Quadratic Residues and the Distribution of Primes

By Daniel Shanks

1. Introduction. Chebyshev stated, [1], that there were "many more" primes of the form $4m - 1$ than of the form $4m + 1$ and he indicated, [2], at least three senses in which this assertion was to be understood. If the numbers of primes in these two classes which are $\leqq n$ are designated as $\pi_-(n)$ and $\pi_+(n)$ respectively, and if

$$\Delta(n) = \pi_-(n) - \pi_+(n), \tag{1}$$

and

$$t(n) = \frac{\Delta(n)}{\sqrt{n}/\log n}, \tag{2}$$

then Chebyshev stated that for a certain *subsequence*, n_i, of the integers,

$$\lim_{i \to \infty} t(n_i) = +1. \tag{3}$$

Since $\sqrt{n}/\log n \to \infty$, $\pi_-(n)$ could thus be made to *exceed* $\pi_+(n)$ by any amount by an appropriate choice of n.

This theorem (which was one of the above-mentioned three senses) was proven by Phragmén, [3], and later by Landau, [4], [5]. The other two "senses" have never been proven, [5, p. 647]. Since the theorem, (3), is entirely noncommittal as to the behaviour of $\Delta(n)$, or of $t(n)$, for values of n other than those of the subsequence n_i, it is however not a very convincing argument in favor of Chebyshev's "many more" primes assertion. Littlewood, in fact, showed, [6], that there is a constant K such that for two other sequences n_j and n_k we could have

$$t(n_j) > K \log \log \log n_j \tag{4}$$

and

$$t(n_k) < -K \log \log \log n_k. \tag{5}$$

Since $\log \log \log n \to \infty$ (with great dignity) these inequalities imply that the right side of (3) may be changed from $+1$ to *any* real number, positive or *negative*, for some sequence, n_l. Because of this Cramér wrote, [7]:

> "Chebyshev's general assertion—'there are many more primes of the form $4n + 3$ than of the form $4n + 1$'—can therefore be true only in a rather limited region."

The purpose of this paper, [8], is to determine the sense in which Chebyshev's general assertion is (nonetheless) probably true, and to identify the nature of the cause which tends to diminish the class of $4n + 1$ primes relative to the $4n - 1$ primes. The problem is generalized to primes, and also to composites, in any arith-

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metic progression. Finally, there are some comments concerning the Littlewood phenomena, $\pi(n) > Li(n)$.

2. The Function $\Delta(n)$. Tables of $\pi_-(n)$ and of $\pi_+(n)$, [9], [10], [11], [12], published prior to the short note of John Leech, [13], were not of sufficient completeness (or, in one case, of sufficient accuracy) to correctly identify Cramér's "rather limited region." Leech computed (on EDSAC) $\Delta(n)$ for $n \leqq 3{,}000{,}000$ and determined that 26,861 was the smallest* n for which $\pi_+(n) > \pi_-(n)$:

$$\Delta(26{,}861) = -1. \tag{6}$$

The author, unaware of Leech's note, carried out a similar computation (on an IBM 704). The following description of $\Delta(n)$ in this range, 1 to $3\cdot 10^6$, agrees (where it overlaps) with the shorter account of Leech. In Table 1 we divide this range into six regions and in each region give the maximum and minimum values of $\Delta(n)$, the number of intervals during which $\Delta(n)$ remains zero, positive, or negative, and the total number of n's for which $\Delta(n)$ is zero, positive, or negative. The final figures show that $\Delta(n) > 0$ for 99.84 % of the $n \leqq 3\cdot 10^6$. Table 2 identifies some extrema (and some zeros) of $\Delta(n)$.

