# A Short Table of $\int_x^\infty J_0(t)t^{-n}\,dt$ and $\int_x^\infty J_1(t)t^{-n}\,dt$

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**1. Introduction.** In various physical applications, and in particular in geophysics, there arises a need for the numerical evaluation of integrals of the type

$$L = \int_x^\infty J_0(t)g(t)\,dt \tag{1}$$

and

$$M = \int_x^\infty J_1(t)g(t)\,dt \tag{2}$$

where $x$ is a positive number, and where $g(t)$ is, at least for sufficiently large $t$, a monotonically decreasing function of $t$. Here $J_0(t)$, $J_1(t)$ denote the Bessel functions (of the first kind) of orders zero and one, respectively.

A method of computation of (1) and (2) has been given by the author [1] for the case $x = 0$, and this method can also be applied to the evaluation of (1) and (2) for small values of $x$ (such that the interval $0 - x$ is within the first few cycles of $J_0$ and $J_1$ as the case may be), even if the integrals diverge for $x = 0$. The present paper presents an alternate method (which is easily extended to large values of $x$) for the case where $g(t)$ is a function of the type that can be expanded as a series of inverse powers of $t$

$$g(t) = a_0 + a_1t^{-1} + a_2t^{-2} + \cdots, \tag{3}$$

when $t$ is sufficiently large.

**2. Description of the Method.** Suppose, then, that we wish to evaluate an integral of the type (1) or (2), where for simplicity we will suppose that an expansion of the form (3) is valid when $t > x$. Then our integrals can be expanded in the forms

$$L = a_0\int_x^\infty J_0(t)\,dt + a_1\int_x^\infty J_0(t)t^{-1}\,dt + a_2\int_x^\infty J_0(t)t^{-2}\,dt + \cdots, \tag{4}$$

$$M = a_0\int_x^\infty J_1(t)\,dt + a_1\int_x^\infty J_1(t)t^{-1}\,dt + a_2\int_x^\infty J_1(t)t^{-2}\,dt + \cdots, \tag{5}$$

so that (1), (2) can be evaluated if we can evaluate integrals of the type

$$L_n = \int_x^\infty J_0(t)t^{-n}\,dt, \qquad n = 0, 1, 2, \cdots \tag{6}$$

and

$$M_n = \int_x^\infty J_1(t)t^{-n}\,dt, \qquad n = 0, 1, 2, \cdots. \tag{7}$$

We will see later that the series (4), (5) are quite rapidly convergent for $x > 1$, the rapidity of convergence increasing with $x$. However it should be noted that in

Received March 10, 1959; Institute of Geophysics Publication No. 136.

a number of applications (for example propagation in layered media, seismology problems) the integrals we need to evaluate have $\rho t$ in place of $t$ in the arguments of the Bessel functions, where $\rho$ is a positive number which may be as high as 20. After making the appropriate scale-factor transformation we obtain the integrals (6) and (7) with a factor $\rho^{n-1}$ in front of each integral. For this reason the integrals (6) and (7) are often required with high precision and at least ten place accuracy would be needed for such application.

With regard to the integrals $L_n$, $M_n$, the only published tables known to the author are one of

$$\int_0^x J_0(t)\,dt, \qquad x = 0(0.01)10$$

by Lowan and Abramowitz [2], from which we can obtain $L_0$ as

$$L_0 = 1 - \int_0^x J_0(t)\,dt,$$

and one of $L_1$, for $x = 0\ (0.1)\ 10\ (1)\ 22$, by Lowan, Blanch and Abramowitz, [3]. Watson [4] gives a table of the maxima and minima of

$$\int_0^x J_0(t)\,dt$$

and this table has been extended by the author [1], who has also pointed out an error in Watson's table. Also Smith [5] and Lowan, Blanch and Abramowitz [3] give asymptotic expressions for $L_1$. For large values of $x$ beyond the range of the published tables we can use the series

$$\int_0^x J_0(t)\,dt = 2[J_1(x) + J_3(x) + J_5(x) + \cdots]. \tag{8}$$

Also we have the obvious result

$$M_0 = J_0(x). \tag{9}$$

It is not the purpose of this paper to give extensive tables, but rather to present a general method for their computation, and to illustrate their use by way of an example.

**3. Method of Computation.** Integration by parts show that the $L_n$, $M_n$ satisfy the following recurrence relations for a given value of $x$:

$$L_n = -x^{-n}J_1(x) + (n + 1)M_{n+1} \tag{10}$$

$$M_n = x^{-n}J_0(x) - nL_{n+1} \tag{11}$$

from which we deduce

$$L_{n+2} = \frac{J_0(x)}{(n+1)x^{n+1}} - \frac{J_1(x)}{(n+1)^2x^n} - \frac{L_n}{(n+1)^2} \tag{12}$$

and

$$M_{n+2} = \frac{J_0(x)}{n(n+2)x^n} + \frac{J_1(x)}{(n+2)x^{n+1}} - \frac{M_n}{n(n+2)}. \tag{13}$$

Equation (12) has been given by Smith [5] where he uses it to obtain an asymptotic expansion for $L_1$. Thus starting from $L_0$ and $L_1$, which are either obtained from existing tables, or must be calculated, we can use (12) to compute successive $L_n$'s. Furthermore the form (12) shows that for even moderately large $x$, $L_n$ tends to zero very rapidly as $n$ increases. This makes the calculation of the series (4) a very rapid process.

