Evaluation at Half Periods of Weierstrass' Elliptic Function with Rhombic Primitive Period-Parallelogram

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In a recent note [1], the senior author evaluated the Weierstrass' elliptic function at half periods to 16D for the case of rectangular primitive period-parallelogram, i.e., the three functions

$$(1) \qquad e_1 = \wp(\omega_1), \qquad e_2 = \wp(\omega_2), \qquad e_3 = \wp(\omega_3)$$

with double periods $2\omega_1 = 1$ and $2\omega_2 = ai$; ω_3 being defined by

$$(2) \qquad \omega_1 + \omega_2 + \omega_3 = 0.$$

In this note, these three functions will be evaluated, also to 16D, for the case of rhombic primitive period-parallelogram. Two methods of evaluation are given. One is similar to that described previously [1] and the other is in terms of Jacobian Theta functions. As no generality is lost, here the double periods are represented by $2\omega_1 = 1$ and $2\omega_2 = \frac{1}{2} + ci$, where c is a positive real quantity.

It is known that these three functions form a set of distinct roots of the following cubic [2],

$$(3) \qquad x^3 - (15\sigma_4)x - (35\sigma_6) = 0$$

where, in the case under consideration,

$$(4) \qquad \begin{aligned} \sigma_{2k} &= \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(2m\omega_1 + 2n\omega_2)^{2k}} \\ &= 2\sum_{m=1}^{\infty} \frac{1}{m^{2k}} + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\{m + n(\frac{1}{2} + ci)\}^{2k}} \\ &= 2\sum_{m=1}^{\infty} \frac{1}{m^{2k}} + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\{m + \frac{1}{2} + (2n-1)ci\}^{2k}} \\ &\qquad + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{(m + 2nci)^{2k}}. \end{aligned}$$

The accent on the first summation sign denotes the omission of simultaneous zeros of m and n. The cubic (3) indicates that the sum of the three functions vanishes identically, i.e.,

$$(5) \qquad e_1 + e_2 + e_3 = 0.$$

In a similar manner as described previously [1], it can be shown that

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$$
(6) \quad
\begin{aligned}
\frac{1}{\pi^4}\sum_{m=-\infty}^{\infty}\frac{1}{(m+ix)^4} &= \frac{2}{3\sinh^2\pi x}+\frac{1}{\sinh^4\pi x},\\
\frac{1}{\pi^6}\sum_{m=-\infty}^{\infty}\frac{1}{(m+ix)^6} &= -\frac{2}{15\sinh^2\pi x}-\frac{1}{\sinh^4\pi x}-\frac{1}{\sinh^6\pi x},\\
\frac{1}{\pi^4}\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2}+ix)^4} &= -\frac{2}{3\cosh^2\pi x}+\frac{1}{\cosh^4\pi x},\\
\frac{1}{\pi^6}\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2}+ix)^6} &= \frac{2}{15\cosh^2\pi x}-\frac{1}{\cosh^4\pi x}+\frac{1}{\cosh^6\pi x}.
\end{aligned}
$$

Consequently, we have

$$
(7) \qquad \sigma_4 = \frac{4\pi^4}{3}\left(\frac{1}{60}+K_1+K_3\right), \qquad \sigma_6 = \frac{4\pi^6}{15}\left(\frac{1}{126}-K_2+K_4\right)
$$

where

$$
(8) \quad
\begin{aligned}
K_1 &= \sum_{n=2,4,6,\cdots}^{\infty}\left(\frac{1}{\sinh^2 n\pi c}+\frac{3}{2\sinh^4 n\pi c}\right),\\
K_2 &= \sum_{n=2,4,6,\cdots}^{\infty}\left(\frac{1}{\sinh^2 n\pi c}+\frac{15}{2\sinh^4 n\pi c}+\frac{15}{2\sinh^6 n\pi c}\right),\\
K_3 &= \sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{1}{\cosh^2 n\pi c}-\frac{3}{2\cosh^4 n\pi c}\right),\\
K_4 &= \sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{1}{\cosh^2 n\pi c}-\frac{15}{2\cosh^4 n\pi c}+\frac{15}{2\cosh^6 n\pi c}\right).
\end{aligned}
$$

The evaluation of e_1, e_2, and e_3 requires the solution of the cubic (3). The real root gives the value of e_1 and the pair of complex conjugate roots the values of e_2 and e_3. The former can be found from the cubic (3) directly by using Newton's method and the latter from the depressed equation of the cubic.