This detailed description makes it highly plausible that the *predominantly* positive character of $\Delta(n)$ in this range of n is not merely a passing fancy of the integers (as is almost implied by Cramér's remark) but a permanent phenomenon

TABLE 1

Description of $\Delta(n)$ for $1 \leqq n \leqq 3{,}000{,}000$

Regions of n	Max	Min	0 Intervals	+ Int.	− Int.	0 n's	+ n's	− n's
1 – 462	+6	0	5	4	0	10	452	0
463 – 26,832	+31	+1	0	1	0	0	26,370	0
26,833 – 26,926	+1	−1	6	4	1	60	32	2
26,927 – 616,768	+105	+1	0	1	0	0	589,842	0
616,769 – 633,882	+12	−8	101	48	52	1,282	12,428	3,404
633,883 – 3,000 000 +	+256	+1	0	1	0	0	2,366,118	0
						1,352 00.05 %	2,995,242 99.84 %	3,406 00.11 %

TABLE 2

Some Special Values

$$\begin{aligned}
&\Delta(227) = +6\\
\Delta(461) = {}&\Delta(462) = 0\\
&\Delta(17{,}971) = +31\\
\Delta(26{,}861) = {}&\Delta(26{,}862) = -1\\
&\Delta(359{,}327) = +105\\
&\Delta(623{,}681) = -8\\
\Delta(627{,}859) = \cdots = {}&\Delta(627{,}900) = 0\\
&\Delta(2{,}951{,}071) = +256
\end{aligned}$$

* This first axis crossing of $\Delta(n)$ was discovered independently by J. W. Wrench, Jr.

for which a sufficient number-theoretic cause should be assigned and of which a more precise formulation is desirable.

3. The Function $\tau(n)$. We therefore return to $t(n)$ and seek a better insight into this function. We do not compute $t(n)$ itself, but instead, the somewhat simpler:

$$\tau(n) = \Delta(n)\cdot\sqrt{n}/\pi(n) \tag{7}$$

where $\pi(n)$ is the *total* number of primes $\leqq n$. From the prime number theorem, $\pi(n) \sim n/\log n$, we have

$$\tau(n) \sim t(n). \tag{8}$$

For example, for $n = 2{,}000{,}000$ we have

$$\pi_-(n) = 74{,}516; \quad \pi_+(n) = 74{,}416; \quad \pi(n) = 148{,}933;$$
$$\Delta(n) = 100; \quad \tau(n) = 0.9496; \quad \text{and} \quad t(n) = 1.0259.$$

The function $\tau(1000\,k)$ was computed for the 2000 values $k = 1, 2, \cdots, 2000$. The minimum, the mean, and the maximum of these 2000 values are

$$\begin{aligned} \tau(629{,}000) &= -0.0464, \\ \frac{1}{2000}\sum_1^{2000} \tau(1000k) &= 1.0613, \\ \tau(127{,}000) &= 2.0961, \end{aligned} \tag{9}$$

respectively. Only slightly greater extremes would have been obtained had we computed $\tau(n)$ for *all* the $2\cdot 10^6$ n in this range. The distribution of these 2000 values of τ between the extremes was determined by counting the k's for which

$$\frac{m}{16} \leqq \tau(1000k) < \frac{m+1}{16} \qquad (m = -1, 0, 1, \cdots, 33). \tag{10}$$

These counts, $\nu(m)$, are tabulated in Table 3 and plotted in a bar graph in Fig. 1.

The following comments are now in order:

a.) The distribution is roughly normal with a mean of (nearly) $+1$.

b.) The rare cases of $\Delta(n) < 0$ (i.e., $\tau(n) < 0$) are now to be thought of as no more unusual than the equally rare other extreme: $\Delta(n) > 2\pi(n)/\sqrt{n}$ (i.e., $\tau(n) > 2$).

c.) The implication of Littlewood's inequalities, (4) and (5), is that the distribution function has tails of infinite extent. Of course, occurrences of τ far out in a tail will be very rare.

TABLE 3

m	-1	0	1	2	3	4	5	6	7	8	9	10
$\nu(m)$	4	7	9	7	10	4	3	14	11	48	55	80
m	11	12	13	14	15	16	17	18	19	20	21	22
$\nu(m)$	76	114	139	156	139	151	152	136	129	109	107	77
m	23	24	25	26	27	28	29	30	31	32	33	
$\nu(m)$	52	51	55	29	21	19	17	12	5	0	2	

