dx^{n+k}}\left[(x^2-1)^n \log\frac{x+1}{x-1}\right] - \frac{1}{2}\frac{d^k}{dx^k}\left[\log\frac{x+1}{x-1}\cdot\frac{d^n}{dx^n}(x^2-1)^n\right].$$

These functions can be written as follows:

$$(1) \qquad Q_n^k(x) = \begin{cases} \dfrac{(x^2-1)^{-k/2}}{(n-k)!}\left[\frac{1}{2} A_{nk}^{(1)}(x) \log\dfrac{x+1}{x-1} + A_{nk}^{(2)}(x)\right] & (n \geqq k) \\ \dfrac{(-1)^k}{(x^2-1)^{k/2}} A_{nk}^{(2)}(x) & (n < k) \end{cases}$$

where $A_{nk}^{(i)}(x)$ $(i = 1, 2)$ is a polynomial and

$$\text{degree of } A_{nk}^{(1)}(x) = \begin{cases} n+k & \text{for } n \geqq k \\ \text{no degree for } n < k & (A_{nk}^{(1)}(x) \equiv 0) \end{cases}$$

$$\text{degree of } A_{nk}^{(2)}(x) = \begin{cases} n+k-1 & \text{for } n \geqq k \\ k-n-1 & \text{for } n < k \end{cases}$$

(2) If the degree of $A_{nk}^{(i)}(x)$ is even (uneven), $A_{nk}^{(i)}(x)$ contains only even (uneven) powers of $x$.

(3) The coefficients of $A_{nk}^{(i)}(x)$ are integers.

(4) For each fixed triple $(n, k, i)$ and $n \geqq k$ the coefficients of $A_{nk}^{(i)}(x)$ have alternating signs and the coefficient of the highest power in $A_{nk}^{(1)}(x)$ is always positive and in $A_{nk}^{(2)}(x)$ always negative. For $n < k$ the coefficients of $A_{nk}^{(2)}(x)$ are all positive.

(5) If $n \geqq k > 0$, $A_{nk}^{(1)}(x)$ has a zero of $n$th order for $x = 1$.

The formulae are as follows: Let $\tau = \frac{1}{2}\log(x+1)/(x-1)$. Then

$$Q_0^0(x) = \tau$$

$$Q_1^0(x) = x\tau - 1$$

$$Q_2^0(x) = \tfrac{1}{2}[(3x^2-1)\tau - 3x]$$

$$Q_3^0(x) = \frac{1}{3!}[(15x^3 - 9x)\tau - 15x^2 + 4]$$

$$Q_4^0(x) = \frac{1}{4!}[(105x^4 - 90x^2 + 9)\tau - 105x^3 + 55x]$$

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$$Q_0^1(x) = -\frac{1}{\sqrt{x^2-1}}$$

$$Q_1^1(x) = \frac{1}{\sqrt{x^2-1}}[(x^2-1)\tau - x]$$

$$Q_2^1(x) = \frac{1}{\sqrt{x^2-1}}[(3x^3-3x)\tau - 3x^2 + 2]$$

$$Q_3^1(x) = \frac{1}{2\sqrt{x^2-1}}[(15x^4-18x^2+3)\tau - 15x^3 + 13x]$$

$$Q_4^1(x) = \frac{1}{3!\sqrt{x^2-1}}[(105x^5-150x^3+45x)\tau - 105x^4 + 115x^2 - 16]$$

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$$Q_0^2(x) = \frac{2x}{x^2-1}$$

$$Q_1^2(x) = \frac{2}{x^2-1}$$

$$Q_2^2(x) = \frac{1}{x^2-1}[(3x^4-6x^2+3)\tau - 3x^3 + 5x]$$

$$Q_3^2(x) = \frac{1}{x^2-1}[(15x^5-30x^3+15x)\tau - 15x^4 + 25x^2 - 8]$$

$$Q_4^2(x) = \frac{1}{2(x^2-1)}[(105x^6-225x^4+135x^2-15)\tau - 105x^5 + 190x^3 - 81x]$$

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$$Q_0^3(x) = -\frac{6x^2+2}{(x^2-1)^{3/2}}$$

$$Q_1^3(x) = -\frac{8x}{(x^2-1)^{3/2}}$$

$$Q_2^3(x) = -\frac{8}{(x^2-1)^{3/2}}$$

$$Q_3^3(x) = \frac{1}{(x^2-1)^{3/2}}[(15x^6-45x^4+45x^2-15)\tau - 15x^5 + 40x^3 - 33x]$$

$$Q_4^3(x) = \frac{1}{(x^2-1)^{3/2}}[(105x^7-315x^5+315x^3-105x)\tau - 105x^6 + 280x^4 - 231x^2 + 48]$$

$$Q_0^4(x) = \frac{24x^3 + 24x}{(x^2 - 1)^2}$$

$$Q_1^4(x) = \frac{40x^2 + 8}{(x^2 - 1)^2}$$

$$Q_2^4(x) = \frac{48x}{(x^2 - 1)^2}$$

$$Q_3^4(x) = \frac{48}{(x^2 - 1)^2}$$

$$Q_4^4(x) = \frac{1}{(x^2 - 1)^2}[(105x^8 - 420x^6 + 630x^4 - 420x^2 + 105)\tau - 105x^7 + 430x^5 - 511x^3 + 231x].$$

The above formulae have been calculated by use of the recurrence relations [2]

$$Q_n^{k+2}(x) = (n - k)(n + k - 1)Q_n^k(x) - 2(k + 1)\frac{x}{\sqrt{x^2 - 1}}Q_n^{k+1}(x)$$

$$(n - k + 1)Q_{n+1}^k(x) = (2n + 1)xQ_n^k(x) - (n + k)Q_{n-1}^k(x)$$

$$\sqrt{x^2 - 1}\,Q_n^{k+1}(x) = (n - k)xQ_n^k(x) - (n + k)Q_{n-1}^k(x)$$

starting with

$$Q_0^0(x) = \tau, \qquad Q_0^1(x) = -\frac{1}{\sqrt{x^2 - 1}}, \qquad Q_1^0(x) = x\tau - 1.$$

Mathematisches Institut der Technischen Hochschule,
München, Germany

1. W. Magnus & F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen der Mathematischen Physik*, Springer-Verlag, Berlin, 1948.

2. A. Erdelyi, Editor, *Higher Transcendental Functions*, vol. I, McGraw-Hill Book Co., Inc., New York, 1953.