(3) All calculations are mod m_i so that the digits of the mixed radix representation of $[x]_M$ can be obtained using only calculations mod m_i.

(4) The value $x - x^{(a)}$ can be obtained from $x - x^{(a)} = x - x^{(a-1)} + x^{(a-1)} - x^{(a)}$ so that the modular number x need not be remembered during the whole process.

(5) The matrices (a_{ij}) and (x_{ij}) are computed preliminary to the iteration procedure and are not part of it.

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Generation of Permutations by Transposition

By Mark B. Wells

1. Introduction. As discussed by Tompkins [1], many problems require the generation of all $n!$ permutations of n marks (henceforth called arrangements). This note presents a generation scheme whereby each step consists of merely transposing two of the marks. The bookkeeping is quite simple, thus this scheme is somewhat faster than either the usual dictionary order method or the Tompkins-Paige method [1]. Also, the important property of leaving the $(j + 1)$st position alone until all $j!$ arrangements of the marks in the first j positions have been generated is preserved.

2. Notation. An arrangement of n marks will be given by an n-tuple, $(m_1, m_2, \cdots m_n)$. A permutation, that is, an operation of permuting an arrangement of marks, will be given in cyclic form, with P's modified by subscripts as entries. The subscripts indicate the position of the marks to be moved in the n-tuple on which the permutation is operating. For example, if $a = (1, 2, 5, 4, 3)$ is an arrangement of five marks and $\rho = (P_1P_3P_2)(P_4P_5)$ is a permutation, then $\rho(a) = (2, 5, 1, 3, 4)$.

The bookkeeping for this generation scheme is handled, as in most schemes of this type, by an ordered set of indices t_k, $k = 2, 3, \cdots, n$, where each t_k assumes the values 1 through k and indicates the progress of the subgeneration of the arrangements of marks in positions 1 to k. (This is essentially the "signature" discussed in [1].) Thus there are $n!$ sets of values for the t_k's, one set for each arrangement of the n marks. The set $t_k = 1$ for all k corresponds to the initial arrangement, and successive sets are formed in dictionary order (assuming increasing significance with increasing subscript). An index k' gives at each step the smallest subscript k for which $t_k \neq k$.

3. The Generation Rules. The transposition required at each step depends on the current value of the index k' and on the corresponding value of $t_{k'+1}$ (t_{n+1} is assumed $= 1$). The rules are:

I. If k' is even, then interchange the marks in positions k' and $k' - 1$.

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II. a. If k' is odd and $t_{k'+1} \leqq 2$ then interchange the marks in positions k' and $k' - 1$.

b. If k' is odd and $2 < t_{k'+1} < k'$, then interchange the marks in positions k' and $k' - t_{k'+1} + 1$.

c. If k' is odd and $t_{k'+1} \geqq k$, then interchange the marks in positions k' and 1.

Before proving that these rules yield all $n!$ arrangements in $n! - 1$ applications (starting with a given arrangement), let us illustrate their application for $n = 5$. The rules apply at step s to yield the arrangement given at step $s + 1$.

Step	t_2	t_3	t_4	t_5	t_6	k'	Arrangement 1	2	3	4	5
1	1	1	1	1	1	2	(1,	2,	3,	4,	5)
2	2	1	1	1	1	3	(2,	1,	3,	4,	5)
3	1	2	1	1	1	2	(2,	3,	1,	4,	5)
4	2	2	1	1	1	3	(3,	2,	1,	4,	5)
5	1	3	1	1	1	2	(3,	1,	2,	4,	5)
6	2	3	1	1	1	4	(1,	3,	2,	4,	5)
7	1	1	2	1	1	2	(1,	3,	4,	2,	5)
⋮	⋮	⋮				⋮	⋮	⋮	⋮		
12	2	3	2	1	1	4	(1,	4,	3,	2,	5)
13	1	1	3	1	1	2	(1,	4,	2,	3,	5)
⋮	⋮	⋮				⋮	⋮	⋮	⋮		
18	2	3	3	1	1	4	(2,	4,	1,	3,	5)
19	1	1	4	1	1	2	(2,	4,	3,	1,	5)
⋮	⋮	⋮				⋮	⋮	⋮	⋮		
24	2	3	4	1	1	5	(3,	4,	2,	1,	5)
25	1	1	1	2	1	2	(3,	4,	2,	5,	1)
⋮	⋮	⋮	⋮			⋮	⋮	⋮	⋮	⋮	
48	2	3	4	2	1	5	(2,	5,	4,	3,	1)
49	1	1	1	3	1	2	(2,	5,	4,	1,	3)
⋮	⋮	⋮	⋮			⋮	⋮	⋮	⋮	⋮	
72	2	3	4	3	1	5	(4,	1,	5,	2,	3)
73	1	1	1	4	1	2	(4,	1,	5,	3,	2)
⋮	⋮	⋮	⋮			⋮	⋮	⋮	⋮	⋮	
96	2	3	4	4	1	5	(5,	3,	1,	4,	2)
97	1	1	1	5	1	2	(5,	3,	1,	2,	4)
⋮	⋮	⋮	⋮			⋮	⋮	⋮	⋮	⋮	
120	2	3	4	5	1	6	(1,	2,	3,	5,	4)

A close inspection of the above example will reveal the mechanism at work. Following a transposition (P_iP_k) with $i < k$ all $(k - 1)!$ arrangements involving change only in positions 1 through $k - 1$ are generated before P_k appears again. During a complete subgeneration of the $k!$ arrangements of the k leftmost positions, the transposition (P_iP_k), for some particular $i < k$, occurs $k - 1$ times, each time $k' = k$. The particular value of i will be $k - 1$, $k - t_{k+1} + 1$, or 1, according to the rule in force. To insure that no duplicate arrangements appear, the mark initially (at the time the subgeneration begins) in position k and the marks successively (each time (P_iP_k) is performed) in position i must all be distinct. This is accomplished in two ways according as k is even or odd. For an example with $k = 4$, compare the marks in position 4 at step 1, and in position 3 at steps 6, 12, and 18.