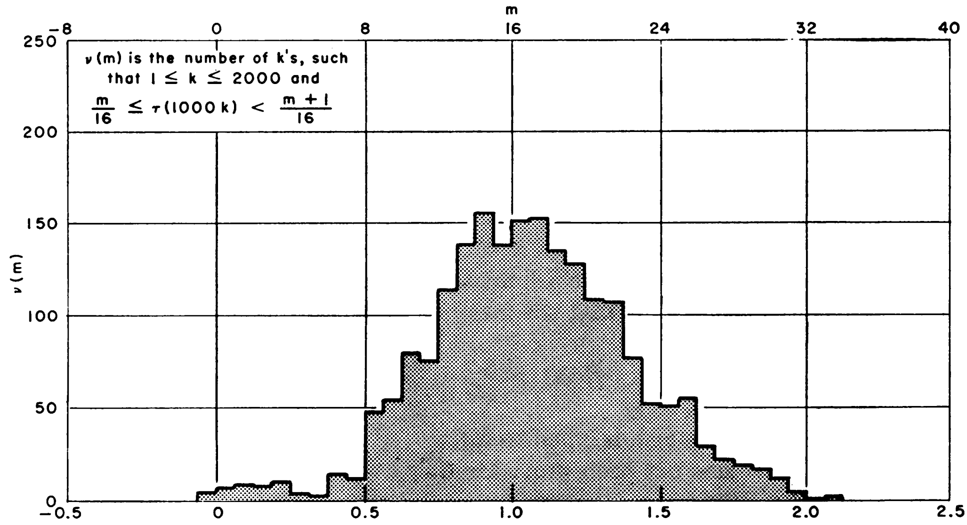

FIG. 1—The distribution of 2000 values of $\tau(1000\ k)$

d.) Chebyshev's function $\sqrt{n}/\log n$ (or its equivalent $\pi(n)/\sqrt{n}$) is seen to be an appropriate normalizing factor—it reduces the function $\Delta(n)$ to a function of n with relatively small variation. For example, throughout most of the second million, (1,015,000 to 2,000,000), $\tau(1000k)$ has the following very modest behaviour:

(11) $$\tau(1{,}811{,}000) = 0.426 \leqq \tau(1000k) \leqq 1.610 = \tau(1{,}521{,}000).$$

e.) The function

$$\tau_s = \frac{1}{50s} \sum_{k=1}^{k=50s} \tau(1000k) \qquad (s = 1, 2, \cdots, 40)$$

provides a running history of the mean value of τ, and is found to change very little:

(12) $$\tau_{15} = 1.0222 \leqq \tau_s \leqq 1.1799 = \tau_3 \qquad (s = 1, 2, \cdots, 40).$$

4. A Conjecture. The above discussion suggests that the proper formulation of Chebyshev's "many more" primes assertion is the following

CONJECTURE. The mean value of $\tau(n)$,

$$\frac{1}{N-1} \sum_{2}^{N} \tau(n),$$

has a limit as $N \to \infty$ and this limit equals $+1$;

(13) $$\lim_{N\to\infty} \frac{1}{N-1} \sum_{2}^{N} \tau(n) = +1.$$

The conjectured limit, $+1$, may seem a little rash in view of the limited data, (12). However, there is other evidence. The remark made after (5) may suggest that the $+1$ right side of (3) is of no significance since it may be replaced by any real number. Nonetheless, from Landau's proof, [5], of (3) it is readily seen that $+1$ has a very special role. For every a the Dirichlet series

$$F_a(s) = \sum_{n=2}^{\infty} \frac{\Delta(n) - a\sqrt{n}/\log n}{n^{s+1}} \tag{14}$$

defines a function which is regular in the half plane $\mathfrak{R}(s) > 1$ and for the real values $\frac{1}{2} < s \leqq 1$, but is singular for $s = \frac{1}{2}$. However, only for $a = 1$, that is, only for $F_1(s)$, does the function have a limit as $s \to \frac{1}{2}+$. Using this fact, and $\tau(n) \sim t(n)$, what is to be proved is that the mean value of the a_n in

$$F_1(s) = \sum_{n=2}^{\infty} \frac{a_n}{n^{s+\frac{1}{2}} \log n} \tag{14a}$$

has a limit as $n \to \infty$ which is equal to zero.

The conjecture has not been proved, either as its stands, or in one of the two following weaker forms:

Conjecture. If no zero of

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \cdots$$

has a real part $> \frac{1}{2}$, then (13) is true.

Conjecture. If

$$\operatorname*{Lim}_{N\to\infty} \frac{1}{N-1} \sum_{2}^{N} \tau(n)$$

exists, it equals $+1$.

