On the Computation of Euler's Constant

By Dura W. Sweeney

1. Introduction. The computation of Euler's constant, γ , to 3566 decimal places by a procedure not previously used is described. As a part of this computation, the natural logarithm of 2 has been evaluated to 3683 decimal places. A different procedure was used in computations of γ performed by J. C. Adams in 1878 [1] and J. W. Wrench, Jr. in 1952 [2], and recently by D. E. Knuth [3]. This latter procedure is critically compared with that used in the present calculation. The new approximations to γ and $\ln 2$ are reproduced *in extenso* at the end of this paper.

2. Evaluation of γ . A new procedure based upon the expansion of the exponential integral, $-E_i(-x)$, was used to evaluate γ rather than the classical approach used by Adams, Wrench, and Knuth. This new procedure was chosen so as to avoid the more complex programming required in the computation of high orders of Bernoulli numbers.

The exponential integral is given as

$$(1) \quad -E_i(-x) = \int_x^\infty \frac{e^{-t} dt}{t} = -\gamma - \ln x + x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \cdots = -\gamma - \ln x + S(x).$$

Its asymptotic expansion is

$$(2) \quad -E_i(-x) = \int_x^\infty \frac{e^{-t} dt}{t} \cong \frac{e^{-x}}{x} \left(1 - \frac{1}{x} + \frac{2!}{x^2} - \cdots \right) = R(x).$$

Equating these and moving γ to the left, we have

$$(3) \quad \gamma \cong S(x) - \ln x - R(x).$$

Since the asymptotic form behaves as e^{-x}/x for large x , the difference between $S(x)$ and $\ln x$ will approximate γ to the accuracy of the number of leading zeros in the value of $R(x)$.

$$(4) \quad \text{For } x = 8192, \quad R(x) = 0.22190 \cdots 10^{-3561}.$$

The value of x was chosen as a power of 2 to simplify the calculation of $\ln x$. Also, since a binary computer was to be used, many of the multiplications in the terms of $S(x)$ could be reduced to shifting operations.

3. Method of Computation. The computation of $\ln 2$ is very rapid and straightforward on a binary computer using one of the forms of the expansion

$$(5) \quad \ln 2 = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$$

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The computation of $S(x)$ is also straightforward, but requires substantially more computer time since for $x = 8192$ almost 30,000 terms are required for convergence, and up to three times the number of digits in the final answer are required during intermediate computations to avoid truncation errors and to compensate for the loss of significant figures arising from subtractions.

The computer program was written to do the computation two different ways to establish the accuracy of the analysis, the programming, and the system operation. Two different binary-to-decimal conversion routines were also used, one with each of the computations.

The first part of the computer run used the following procedure. The individual terms of the expansion,

$$(6) \quad 13 \ln 2 = 13 \left[\frac{5}{1 \cdot 1 \cdot 2^3} + \frac{11}{2 \cdot 3 \cdot 2^5} + \frac{17}{3 \cdot 5 \cdot 2^7} + \cdots \right],$$

were evaluated and summed to form $\ln 8192$. $S(x)$ was evaluated by summing the odd and even terms of the expansion separately to avoid subtractions, thus:

$$(7) \quad S(x) = \left(x + \frac{x^3}{3 \cdot 3!} + \cdots \right) - \left(\frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} + \cdots \right) = x + \sum D_{2n+1} - \sum D_{2n}.$$

The individual terms of these sums were computed from an intermediate value, C_{2n} , as follows:

$$(10) \quad C_{2n} = \frac{x^{2n+1}}{(2n)!} = \frac{x^2}{2n(2n-1)} D_{2n-2}, \quad \text{where}$$

$$(11) \quad D_{2n+1} = \frac{C_{2n}}{(2n+1)^2} \quad \text{and} \quad D_{2n} = \frac{C_{2n}}{2nx}.$$

The second part of the computer run used the following procedure. $\ln 8192$ was evaluated from the following recursion starting at $n = 12,300$:

$$(12) \quad \ln 8192 = 13B_1, \quad B_n = \frac{1}{2} \left(\frac{1}{n} + B_{n+1} \right).$$

$S(x)$ was evaluated by the following recursion starting at $n = 30,000$:

$$(13) \quad S(x) = xA_1, \quad A_n = 1 - \frac{nx}{(n+1)^2} A_{n+1}.$$

The complete computation was performed on the engineering model of the IBM 7094 in 58 minutes. The first part of the computer run took approximately 20 minutes. The second part took approximately 35 minutes. The remaining time was required for non-overlapped printing and punching of results. The same computation was performed again on an IBM 7090 in 114 minutes as part of the tests of the speed and compatibility of the two systems.