With regard to the $M_n$'s, starting with the relations

$$M_0 = J_0(x)$$
$$M_1 = L_0 + J_1(x)$$
$$M_2 = \tfrac{1}{2}(L_1 + x^{-1}J_1(x))$$

(which are obtainable from (10), (11)), successive $M_n$'s can be computed by means of (13). Some $L_n$'s and $M_n$'s have been computed in this way, and these are given in tables 1 and 2.

With regard to $L_0$ and $L_1$, these are obtainable (by interpolation if necessary) from existing tables ([1], [2], [3], [4], [5]), or, for large $x$, by means of asymptotic expansions. The interpolation can easily be carried out accurately using Taylor's theorem. Examples of such interpolation are given in Longman [1]. Asymptotic series for $L_1$ are given by Smith [5] and by Lowan, Blanch and Abramowitz [3], while for $L_0$ we can apply Smith's method to obtain the result

$$\int_x^\infty J_0(t)\,dt \sim J_0(x)\left[\frac{1}{x} - \frac{1^2.3}{x^3} + \frac{1^2.3^2.5}{x^5} - \frac{1^2.3^2.5^2.7}{x^7} + \cdots\right] - J_1(x)\left[1 - \frac{1^2}{x^2} + \frac{1^2.3^2}{x^4} - \frac{1^2.3^2.5^2}{x^6} + \cdots\right], \tag{14}$$

and for large $x$ this can be used to compute $L_0$ using the known asymptotic series for $J_0(X)$, $J_1(X)$.

**4. Tables.** Tables 1 and 2 were computed from equations (10), (11), (12) and (13) and existing tables [2] and [3] of $L_0$ and $L_1$ by the use of an IBM 709 computer at the Western Data Processing Center of the University of California, Los Angeles.

**5. Example of the Use of the Tables.** Suppose we wish to compute

$$I = \int_{10}^\infty [J_0(t)/(t^2 + 1)]\,dt.$$

Using the expansion

$$(t^2 + 1)^{-1} = t^{-2} - t^{-4} + t^{-6} - \cdots, \qquad (t > 1)$$

we have

$$I = L_2 - L_4 + L_6 - L_8 + \cdots,$$

from which we obtain

$$I = -0.001042.$$

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TABLE 1. $L_n = \int_x^\infty J_0(t)t^{-n}\,dt$

| | $x = 1$ | $x = 2$ | $x = 3$ | $x = 4$ | $x = 5$ | $x = 6$ | $x = 7$ | $x = 8$ | $x = 9$ | $x = 10$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_0$ | 0.080270 | −0.425770 | −0.387567 | −0.024734 | 0.284688 | 0.293779 | 0.045360 | −0.210747 | −0.252266 | −0.067011 |
| $L_1$ | 0.237097 | −0.135296 | −0.126534 | −0.023126 | 0.046841 | 0.049422 | 0.011431 | −0.022961 | −0.028159 | −0.008787 |
| $L_2$ | 0.244878 | −0.039009 | −0.038176 | −0.008510 | 0.007372 | 0.008013 | 0.002192 | −0.002433 | −0.003082 | −0.001055 |
| $L_3$ | 0.213312 | −0.010280 | −0.011069 | −0.002502 | 0.001117 | 0.001265 | 0.000372 | −0.000251 | −0.000332 | −0.000120 |
| $L_4$ | 0.178963 | −0.002357 | −0.003155 | −0.000664 | 0.000163 | 0.000196 | 0.000059 | −0.000025 | −0.000035 | −0.000013 |
| $L_5$ | 0.150464 | −0.000365 | −0.000896 | −0.000167 | 0.000023 | 0.000030 | 0.000009 | −0.000002 | −0.000004 | −0.000001 |
| $L_6$ | 0.128279 | +0.000052 | −0.000255 | −0.000041 | 0.000003 | 0.000005 | 0.000001 | −0.000000 | −0.000000 | −0.000000 |
| $L_7$ | 0.111130 | 0.000093 | −0.000073 | −0.000010 | 0.000000 | 0.000001 | 0.000000 | | | |
| $L_8$ | 0.097715 | 0.000065 | −0.000021 | −0.000002 | | 0.000000 | | | | |
| $L_9$ | 0.087037 | 0.000037 | −0.000006 | −0.000001 | | | | | | |
| $L_{10}$ | 0.078383 | 0.000020 | −0.000002 | −0.000000 | | | | | | |
| $L_{11}$ | 0.071249 | 0.000010 | −0.000001 | | | | | | | |
| $L_{12}$ | 0.065279 | 0.000005 | −0.000000 | | | | | | | |
| $L_{13}$ | 0.060216 | 0.000002 | | | | | | | | |
| $L_{14}$ | 0.055871 | 0.000001 | | | | | | | | |
| $L_{15}$ | 0.052105 | 0.000001 | | | | | | | | |
| $L_{16}$ | 0.048809 | 0.000000 | | | | | | | | |
| $L_{17}$ | 0.045902 | | | | | | | | | |
| $L_{18}$ | 0.043320 | | | | | | | | | |
| $L_{19}$ | 0.041011 | | | | | | | | | |
| $L_{20}$ | 0.038935 | | | | | | | | | |
| $L_{21}$ | 0.037057 | | | | | | | | | |
| $L_{22}$ | 0.035352 | | | | | | | | | |