However, even the original conjecture seems sufficiently likely to be true to merit recording. The closest thing to it, in the literature known to the author, is Theorem 2.351 of Hardy and Littlewood, [6, p. 151]. This, together with their Theorem 2.34 and a third result due to Landau, [7], may be combined and entirely rewritten to read:

Theorem. If no zero of $L(s)$ has a real part $> \frac{1}{2}$, then there is a positive K such that

$$\frac{1}{N} \sum_{1}^{N} \Delta(n) > K \frac{\sqrt{N}}{\log N}$$

for all sufficiently large N, and conversely.

From this theorem, Abel's lemma on partial summation, [14], and (8), we can easily prove the following

Corollary. If no zero of $L(s)$ has a real part $> \frac{1}{2}$, then there is a positive K such that

$$\frac{1}{N-1} \sum_{2}^{N} \tau(n) > K$$

for all sufficiently large N.

We will present further evidence for the conjecture, in section 9 below, after we have determined the number-theoretic cause of the $4m + 1$ prime deficiency.

5. Localization of the Deficiency. If the classes $4m + 1$ and $4m - 1$ are split, modulo 8, we have

$$\overbrace{8m + 1 \qquad 8m + 5}^{4m+1} \qquad\qquad \overbrace{8m + 3 \qquad 8m + 7}^{4m-1}$$

and we now inquire whether the deficiency exhibited by the $4m + 1$ primes is shared equally by the $8m + 1$ and the $8m + 5$ primes. Landau's generalization of (3) for any modulus, [5, p. 704–711], and other simpler considerations given below, suggest, on the contrary, that the deficiency is confined entirely to $8m + 1$.

Let $\pi_{a,b}(n)$ be the number of primes of the form $am + b$ which are $\leqq n$. A table of $\pi_{8,b}(1000k)$ was computed for $b = 1, 3, 5$, and 7 and $k = 1, 2, \cdots, 1000$. Examination of this table shows that while the numbers of primes of the forms $8m + 3$, $8m + 5$, and $8m + 7$ are continually jockeying for first place, the number of $8m + 1$ primes is always in last place. Let $f_b(n)$ be the fraction of the *tabular values*, with argument $\leqq n$, for which $am + b$ is in first place. (Two-way or three-way ties for first place are pro-rated $\frac{1}{2}$ or $\frac{1}{3}$ of a point respectively.) Similarly let $l_b(n)$ be the fraction of the tabular values for which $am + b$ is in last place. In the present case, since $8m + 1$ is in last place throughout the entire range of the table, we have for every $k = 1, 2, \cdots, 1000$:

$$f_1(1000k) = l_3(1000k) = l_5(1000k) = l_7(1000k) = 0. \tag{15}$$

A brief summary of $\pi_{8,b}$ and the four other f_b and l_b functions is shown in Table 4. No significant deviations from equality, [15], are to be noted among $8m + 3$, $8m + 5$, and $8m + 7$,— the slightly low $f_5(1{,}000{,}000)$ is compensated by the slightly high $f_5(750{,}000)$. We conclude that the deficiency of the $4m + 1$ primes resides solely in the $8m + 1$ primes. On those rare occasions (regions 3 and 5 of Table 1) when $\pi_+(n) > \pi_-(n)$, we invariably find that $8m + 5$ is in first place with a sufficient lead over $8m + 3$ and $8m + 7$ to overcompensate for the leads which these latter classes hold over $8m + 1$. It is not known whether $8m + 1$ itself can ever take the lead but in view of what is written below about $24m + 1$, this is probable — for sufficiently large n.

TABLE 4

Distribution of the Primes Modulo 8

n	$\pi_{8,1}$	$\pi_{8,3}$	$\pi_{8,5}$	$\pi_{8,7}$	l_1	f_3	f_5	f_7
250,000	5473	5525	5523	5522	1.000	0.446	0.322	0.232
500,000	10334	10418	10397	10388	1.000	0.337	0.344	0.319
750,000	14998	15087	15078	15074	1.000	0.380	0.382	0.237
1,000,000	19552	19653	19623	19669	1.000	0.401	0.287	0.312