The computed values of $\ln 2$ agreed to 3683 decimal places, and the tabulation is believed accurate to that number of decimals. The value of $\ln 2$ confirms the value calculated by H. S. Uhler [4] to 330 decimal places.

The computed values of γ agreed to the same number of decimal places as $\ln 2$, but the accuracy is limited by the value of x to 3561 decimal places. The value of $R(8192)$ given in (4) was subtracted to give the additional five decimal places shown in parentheses in the tabulation. This value of γ is believed accurate to 3566 decimal places and confirms the value calculated by D. E. Knuth to 1270 decimal places.

4. Comparison of Methods. The operating times reported by Knuth presented an opportunity to compare the two methods to determine which might be more useful in extending the value of γ to greater accuracy. An estimate of the time required shows that if the expansion of the exponential integral had been used it would have been substantially faster than the classical method for the evaluation of γ to 1271 decimal places on the Burroughs 220.

$$(14) \quad \text{For } x = 3000, \quad R(x) < 10^{-1300}, \quad \text{and}$$

$$(15) \quad \begin{aligned} \ln 3000 &= \frac{7}{8} \ln 10000 + \frac{1}{2} \ln \left(1 - \frac{1}{10}\right) \\ &= \frac{7}{8} \ln 10000 - \frac{1}{2} \left(\frac{21}{1 \cdot 2 \cdot 10^2} + \frac{43}{3 \cdot 4 \cdot 10^4} + \cdots \right). \end{aligned}$$

Knuth reported a time for the evaluation of $\ln 10000$ of approximately 18 minutes. The additional logarithm would take approximately 4 minutes more.

$S(x)$ would require approximately 10,800 terms for convergence and could probably be most efficiently computed as follows:

$$(16) \quad S(x) = \left(x + \frac{x^3}{3 \cdot 3!} + \cdots \right) - \left(\frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} + \cdots \right) = \sum D_{2n-1} - \sum D_{2n},$$

where

$$(17) \quad D_{2n-1} = \frac{(2n-2)x}{(2n-1)^2} D_{2n-2} \quad \text{and} \quad D_{2n} = \frac{(2n-1)x}{(2n)^2} D_{2n-1}.$$

The evaluation of each D_n would require a multiplication loop, a division loop, and a summation loop which would be used to evaluate each storage word of accuracy for each of the terms required for the convergence of $S(x)$. These loop operations would require less than 10 milliseconds for each word of storage. Since there are 10,800 terms, all that is required is an estimate of the accuracy or number of storage words for each term.

The upper curve in Figure 1 shows the value of $r = \log_{10}(3000^n/n \cdot n!)$. For $n = 3000$, r reaches its maximum value of almost 1300. At this value of n , the value of D_n must be known to 2600 decimal places (260 words of storage). To avoid truncation errors, at least 2600 decimal places must be carried for each $n < 3000$. As n becomes larger, the required accuracy decreases, reaching 1300 decimal places at $n \cong 8200$, and going to zero at $n \cong 10,800$. This is shown as the difference at a particular n between the upper and lower curves in Figure 1. If the accuracy is carried to 2600 decimal places throughout, as shown by the area between the two curves plus the area outlined by the dotted line, the computation of $S(x)$ would have taken 7.8 hours, i.e.,

$$\left(\frac{260 \times 10,800 \times .010}{3600} \right).$$

FIG. 1. $r = \log_{10} \left(\frac{3000^n}{n \cdot n!} \right)$

Since the area bounded by the two curves is less than 75% of the total area considered above, an achievable and still faster time for the evaluation of $S(x)$ would be approximately 5.8 hours. This is to be compared to the approximately 9 hours reported by Knuth for the evaluation of the sum of the first 10,000 reciprocals and the first 250 Bernoulli numbers. When the times for the evaluation of the logarithms are added, the comparison shows that the evaluation of γ by this new method would have required about two-thirds of the time reported by Knuth.

