this to

$$\begin{aligned}
 a_n(k) &= (-1)^n 2 \int_0^\infty e^{-t^2} J_{2n+1}(2kt) dt \\
 (5) \qquad &= (-1)^n \sqrt{\pi} e^{-k^2/2} I_{n+(1/2)}(k^2/2) \\
 &= \sum_{r=0}^n \frac{(n+r)!}{r!(n-r)!} k^{-2r-1} [(-1)^{r+n} - e^{-k^2}], \qquad n = 0, 1, 2, \dots
 \end{aligned}$$

This expression may easily be seen to be consistent with (4).

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First One Hundred Zeros of $J_0(x)$ Accurate to 19 Significant Figures

By Henry Gerber

1. Introduction. Some physical investigations require a knowledge of accurate values of the zeros of the Bessel function $J_0(x)$. The most extensive values previously published are those of the British Association for the Advancement of Science [1], which consist of 10 decimal places. More accurate values have now been computed, and are presented in Table 1. The minimum accuracy of the tabulated zeros is 19 significant figures.

2. Method of Computation. Two methods were used to compute the roots. The first twelve roots were computed by the method of "false position." The values of

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TABLE 1
The first one hundred roots of $J_0(x) = 0$
($n = \text{Number of Zero}$)

n	x_n				n	x_n			
1	2.40482	55576	95772	769	51	159.43661	11642	63146	32
2	5.52007	81102	86310	649	52	162.57818	86689	46677	52
3	8.65372	79129	11012	216	53	165.71976	67479	55020	87
4	11.79153	44390	14281	615	54	168.86134	53692	35825	69
5	14.93091	77084	87785	948	55	172.00292	45030	78200	21
6	18.07106	39679	10922	545	56	175.14450	41219	02743	06
7	21.21163	66298	79258	960	57	178.28608	42000	73770	68
8	24.35247	15307	49302	736	58	181.42766	47137	31050	79
9	27.49347	91320	40254	79	59	184.56924	56406	38718	14
10	30.63460	64684	31975	12	60	187.71082	69600	49359	78
11	33.77582	02135	73568	69	61	190.85240	86525	81522	32
12	36.91709	83536	64043	98	62	193.99399	07001	09119	79
13	40.05842	57646	28239	29	63	197.13557	30856	61414	74
14	43.19979	17131	76730	36	64	200.27715	57933	32411	78
15	46.34118	83716	61814	02	65	203.41873	88081	98646	17
16	49.48260	98973	97817	17	66	206.56032	21162	44473	65
17	52.62405	18411	14996	03	67	209.70190	57042	94075	20
18	55.76551	07550	19979	31	68	212.84348	95599	49482	75
19	58.90698	39260	80942	13	69	215.98507	36715	34013	16
20	62.04846	91902	27169	88	70	219.12665	80280	40567	46
21	65.18996	48002	06860	44	71	222.26824	26190	84314	34
22	68.33146	93298	56798	27	72	225.40982	74348	59329	90
23	71.47298	16035	93732	82	73	228.55141	24660	98813	30
24	74.61450	06437	01837	88	74	231.69299	77040	38538	78
25	77.75602	56303	88055	04	75	234.83458	31403	83241	02
26	80.89755	58711	37627	86	76	237.97616	87672	75662	86
27	84.03909	07769	38190	16	77	241.11775	45772	68022	51
28	87.18062	98436	41153	65	78	244.25934	05632	95682	56
29	90.32217	26372	10480	06	79	247.40092	67186	52824	85
30	93.46371	87819	44774	17	80	250.54251	30369	69955	47
31	96.60526	79509	96268	78	81	253.68409	95121	93081	00
32	99.74681	98586	80596	47	82	256.82568	61385	64413	02
33	102.88837	42541	94794	60	83	259.96727	29106	04471	57
34	106.02993	09164	51615	51	84	263.10885	98230	95470	69
35	109.17148	96498	05383	55	85	266.25044	68710	65880	12
36	112.31305	02804	94909	63	86	269.39203	40497	76067	14
37	115.45461	26536	66939	63	87	272.53362	13547	04931	45
38	118.59617	66308	72531	72	88	275.67520	87815	37453	85
39	121.73774	20879	50962	96	89	278.81679	63261	53086	58
40	124.87930	89132	32946	04	90	281.95838	39846	14919	85
41	128.02087	70060	08324	08	91	285.09997	17531	59564	54
42	131.16244	62752	13914	61	92	288.24155	96281	87696	44
43	134.30401	66383	05466	10	93	291.38314	76062	55212	24
44	137.44558	80202	84277	79	94	294.52473	56840	64951	46
45	140.58716	03528	54296	55	95	297.66632	38584	58942	52
46	143.72873	35736	89732	53	96	300.80791	21264	11134	77
47	146.87030	76257	96649	59	97	303.94950	04850	20581	11
48	150.01188	24569	54757	49	98	307.09108	89315	05039	11
49	153.15345	80192	27892	49	99	310.23267	74631	94960	95
50	156.29503	42685	33523	82	100	313.37426	60775	27844	72