6. Distinctions between Modulo 12 and Modulo 10. Alternatively, we could have split $4m + 1$ and $4m - 1$ modulo 12:

$$\overbrace{12m + 1 \qquad 12m + 5}^{4m+1} \qquad\qquad \overbrace{12m + 7 \qquad 12m + 11}^{4m-1}$$

TABLE 5
Distribution of the Primes Modulo 12

n	$\pi_{12,1}$	$\pi_{12,5}$	$\pi_{12,7}$	$\pi_{12,11}$	l_1	f_5	f_7	f_{11}
252,000	5504	5567	5564	5566	1.000	0.352	0.420	0.228
504,000	10404	10476	10480	10472	1.000	0.231	0.274	0.495
756,000	15100	15196	15204	15186	1.000	0.396	0.258	0.346
1,008,000	19715	19771	19812	19797	1.000	0.338	0.402	0.260

TABLE 6
Distribution of the Primes Modulo 10

n	$\pi_{10,1}$	$\pi_{10,3}$	$\pi_{10,7}$	$\pi_{10,9}$	l_1	f_3	f_7	l_9
250,000	5495	5520	5517	5510	0.492	0.556	0.444	0.508
500,000	10386	10382	10403	10365	0.555	0.439	0.561	0.445
750,000	15027	15084	15073	15052	0.533	0.445	0.555	0.467
1,000,000	19617	19665	19621	19593	0.502	0.582	0.418	0.498

A table of $\pi_{12,b}(1008k)$ was computed for $b = 1, 5, 7$, and 11 and $k = 1, 2, \cdots, 1000$. (An interval of 1008 was chosen, since its divisibility by 12 simplified the computer program and eliminated even slight inequalities.) The results are similar: the deficiency of $4m + 1$ resides solely in $12m + 1$, and when $\Delta(n) < 0$ we now find $12m + 5$ in the lead. Again,

$$f_1 = l_5 = l_7 = l_{11} = 0 \tag{16}$$

throughout the range computed. Again, the three stronger classes are all equally strong—see Table 5.

In *contrast* now consider the four classes of primes modulo 10, $10m + 1$, $+3$, $+7$, and $+9$. This time we find

$$f_1(1000k) = l_3(1000k) = l_7(1000k) = f_9(1000k) = 0 \tag{17}$$

for $k = 1, 2, \cdots, 1000$ so that for *these* arguments *neither* $10m + 1$ *nor* $10m + 9$ is ever in first place.* We find that $10m \pm 3$ take turns in first place while $10m \pm 1$ take turns in last place. From Table 6 we conclude that this time we have two strong classes (which are equally strong), $10m \pm 3$, and two weak classes (which are equally weak), $10m \pm 1$.

These striking distinctions between modulo 8 and 12 on the one hand and modulo 10 on the other are easily explained in terms of the corresponding modulo multiplication groups, and this explanation provides us with a simple, number—theoretic, sufficient cause.

7. Group Multiplication and Fluctuations. Consider the multiplication table modulo 8 of the four residue classes 1, 3, 5, and 7

* At some *intermediate* arguments, e.g., $n = 135852$, $10m + 1$ is *tied* for first place and similarly, for $n = 969240$, $10m + 7$ is *tied* for last place.

1	3	5	7
3	1	7	5
5	7	1	3
7	5	3	1

and assume that in an interval centered around n there are fewer primes of the form $8m + 1$ than of the other three types. Consider the *excess* primes of the other three classes. Their products with themselves and with each other are composites (of order n^2) whose residue classes are contained in the lower right 3 x 3 box. Now note that these products are distributed into the four residue classes in proportions 3:2:2:2—there is an *extra* product congruent to 1. Assume on the contrary an excess number of primes of the form $8m + 1$. The composites from these primes lie in the upper left 1 x 1 box. Again we find an excess of composites congruent to 1. Similarly, too few or too many primes congruent to 3, 5, or 7 will also lead to an excess number of composites congruent to 1. Briefly, *any fluctuation from equality in the distribution of the primes into the four residue classes will create an excess number of composites and therefore a diminished number of primes congruent to 1 modulo 8.* This idea, that the phenomenon in question is essentially a fluctuation phenomenon (with analogies to the fluctuation phenomena of physics), arose during a discussion of this problem between the author and T. S. Walton. The idea is thus partly due to him.