A similar comparison was attempted for the evaluation of γ to 3566 decimal places on an IBM 7094 using the classical method. This would have required the evaluation of the sum of the first 65,536 reciprocals and the first 610 Bernoulli numbers. This approach was abandoned since a "good" lower bound of the time required could not be established with reasonable effort because of the complexity in establishing the accuracy (number of words of storage) needed for each of the Bernoulli numbers used in the recursion for evaluating the next higher Bernoulli number. It also appeared that the storage capacity of the system would have been exceeded, requiring additional time and programming complexity. No auxiliary storage is required for the evaluation of γ using the expansion of the exponential integral on either computer.

It should be noted that there exists a still faster method which remains to be tried. This method will require additional programming effort, but substantially less computer time will be required. For a given x evaluate $\ln x$, $S(x)$ and e^{-x} to twice the number of decimal places which would be expected from the value of $R(x)$. Then evaluate the semi-convergent portion of $R(x)$ and multiply by the value of e^{-x} . When this value of $R(x)$ is subtracted from $S(x) - \ln x$, the accuracy of γ will be extended to that expected from the value of $R(2x)$. This method will be faster since $S(x)$ will require far fewer terms for convergence to a certain accuracy than $S(2x)$; e.g., $S(8192)$ will require approximately 36,000 terms for convergence to about 7200 decimal places, while $S(16384)$ will require almost 60,000 terms to achieve the same accuracy.

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1. J. C. ADAMS, "On the value of Euler's constant," *Proc. Roy. Soc. London*, v. 27, 1878, p. 88-94.
2. J. W. WRENCH, JR., "A new calculation of Euler's constant," *MTAC*, v. 6, 1952, p. 255.
3. D. E. KNUTH, "Euler's constant to 1271 places," *Math. Comp.*, v. 16, 1962, p. 275-281.
4. H. S. UHLER, "Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7, and 17," *Proc. Nat. Acad. Sci.*, v. 26, 1940, p. 205-212.

$\gamma =$	.57721	56649	01532	86060	65120	90082	40243	10421	59335	93992
	35988	05767	23488	48677	26777	66467	09369	47063	29174	67495
	14631	44724	98070	82480	96050	40144	86542	83622	41739	97644
	92353	62535	00333	74293	73377	37673	94279	25952	58247	09491
	60087	35203	94816	56708	53233	15177	66115	28621	19950	15079
	84793	74508	57057	40029	92135	47861	46694	02960	43254	21519
	05877	55352	67331	39925	40129	67420	51375	41395	49111	68510
	28079	84234	87758	72050	38431	09399	73613	72553	06088	93312
	67600	17247	95378	36759	27135	15772	26102	73492	91394	07984
	30103	41777	17780	88154	95706	61075	01016	19166	33401	52278
	93586	79654	97252	03621	28792	26555	95366	96281	76388	79272
	68013	24310	10476	50596	37039	47394	95763	89065	72967	92960
	10090	15125	19595	09222	43501	40934	98712	28247	94974	71956
	46976	31850	66761	29063	81105	18241	97444	86783	63808	61749
	45516	98927	92301	87739	10729	45781	55431	60050	02182	84409
	60537	72434	20328	54783	67015	17739	43987	00302	37033	95183
	28690	00155	81939	88042	70741	15422	27819	71652	30110	73565
	83396	73487	17650	49194	18123	00040	65469	31429	99297	77956
	93031	00503	08630	34185	69803	23108	36916	40025	89297	08909
	85486	82577	73642	88253	95492	58736	29596	13329	85747	39302
	37343	88470	70370	28441	29201	66417	85024	87333	79080	56275
	49984	34590	76164	31671	03146	71072	23700	21810	74504	44186
	64759	13480	36690	25532	45862	54422	25345	18138	79124	34573
	50136	12977	82278	28814	89459	09863	84600	62931	69471	88714
	95875	25492	36649	35204	73243	64109	72682	76160	87759	50880
	95126	20840	45444	77992	29915	72482	92516	25127	84276	59657
	08321	46102	98214	61795	19579	59095	92270	42089	89627	97125
	53632	17948	87376	42106	60607	06598	25619	90102	88075	61251
	99137	51167	82176	43619	05705	84407	83573	50158	00560	77457
	93421	31449	88500	78641	51716	15194	56570	61704	32450	75008
	16870	52307	89093	70461	43066	84817	91649	68425	49150	49672
	43121	83783	87535	64894	95086	84541	02340	60162	25085	15583
	86723	49441	87880	44094	07701	06883	79511	13078	72023	42639
	52269	20971	60885	69083	82511	37871	28368	20491	17892	59447
	84861	99118	52939	10293	09905	92552	66917	27446	89204	43869
	71114	71745	71574	57320	39352	09122	31608	50868	27558	89010
	94516	81181	01687	49754	70969	36667	12102	06304	82716	58950
	49327	31486	08749	40207	00674	25909	18248	75962	13738	42311
	44265	31350	29230	31751	72257	22162	83248	83811	24589	57438
	62398	70375	76628	55130	33143	92999	54018	53134	14158	62127
	88648	07611	00301	52119	65780	06811	77737	63501	68183	89733
	89663	98689	57932	99145	63886	44310	37060	80781	74489	95795
	83245	79418	96202	60498	41043	92250	78604	60362	52772	60229
	19682	99586	09883	39013	78717	14226	91788	38195	29844	56079
	16051	97279	73604	75910	25109	95779	13351	57917	72251	50254
	92932	46325	02874	76779	48421	58405	07599	29040	18557	64599
	01862	69267	76437	26605	71176	81336	55908	81554	81074	70000
	62336	37252	88949	55463	69714	33012	00791	30855	52639	59549
	78230	23144	03914	97404	94746	82594	73208	46185	24605	87766
	94882	87953	01040	63491	72292	18580	08706	77069	04279	26743