$J_0(x)$ corresponding to a given trial root x were calculated by direct interpolation of the Harvard tables [2], which give $J_0(x)$ accurate to 18 decimal places. For $0 \leq x \leq 25$ the argument increment h is 0.001; for $25 < x \leq 100$ the increment is 0.01. Seven terms of the Newton-Bessel central difference formula [3] were used in the interpolation. This formula requires eight tabulated values of $J_0(x_0 + mh)$, where

$$x_0 = \text{greatest tabulated argument not exceeding } x$$

$$m = \pm 1, \pm 2, \pm 3, -4.$$

This method of computation has two advantages. First, in the vicinity of a zero of $J_0(x)$ the tabulated values consist of only 14 to 16 significant figures. The double-precision method of programming the IBM 7090 computer permits calculations with 17 significant digits. Thus, the above values of $J_0(x_0 + mh)$, which serve as "constants" for the interpolation process, can be entered into the computer without error.

Secondly, the interpolation variable u is given by the relationship

$$(1) \quad u = (x - x_0)/h$$

where

$$(2) \quad x_0 < x < x_0 + h.$$

The number of significant figures in the root, x , is thus equal to the sum of the number of significant figures in u and x_0 . An examination of the interpolation formula shows that fewer than two significant digits are lost because of round-off error. Consequently the variable, u , can be calculated accurate to 15 significant figures. Interpolation of the Harvard tables by means of double-precision computation thus gives the roots accurate to 18 decimal places for $x \leq 25$, and 17 decimal places for $25 < x < 100$.

The roots of $J_0(x)$ can also be computed by the following asymptotic series given by Bickley and Miller [4]. Let

$$(3) \quad c_n = 1/(4n - 1)\pi \quad n = 1, 2, 3 \dots$$

The n th root $j_{0,n}$ is then given by the expression

$$(4) \quad j_{0,n} = \left(n - \frac{1}{4}\right)\pi + \frac{c_n}{2} - \frac{31c_n^3}{6} + \frac{3779c_n^5}{15} - \frac{62\,77237c_n^7}{210} \\ + \frac{20921\,63573c_n^9}{315} - \frac{824\,97257\,36393c_n^{11}}{3465} \\ + \frac{847\,49688\,72511\,28654c_n^{13}}{6\,75675} \dots$$

The first one hundred roots were computed by means of Eq. (4). For n equal to or larger than 11, roots calculated by the two methods agree to 17 decimal places. This agreement confirms the validity of Eq. (4), and confirms the accuracy of the corresponding zeros in Table 1. It is interesting to note that discrepancies in the 10th decimal place of x_n occur between the data of Table 1 and the earlier tables at $n = 4, 5, 8, 41, 45, 85, 95$, and 100. These differences, which are all less than 1.2×10^{-10} , are presumably due to errors in the previous calculations.

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Polylogarithms, Dirichlet Series, and Certain Constants

By Daniel Shanks

The *polylogarithms* $F_s(z)$ are defined by

$$(1) \quad F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s}$$

for $|z| < 1$ and for the real part of $s \geq 0$, and by analytic continuation for other values of z and s . They can be regarded as functions of z , with a parameter s , given by the power series (1), or as functions of s , with a parameter z , given by the Dirichlet series (1).

Recently [1] we discussed the Dirichlet series defined by

$$(2) \quad L_a(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{-a}{2k+1}\right)}{(2k+1)^s}$$

and its analytic continuation, where $\left(\frac{-a}{2k+1}\right)$ is the Jacobi symbol. It is expressible in closed form for three-quarters of all combinations of integers a and s ; namely, for $s \leq 1$ and all a , for s even and > 1 if $a < 0$, and for s odd and > 1 if $a > 0$.

The remaining, non-closed form $L_a(n)$ for $a = \pm 2, \pm 3$, and ± 6 , with $n \leq 10$, were computed [1] by a device, which (in essence) is based on the fact that all of the so-called *characters* modulo 8, 12, or 24 are real. In contrast, the corresponding $L_a(n)$ for $a = \pm 5, \pm 7$, and ± 10 , say, which were also desired, are not obtainable by that method, unless it is modified, since now some of the characters are complex.

We did, however, express $L_a(s)$ as a linear combination of the functions $S_s(x)$ or $C_s(x)$ for various values of x determined by the integer a [1, equations (24)–(27)]. These functions [1, equation (18)] are defined by

$$(3) \quad \begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s}, \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s}. \end{aligned}$$