Since the multiplication group modulo 12 is isomorphic to that modulo 8, the behaviour found modulo 12 is similar. But for 10 we do not have the "vier" group but the cyclic group

1	3	7	9
3	9	1	7
7	1	9	3
9	7	3	1

and this time we find that fluctuations in $10m + 3$ or $10m + 7$ diminish the $10m + 9$ primes while fluctuations in $10m + 1$ or $10m + 9$ diminish the $10m + 1$ primes.

8. Quadratic Residues and the Distribution of Primes. In general, it is readily seen, those residue classes which occur on the principal diagonal are the ones whose numbers of primes are diminished. These are the quadratic residues. We expect then, for any modulus, that the prime race will separate (in the mean) into two races—that between the non-residues, up front, and that between the residues, in the rear. In particular, since $+1$ is the only quadratic residue of 24, the deficiency of $4m + 1$, which was found to reside solely in $8m + 1$ or $12m + 1$, can be further localized to their intersection, $24m + 1$.

For most larger moduli, unlike the unusual 24, there will be many quadratic residues and the deficiency, $\sqrt{n}/\log n$, will be shared among all of them. Thus the separation between the residues and the non-residues will not be as sharp, and extensive interplay is to be expected.

9. Propagation of the Deficiency and Higher Order Effects. The quadratic fluctuation effect just discussed is not merely strong enough to maintain the mean deficiency in the $4m + 1$ primes, but, as we shall see presently, as n increases it becomes *too* strong and we must examine the compensating cubic and higher order effects in order to obtain an accurate picture. We confine ourselves to the modulus 4, although the generalization offers little difficulty. We have seen, (9), that the mean value of τ up to $n = 2 \cdot 10^6$ is nearly one and we wish to show how a mean value equal to one can propagate itself to larger values of n. Our computation, however, is only approximate, and while the result gives further evidence for the truth of (13), it is not a proof of that conjecture.

Let a be a positive integer and let $\pi_+^{(a)}(n)$ be the number of positive integers of the form $4m + 1$ which are $\leqq n$ and which are the products of a (not necessarily distinct) primes. Our previous $\pi_+(n)$ is now $\pi_+^{(1)}(n)$. With a similar definition for $\pi_-^{(a)}(n)$, let

$$\Delta^{(a)}(n) = \pi_-^{(a)}(n) - \pi_+^{(a)}(n), \tag{18}$$

$$\Sigma^{(a)}(n) = \pi_-^{(a)}(n) + \pi_+^{(a)}(n), \tag{19}$$

and

$$\tau^{(a)}(n) = \frac{\Delta^{(a)}(n)}{\Sigma^{(a)}(n)} \sqrt{n}. \tag{20}$$

Note the slight difference between τ and $\tau^{(1)}$—the new denominator, $\Sigma^{(1)}(n) = \pi(n) - 1$, omits the count of the prime 2.

We wish to compute the mean value of $\tau^{(1)}(n)$ and we assume that the mean values of $\tau^{(1)}(x)$ for $x = n^{1/a}$ $(a = 2, 3, 4, \cdots)$ are all equal to 1. It follows that of all the odd primes of order $n^{1/a}$, a fraction equal to $\frac{1}{2}(1 + n^{-1/2a})$ is of the form $4m - 1$ and a fraction equal to $\frac{1}{2}(1 - n^{-1/2a})$ is of the form $4m + 1$. The composites of order n, which are the products of a primes, have as prime factors primes whose (geometric) mean order is $n^{1/a}$. With reference to the multiplication table (mod 4):

+1	−1
−1	+1

we now find, by induction on a, that among all possible a—fold products of these primes, a fraction equal to $\frac{1}{2}(1 + (-1)^{a+1}n^{-\frac{1}{2}})$ is of the form $4m - 1$ and a fraction equal to $\frac{1}{2}(1 - (-1)^{a+1}n^{-\frac{1}{2}})$ is of the form $4m + 1$. In other words, on the average, we will have, for $a = 2, 3, \cdots$,

$$\bar{\tau}^{(a)}(n) = (-1)^{a+1}. \tag{21}$$

Thus we find a simple and interesting generalization of the Chebyshev phenomenon. There will be, in the mean, an excess of $4m - 1$ composites of odd degree a, and an excess of $4m + 1$ composites of even degree. Further, the mean fractional excess will be $n^{-\frac{1}{2}}$, independent of the degree a.