γ (continued)

28444	69685	14971	82567	80958	41654	49185	14575	33196	40633
11993	73821	57345	08749	88325	56088	88735	28019	01915	50896
88554	68259	24544	45277	28173	05730	10806	06177	01136	37731
82462	92466	00812	77162	10186	77446	84959	51428	17901	45111
94893	42288	34482	53075	31187	01860	97612	24623	17674	97755
64124	61983	85640	14841	23587	17724	95542	24820	16151	76579
94080	62968	34242	89057	25947	39269	63863	38387	43805	47131
96764	29268	37249	07608	75073	78528	37023	04686	50349	05120
34227	21743	66897	92848	62972	90889	26789	77703	26246	23912
26188	87653	00577	86274	36060	94443	60392	80977	08133	83693
42355	08583	94112	67092	18734	41451	21878	03276	15050	94780
55466	30058	68455	63152	45460	53151	13252	81889	10792	31491
31103	23443	02450	93345	00030	76558	64874	22297	17700	33178
45391	50566	94015	99884	92916	09114	00294	86902	08848	53816
97009	55156	63470	55445	22176	40358	62939	82865	81312	38701
32535	88006	25686	62692	69977	67737	73068	32269	00916	08510
45150	02261	07180	25546	59284	93894	92775	95897	54076	15599
33782	64824	19795	06418	68143	78817	18508	85408	03679	96314
23954	00919	64388	75007	89000	00627	99794	28098	86372	99259
19777	65040	40992	20379	40427	61681	78371	56686	53066	93983
09165	24322	70595	53041	76673	66401	16792	95901	29305	37449
71830	80042	7(58486)							

$\ln 2 =$.69314 71805 59945 30941 72321 21458 17656 80755 00134 36025
 52541 20680 00949 33936 21969 69471 56058 63326 99641 86875
 42001 48102 05706 85733 68552 02357 58130 55703 26707 51635
 07596 19307 27570 82837 14351 90307 03862 38916 73471 12335
 01153 64497 95523 91204 75172 68157 49320 65155 52473 41395

 25882 95045 30070 95326 36664 26541 04239 15781 49520 43740
 43038 55008 01944 17064 16715 18644 71283 99681 71784 54695
 70262 71631 06454 61502 57207 40248 16377 73389 63855 06952
 60668 34113 72738 73722 92895 64935 47025 76265 20988 59693
 20196 50585 54764 70330 67936 54432 54763 27449 51250 40606

 94381 47104 68994 65062 20167 72042 45245 29612 68794 65461
 93165 17468 13926 72504 10380 25462 59656 86914 41928 71608
 29380 31727 14367 78265 48775 66485 08567 40776 48451 46443
 99404 61422 60319 30967 35402 57444 60703 08096 08504 74866
 38523 13818 16767 51438 66747 66478 90881 43714 19854 94231

