Numerical Integration of $\int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\, dx$

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1. Introduction. This integral arises in a certain thermo-elastic study, and although it is similar to many evaluated in Watson's "Bessel Functions," this integral is not listed. Numerical integration techniques were applied. By means of Simpson's rule with the limits of integration set at 0 to 12 and making use of 400 points, the above integral was evaluated to four decimal places for seven different combinations of η and ξ.

The same integral was evaluated by the Gauss-Laguerre formula. Although the integrals in question can be expressed in terms of complicated elliptic integrals, the natural method of evaluation, to avoid lengthy analysis, is by quadrature. It is deemed that the experience with the Gauss-Laguerre formula is worth reporting, even though the integrals are of a special nature. Using the Gauss-Laguerre formula with only six points, the results agreed with Simpson's rule entries to three decimal places, although the upper bound of the truncation error associated with the Gauss-Laguerre formula was greater than unity.

2. Numerical Results. The integral under consideration is

$$I(\eta, \xi, n) = \int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\, dx \tag{1}$$

where η and ξ are integers and n is 0 or 1. A careful analysis of Simpson's rule applied to $I(\eta, \xi, 0)$ indicates accuracy to within a unit in the fifth decimal place in the values listed in Table 1. It is possible to show* that

$$I(\eta, \xi, 0) + I(\eta, \xi, 1) = \frac{1}{\pi\sqrt{\xi}}\,[Q_{-1/2}(x) - \eta Q_{1/2}(x)] \tag{2}$$

where $x = (1 + \xi^2 + \eta^2)/2\eta$ and $Q_m(x)$ is the Legendre function of the second kind. Use of (2) then verifies that the values of $I(\eta, \xi, 1)$ are also correct to the number of places given in Table 1.

TABLE 1

ξ	η	$I(\eta, \xi, 0)$	$I(\eta, \xi, 1)$	Maximum Error in $I(\eta, \xi, 0)$
1	1	.17868	.33880	1.12×10^{-5}
1	3	.01741	.15950	8.37×10^{-6}
1	6	.00229	.08246	7.97×10^{-6}
3	1	.13463	.15486	2.48×10^{-5}
3	3	.05948	.11742	8.37×10^{-6}
6	1	.07850	.08167	1.47×10^{-4}
6	6	.02954	.05887	7.97×10^{-6}

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$I(\eta, \xi, 0)$ was also evaluated for $\xi = \eta = 1$ by means of the Gauss-Laguerre formula. Using four points the results agree to two decimal places with those in Table 1. Using six points the results agree to three places. In both cases, the truncation error found exceeded unity.

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1. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, The Macmillan Company, New York, 1945.