Now, for any n,

$$0 = \Delta^{(1)}(n) + \Delta^{(2)}(n) + \Delta^{(3)}(n) + \cdots$$

with an error of 0 or 1, and therefore, with the same error,

$$\Delta^{(1)}(n) = -\Delta^{(2)}(n) - \Delta^{(3)}(n) - \cdots. \tag{22}$$

As long as n is not too large, say 1000, most of the composites are of second degree, and we have $\Delta^{(1)}(n) \approx -\Delta^{(2)}(n)$. This, essentially, was our picture in sec. 7 above. But as n increases, the other even degree composites would tend to increase the deficiency while the odd degree composites tend to diminish it. To determine the balance we use the known fact, [5, p. 627], that in any arithmetic progression the number of integers with a odd is asymptotically equal to those with a even. Specifically, for the progression $2m + 1$, we have

$$(1 + \epsilon)\, \Sigma^{(1)}(n) = \Sigma^{(2)}(n) - \Sigma^{(3)}(n) + \Sigma^{(4)}(n) - \cdots,$$

where $\epsilon \to 0$ as $n \to \infty$. Therefore from (20), (21), and (22) we have, in the mean,

$$\Delta^{(1)}(n) = (1 + \epsilon)\, \Sigma^{(1)}(n)/\sqrt{n}.$$

Thus

$$\bar{\tau}^{(1)}(n) = 1 + \epsilon, \tag{23}$$

and a mean value of one propagates itself.

There are two remarks of interest concerning equations (21) and (22).

1.) The sequences $\chi(n)$ and $\lambda(n)$ are defined as follows:

$$\lambda(n) = (-1)^a$$

where a is the number of prime factors of n, and $\chi(n) = 0, 1, 0,$ and -1 for numbers of the form $4m$, $4m + 1$, $4m + 2$, and $4m + 3$ respectively. Let

$$b_n = \lambda(n)\chi(n). \tag{24}$$

A generalized Chebyshev assertion now reads: there are "many more" integers m for which $b_m = +1$ than for which $b_m = -1$ and the mean excess is of order $\frac{1}{2}\sqrt{n}$. Since it is known, [5, p. 674], that

$$f(s) = \sum_{n=1}^{\infty} b_n n^{-s} = (1 - 2^{-2s})\zeta(2s)/L(s),$$

it is clear that a function-theoretic formulation of this assertion is concerned with $f(s)$ and with the zeros of $L(s)$.

2.) Let n be sufficiently large such that the number of primes is small compared with the number of composites. Then the terms of like sign in the right side of (22) could be combined so that the right side would be the difference between two nearly equal numbers. We therefore have an analogue of a difficulty in numerical analysis—subtractive loss of significance. In fact, the analogy is very good, since it is known, [15, p. 342], that the normal order of the number of prime divisors of n is log log n and that the number of composites of degree a is given asymptotically by

$$\pi^{(a)}(n) \sim n(\log\log n)^{a-1}/(a-1)!\log n.$$

This means that the numbers of composites of degree a have an approximate Poisson distribution around a mean of log log n and that the equation (22) is comparable to the following equation, which is numerically sensitive when n is large:

$$1 - e^{-\log\log n} = \log\log n - \tfrac{1}{2}(\log\log n)^2 + \cdots.$$

We should therefore expect disturbances in the balance, (23), and more or less random oscillations around this mean. The oscillations should grow as log log n increases and may be expected to show strength when the condition $\Sigma^{(1)} \ll \Sigma^{(2)} \approx \Sigma^{(3)}$ is met. This should occur for $\log\log n = 2.5+$ or $\log\log\log n \approx 1$.

With these remarks we are led to our final topic.