 51997 35488 03751 65861 27535 29166 10007 10535 58249 87941
 47295 09293 11389 71559 98205 65439 28717 00072 18085 76102
 52368 89213 24497 13893 20378 43935 30887 74825 97017 15591
 07088 23683 62758 98425 89185 35302 43634 21436 70611 89236
 78919 23723 14672 32172 05340 16492 56872 74778 23445 35347

 64811 49418 64238 67767 74406 06956 26573 79600 86707 62571
 99184 73402 26514 62837 90488 30620 33061 14463 00737 19489
 00274 36439 65002 58093 65194 43041 19115 06080 94879 30678
 65158 87090 06052 03468 42973 61938 41289 65255 65396 86022
 19412 29242 07574 32175 74890 97705 75268 71158 17051 13700

 91589 42665 47859 59648 90653 05846 02586 68382 94002 28330
 05382 07400 56770 53046 78700 18416 24044 18833 23279 83863
 49001 56312 18895 60650 55315 12721 99398 33203 07514 08426
 09147 90012 65168 24344 38935 72472 78820 54862 71552 74187
 72430 02489 79454 01961 87233 98086 08316 64811 49093 06675

 19339 31289 04316 41370 68139 77764 98176 97486 89038 87789
 99129 65036 19270 71088 92641 05230 92478 39173 73501 22984
 24204 99568 93599 22066 02204 65494 15106 13918 78857 44245
 57751 02068 37030 86661 94808 96412 18680 77902 08181 58858
 00016 88115 97305 61866 76199 18739 52007 66719 21459 22367

 20602 53959 54365 41655 31129 51759 89940 05600 03665 13567
 56905 12459 26825 74394 64831 68332 62490 18038 24240 82423
 14523 06140 96380 57007 02551 38770 26817 85163 06902 55137
 03234 05380 21450 19015 37402 95099 42262 99577 96474 27138
 15736 38017 29873 94070 42421 79972 26696 29799 39312 70693

 57472 40493 38653 08797 58721 69964 51294 46491 88377 11567
 01678 59880 49818 38896 78413 49383 14014 07316 64727 65327
 63591 92335 11233 38933 87095 13209 05927 21854 71328 97547
 07978 91384 44546 66761 92702 88553 34234 29899 32180 37691
 54973 34026 75467 58873 23677 83429 16191 81043 01160 91695

 26554 78597 32891 76354 55567 42863 87746 39871 01912 43175
 42558 88301 20677 92102 80341 20687 97591 43081 28330 72303
 00883 49470 57924 96591 00586 00123 41561 75741 32724 65943
 06843 54652 11135 02154 43415 39955 38185 65227 50221 42456
 64400 06276 18330 32064 72725 72197 51529 08278 56842 13207

In 2 (*continued*)

95988	63896	72771	19552	21881	90466	03957	00977	47065	12619
50527	89322	96088	93140	56254	33442	55239	20620	30343	94177
73579	45592	12590	19925	59114	84402	42390	12554	25900	31295
37051	92206	15064	34583	78787	30020	35414	42178	57580	13236
45166	07099	14383	14500	49858	96688	57722	21486	52882	16941
81270	48860	75897	22032	16663	12837	83291	56763	07498	72985
74638	92826	93735	09840	77804	93950	04933	99876	26475	50703
16221	61390	34845	29942	49172	48373	40613	66226	38349	36811
16841	67056	92521	47513	83930	63845	53718	62687	79732	88955
58871	63442	97562	44755	39236	63694	88877	82389	01749	81027
35655	24050	51854	77306	19440	52423	22125	59024	83308	27788
88890	59629	11972	99545	74415	62451	24859	26831	12607	46797
28163	80902	50005	65599	91461	28332	54358	11140	48482	06064
08242	24792	40385	57647	62350	31100	32425	97091	42501	11461
55848	30670	01258	31821	91534	72074	74111	94009	83557	32728
26144	27382	13970	70477	95625	96705	79023	03384	80617	13455
55368	55375	81065	74973	44479	22511	19654	61618	27896	01006
85129	65395	47965	86637	83522	47362	45460	93585	03605	06784
14391	14452	31457	78033	59179	21127	95570	50555	54514	38788
81881	53519	48593	44672	46429	49864	05062	65184	24475	39566
37833	73482	20753	32944	81306	49336	03546	10101	77464	93267
87716	71986	12073	96832	01235	96077	29024	68304	59403	13056
37763	13240	10804	20285	43590	26945	09403	07400	14933	95076
73160	28502	86973	03187	18239	98433	525			