LEMMA 1. *Let* $\alpha = (P_{i_1}P_{i_2} \cdots P_{i_{k-1}})$ *with* $i_1, i_2, \cdots, i_{k-1} \leqq k - 1$ *be a cycle and let* $j < k$. *Then* $\alpha[(P_jP_k)\alpha]^{k-1} = (P_jP_k)$.

Proof. This is verified by direct permutation multiplication.

The significance of this lemma is the following. Let k be odd and consider any subgeneration of $k!$ arrangements of the marks in the k leftmost positions. During this subgeneration k' will be equal to $k\,k - 1$ times, and we will have $k - 1$ identical applications of rule II interspersed with k identical permutations of the first $k - 1$ positions. If, as Lemma 2 will show, this permutation is a cycle, then Lemma 1 says the effect of the entire subgeneration was as a single application of rule II on the initial arrangement.

LEMMA 2. *For k even,* $(P_1P_{k-1})(P_{k-1}P_k)\,[\prod_{i=1}^{k-2}(P_iP_{k-1})(P_{k-1}P_k)](P_{k-2}P_{k-1}) = \rho_k$, *a single cycle, where* $\rho_2 = (P_1P_2)$, $\rho_4 = (P_1P_4P_2P_3)$ *and in general,* $\rho_k = (P_1P_kP_{k-2}\{P_{k-3}P_{k-5} \cdots P_3\}P_{k-1}\{P_{k-4}P_{k-6} \cdots P_2\})$.

Proof. Again, direct multiplication gives verification.

Thus for k even the effect of complete subgeneration is to permute the k marks by a cycle. For examples of the effects given by these two lemmas, compare the arrangements at steps 1 and 24 and at steps 25 and 48 ($k = 4$) and at steps 1 and 120 ($k = 5$).

Consider now any such subgeneration beginning with the arrangement, say $(m_1, m_2, \cdots, m_k, \cdots m_n)$. With k even, each of the $k - 1$ applications of rule I finds a new mark to put in the kth position, since during this subgeneration t_k is assuming the values $1, 2, \cdots k$, and so by the special construction of rule II (and Lemma 1), position $k - 1$ contains successively, at the time of application of rule I, $m_{k-2}, m_{k-1}, m_{k-3}, m_{k-4}, \cdots, m_1$. With k odd, each of the $k - 1$ applications of rule II finds the $k - 1$ leftmost marks permuted by a $(k - 1)$-cycle (by Lemma 2), and hence also finds a new mark to put in the kth position. A simple induction now shows that such subgenerations yield $k!$ distinct arrangements. We have proved the following:

THEOREM. *The generation scheme as given above yields all* $n!$ *arrangements of* n *marks in exactly* $n! - 1$ *steps (starting from a given arrangement).*

4. Remarks. As in most other generation schemes the property of changing the jth mark only when all arrangements of the previous $j - 1$ marks have been generated allows significant time-savings in some problems. If following a transposition (P_iP_{k+1}) with $i < k + 1$ the problem decides it does not need to use the $k!$ arrangements formed by permuting the present k leftmost marks, then this subgeneration may be skipped by applying Lemma 1 or Lemma 2 according as k is odd or even. This immediately prepares the arrangement for the next application of (P_iP_{k+1}). The permutation of Lemma 2 is not a transposition, but is quite easy to code into the scheme. An interesting question is whether or not an equally simple generation by transposition scheme exists in which block skipping is also always done by transposition.

With the assumption that the marks being permuted are in most problems indices used for address modification, and thus should occupy the address portion of a computer word, a time comparison [2] between this scheme and the Tompkins-Paige method was made on Maniac II. With nine marks the transposition scheme generates arrangements about twenty per cent faster. In addition, the transposition

scheme is advantageous for problems in which minimum mixing of the marks at each step is important.

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Chebyshev Approximations to the Gamma Function

By Helmut Werner and Robert Collinge

In this note several Chebyshev approximations are given for the function $y = \Gamma(x + 2)$ for x in the $0 \leqq x \leqq 1.0$ range. The approximations were obtained from a table of $\Gamma(x + 2)$, employing well-known methods as described in numerous papers; see for instance [1] and the literature quoted there. The table of $\Gamma(x + 2)$ was calculated from the asymptotic expansion of $\log \Gamma(z)$ as given in [2] to provide data accurate to at least 10^{-21}. Compare also [3].

The asymptotic expansion of $\ln \Gamma(z)$ is given by

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \ln \sqrt{2\pi} + \Phi(z)$$

where

$$\Phi(z) = \sum_{r=1}^{n} \frac{(-1)^{r-1} B_r}{2r(2r - 1)} \frac{1}{z^{2r-1}} + R_n(z),$$

and B_r is the rth Bernoulli number.

It can be shown [2] that for $z > 0$ the value of $\Phi(z)$ always lies between the sum of n terms and the sum of $(n + 1)$ terms of the series, for all values of n. In terminating this series with the nth term the error $R_n(z)$ will be less than

$$\frac{B_{n+1}}{2(n + 1)(2n + 1)} \cdot \frac{1}{z^{2n+1}}.$$

By truncating $\Phi(z)$ at the 10th term it is easily shown that for values of $z \geqq 13$, the error in the expansion is less than 5.5×10^{-22}. We therefore replace $\Phi(z)$ by $\sum_{i=1}^{10} A_i/z^{2i-1}$ and calculate $\ln \Gamma(z)$ for values of z in the range $13 \leqq z \leqq 14$.

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