Blank spaces denote zero entries.

TABLE 2. $M_n = \int_x^\infty J_1(t) t^{-n}\, dt$

| | $x = 1$ | $x = 2$ | $x = 3$ | $x = 4$ | $x = 5$ | $x = 6$ | $x = 7$ | $x = 8$ | $x = 9$ | $x = 10$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_0$ | 0.765198 | 0.223891 | −0.260052 | −0.397150 | −0.177597 | 0.150645 | 0.300079 | 0.171651 | −0.090334 | −0.245936 |
| $M_1$ | 0.520320 | 0.150955 | −0.048508 | −0.090778 | −0.042891 | 0.017095 | 0.040677 | 0.023890 | −0.006955 | −0.023539 |
| $M_2$ | 0.338574 | 0.076533 | −0.006757 | −0.019818 | −0.009338 | 0.001654 | 0.005381 | 0.003184 | −0.000451 | −0.002220 |
| $M_3$ | 0.228309 | 0.035057 | −0.000167 | −0.004213 | −0.001910 | 0.000109 | 0.000699 | 0.000411 | −0.000018 | −0.000207 |
| $M_4$ | 0.163341 | 0.015453 | +0.000372 | −0.000883 | −0.000376 | −0.000004 | 0.000089 | 0.000052 | +0.000001 | −0.000019 |
| $M_5$ | 0.123803 | 0.006738 | 0.000206 | −0.000184 | −0.000072 | −0.000003 | 0.000011 | 0.000006 | 0.000000 | −0.000002 |
| $M_6$ | 0.098419 | 0.002943 | 0.000083 | −0.000039 | −0.000014 | −0.000001 | 0.000001 | 0.000001 | | |
| $M_7$ | 0.081190 | 0.001295 | 0.000030 | −0.000008 | −0.000003 | −0.000000 | 0.000000 | 0.000000 | | |
| $M_8$ | 0.068898 | 0.000575 | 0.000010 | −0.000002 | −0.000000 | | | | | |
| $M_9$ | 0.059752 | 0.000258 | 0.000003 | −0.000000 | | | | | | |
| $M_{10}$ | 0.052709 | 0.000116 | 0.000001 | | | | | | | |
| $M_{11}$ | 0.047130 | 0.000053 | 0.000000 | | | | | | | |
| $M_{12}$ | 0.042608 | 0.000024 | | | | | | | | |
| $M_{13}$ | 0.038872 | 0.000011 | | | | | | | | |
| $M_{14}$ | 0.035733 | 0.000005 | | | | | | | | |
| $M_{15}$ | 0.033061 | 0.000002 | | | | | | | | |
| $M_{16}$ | 0.030760 | 0.000001 | | | | | | | | |
| $M_{17}$ | 0.028756 | 0.000001 | | | | | | | | |
| $M_{18}$ | 0.026997 | 0.000000 | | | | | | | | |
| $M_{19}$ | 0.025441 | | | | | | | | | |
| $M_{20}$ | 0.024053 | | | | | | | | | |
| $M_{21}$ | 0.022809 | | | | | | | | | |
| $M_{22}$ | 0.021687 | | | | | | | | | |

Blank spaces denote zero entries

1. I. M. LONGMAN, "Tables for the rapid and accurate numerical evaluation of certain infinite integrals involving Bessel functions," *MTAC*, v. 11, 1957, p. 166.

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3. ARNOLD N. LOWAN, G. BLANCH, & MILTON ABRAMOWITZ, "Table of $Ji_0(x) = \int_x^\infty J_0(t)/t\, dt$ and related functions," *Ibid.*, p. 33.

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5. V. G. SMITH, "An asymptotic expansion of $Ji_0(x) = \int_x^\infty J_0(t)/t\, dt$," *Journal of Mathematics and Physics*, v. 22, 1943, p. 58.