Calculation of Gamma Functions to High Accuracy

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The two constants, $\Gamma\ (\frac{1}{3})$ and $\Gamma\ (\frac{2}{3})$ have been calculated to 35 decimal places on the Cambridge Computer by means of an interpretive routine that treats floating-point numbers of 37 significant digits [1, 2].

Stirling's asymptotic expansion for $\ln \Gamma\ (x)$ can be written

$$(1) \qquad \ln \Gamma(x) \sim (x - \tfrac{1}{2}) \ln x - x + \tfrac{1}{2} \ln 2\pi + \sum_{r=1}^{\infty} \frac{C_r}{x^{2r-1}}.$$

With the remainder of the series made very small, the accuracy of $\ln \Gamma(x)$ depends primarily on the accuracy of $\ln 2\pi$, $\ln x$, and the C_r. Uhler [3, 4] has published the $\ln x$ for all primes through 101, as well as the $\ln \pi$, to more than 100 significant figures. Uhler has also calculated the C_r to over 100 significant figures [5].

The recursion formula for the Gamma function

$$(2) \qquad \Gamma(x + 1) = x\Gamma(x)$$

can be extended to

$$(3) \qquad \Gamma(x + n) = (x)_n\Gamma(x)$$

where $(x)_n$ is the Pochhammer-Barnes symbol

$$(4) \qquad (x)_n = (x + n - 1)(x + n - 2) \cdots (x + 1)x.$$

If we take the logarithm of Eq. (3),

$$(5) \qquad \ln \Gamma(x + n) = \ln \Gamma(x) + \sum_{j=0}^{n-1} \ln (x + j).$$

Substituting Eq. (1) into Eq. (5) gives, after solving for $\ln \Gamma(x)$,

$$(6) \qquad \ln \Gamma\ (x) \sim \lambda(x) + \sum_{r=1}^{\infty} \frac{C_r}{(x + n)^{2r-1}},$$

where

$$(7) \quad \lambda(x) = (x + n - \tfrac{1}{2}) \ln (x + n) - (x + n) + \tfrac{1}{2} \ln 2\pi - \sum_{j=0}^{n-1} \ln (x + j).$$

The term $\lambda(x)$ is calculated quite easily by hand, and it is not difficult to calculate $\ln \Gamma(x)$ from Eq. (6) with a digital computer.

To calculate $\Gamma(x)$, let

$$(8) \qquad \Gamma(x) = e^{\psi} e^{\ln \Gamma(x) - \psi}$$

Received December 31, 1958; revised May 4, 1959.

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where

$$\psi = \pm 0.1n \tag{9}$$

is the exponent of e nearest in value to $\ln \Gamma(x)$. Values of $e^{\pm 0.1n}$, n being any integer from 0 to 10, have been tabulated by Van Orstrand [6].

(10) Since $$|\ln \Gamma(x) - \psi| \leqq 0.05$$

it will always be possible to calculate $e^{\ln \Gamma(x)}$ by means of the exponential power series expansion using very few terms.

These techniques led to the following values:

$$\Gamma(\tfrac{1}{3}) = 2.6789\ 38534\ 70774\ 76336\ 55692\ 94097\ 46776$$

$$\Gamma(\tfrac{2}{3}) = 1.3541\ 17939\ 42640\ 04169\ 45288\ 02815\ 45138$$

$$\ln \Gamma(\tfrac{1}{3}) = .98542\ 06469\ 27767\ 06918\ 71740\ 36977\ 96139$$

$$\ln \Gamma(\tfrac{2}{3}) = .30315\ 02751\ 47523\ 56867\ 58628\ 17372\ 01104.$$

It was possible to examine round-off and truncation errors at each step in the calculation. The final relative error was less than $\pm 3 \times 10^{-35}$ in each case.

As a final independent check, the values of $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$ were put into

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = \frac{2\pi}{\sqrt{3}}. \tag{11}$$

The error analysis was consistent with this identity.

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