10. The Littlewood Phenomena. Analogous to Chebyshev's assertion is the erroneous inequality:

$$\pi(n) < Li(n) \tag{24}$$

which was thought to be correct both by Gauss and by Riemann, [7, p. 791, 795]. Again we have the Littlewood counterexample

$$\pi(n) - Li(n) > K\frac{\sqrt{n}}{\log n}\log\log\log n. \tag{25}$$

And again we obtain, this time from Riemann's prime formula, [7, p. 795],

$$Li(n) = \pi(n) + \tfrac{1}{2}\pi(\sqrt{n}) + \cdots, \tag{26}$$

and from $\pi(\sqrt{n}) \sim 2\sqrt{n}/\log n$, the suggestion to define

$$r(n) = \frac{Li(n) - \pi(n)}{\sqrt{n}/\log n}. \tag{27}$$

Or similarly let us define

$$\rho(n) = \frac{Li(n) - \pi(n)}{\pi(n)}\sqrt{n}. \tag{28}$$

Analogy with (13) now suggests the possibility that

$$\lim_{N\to\infty}\frac{1}{N-1}\sum_{2}^{N}\rho(n) = +1 \tag{29}$$

is the proper formulation of the erroneous (24).

If, indeed, the differences $Li(n) - \pi(n)$ and $\pi_-(n) - \pi_+(n)$ are *usually* of the same sign and order of magnitude we should find that

$$\pi^*(n) = \pi(n) + \pi_-(n) - \pi_+(n) = 2\pi_-(n) + 1 \tag{30}$$

agrees with $Li(n)$ much better than $\pi(n)$ does. This is, in fact, the case as is seen in Table 7, [16]. We therefore regard (29) as plausible. It should be investigated.

A final remark concerning the Littlewood phenomena may be of value. It is sometimes said or implied, [17], that (24) is true "im Bereich der Tabellen", where the tables in question include such entries as $n = 10^8$ and $n = 10^9$. But $\pi(n) - Li(n)$ has not been evaluated for all *intermediate* values of n (say for $10^8 < n < 10^9$) and consequently, it is not certain that (24) holds there. In the analogous case of $\Delta(n)$ discussed above, had we computed $\Delta(10{,}000k)$ up to $k = 300$ we would have *always* found $\Delta > 0$ since all the axis crossings of $\Delta(n)$ up to $n = 3.10^6$ occur at intermediate values of n. Further, since the first crossing of $\pi(n) - Li(n)$ may, like Leech's 26861 above, be of very short duration, it could well be missed unless the computing interval in n were rather small. The author knows of no compelling

TABLE 7

n	$\pi(n)$	$\pi^*(n)$	$\pi(n)/\mathrm{Li}(n)$	$\pi^*(n)/\mathrm{Li}(n)$
10^3	168	175	0.9459	0.9853
$2\cdot10^3$	303	311	0.9625	0.9879
$3\cdot10^3$	430	437	0.9712	0.9870
$4\cdot10^3$	550	561	0.9728	0.9923
$5\cdot10^3$	669	679	0.9777	0.9923
$6\cdot10^3$	783	799	0.9782	0.9982
$7\cdot10^3$	900	915	0.9843	1.0007
$8\cdot10^3$	1007	1015	0.9811	0.9889
$9\cdot10^3$	1117	1125	0.9825	0.9895
10^4	1229	1239	0.9862	0.9943
$2\cdot10^4$	2262	2273	0.9884	0.9932
$3\cdot10^4$	3245	3267	0.9903	0.9970
$4\cdot10^4$	4203	4235	0.9929	1.0005
$5\cdot10^4$	5133	5167	0.9935	1.0001
$6\cdot10^4$	6057	6077	0.9958	0.9990
$7\cdot10^4$	6935	6971	0.9928	0.9980
$8\cdot10^4$	7837	7867	0.9950	0.9988
$9\cdot10^4$	8713	8729	0.9949	0.9968
10^5	9592	9617	0.9961	0.9987
$2\cdot10^5$	17984	18013	0.9971	0.9986
$3\cdot10^5$	25997	26033	0.9965	0.9979
$4\cdot10^5$	33860	33919	0.9981	0.9999
$5\cdot10^5$	41538	41613	0.9983	1.0001
$6\cdot10^5$	49098	49151	0.9985	0.9995
$7\cdot10^5$	56543	56589	0.9982	0.9990
$8\cdot10^5$	63951	64071	0.9986	1.0005
$9\cdot10^5$	71274	71379	0.9988	1.0002
10^6	78498	78645	0.9983	1.0002
$2\cdot10^6$	148933	149033	0.9992	0.9998

reason why the first crossing should occur at anywhere near as large an n as Skewes' fantastic $10^{10^{10^{34}}}$, [18].

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