A second method of evaluation may be as follows. It is known that [3]*

$$
(9) \quad
\begin{aligned}
12\omega_1^2 e_1 &= \pi^2(\theta_3^4+\theta_4^4),\\
12\omega_1^2 e_2 &= -\pi^2(\theta_2^4+\theta_3^4),\\
12\omega_1^2 e_3 &= \pi^2(\theta_2^4-\theta_4^4)
\end{aligned}
$$

where θ_2, θ_3, and θ_4 are Jacobian Theta functions defined by

$$
(10) \quad
\begin{aligned}
\theta_2 &= 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)},\\
\theta_3 &= 1+2\sum_{n=1}^{\infty} q^{n^2},\\
\theta_4 &= 1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2}
\end{aligned}
$$

in which

* e_2 and e_3 have been permuted to conform with the present notation. ω_1 is identified with ω in [3].

$$q = \exp(i\pi\omega_2/\omega_1). \tag{11}$$

It is noted that the preceding three Theta functions are not all independent but are connected by the relation

$$\theta_2^4 + \theta_4^4 = \theta_3^4. \tag{12}$$

Now denote

$$\begin{aligned} p &= e^{-\pi c}, \\ A &= \tfrac{1}{2} + p^4 + p^{16} + p^{36} + \cdots, \\ B &= p + p^9 + p^{25} + \cdots. \end{aligned} \tag{13}$$

We then find, for $\omega_2/\omega_1 = \frac{1}{2} + ci$,

$$\begin{aligned} e_1 &= \frac{32\pi^2}{3}(A^4 - 6A^2B^2 + B^4), \\ e_2 &= -\tfrac{1}{2}e_1 - 64\pi^2(A^3B - AB^3)i, \\ e_3 &= -\tfrac{1}{2}e_1 + 64\pi^2(A^3B - AB^3)i. \end{aligned} \tag{14}$$

TABLE 1

c	σ_4	σ_6
0.05	$2.16464\ 64674\ 10478 \times 10^4$	$-2.03468\ 61239\ 92188 \times 10^6$
0.1	$1.35285\ 51098\ 10968 \times 10^3$	$-3.17943\ 85391\ 87710 \times 10^4$
0.15	$2.65424\ 44894\ 89441 \times 10^2$	$-2.83086\ 37451\ 64231 \times 10^3$
0.2	$7.67059\ 68640\ 15799 \times 10$	$-5.92699\ 26236\ 38137 \times 10^2$
0.25	$1.93710\ 87114\ 46967 \times 10$	$-2.45332\ 24175\ 29611 \times 10^2$
$\sqrt{3}/6$	0	$-1.58301\ 85572\ 24858 \times 10^2$
0.3	$-3.62453\ 58222\ 15253$	$-1.40825\ 71800\ 11392 \times 10^2$
0.35	$-1.29338\ 25507\ 01956 \times 10$	$-8.18148\ 86969\ 71514 \times 10$
0.4	$-1.55320\ 57313\ 46534 \times 10$	$-4.14353\ 39381\ 59347 \times 10$
0.45	$-1.47955\ 79359\ 61305 \times 10$	$-1.52034\ 34879\ 06628 \times 10$
0.5	$-1.26048\ 48008\ 61559 \times 10$	0
0.6	$-7.49518\ 25513\ 33490$	$1.04695\ 23857\ 33236 \times 10$
0.7	$-3.54385\ 93824\ 64415$	$9.96973\ 81357\ 38993$
0.8	$-1.04685\ 36509\ 89181$	$7.37508\ 83157\ 10668$
$\sqrt{3}/2$	0	$5.86303\ 16934\ 25402$
0.9	$4.02787\ 83609\ 98395 \times 10^{-1}$	$5.22023\ 89750\ 45370$
1	$1.21069\ 29446\ 54355$	$3.83331\ 62773\ 90017$
1.25	$1.96367\ 27971\ 88045$	$2.42769\ 61786\ 42181$
1.5	$2.12275\ 22872\ 98572$	$2.11722\ 17565\ 99152$
1.75	$2.15593\ 25105\ 11680$	$2.05187\ 98601\ 55075$
2	$2.16283\ 47970\ 89782$	$2.03826\ 19185\ 60751$
2.5	$2.16456\ 81756\ 97548$	$2.03484\ 06650\ 80135$
3	$2.16464\ 30841\ 25693$	$2.03469\ 28023\ 27625$
4	$2.16464\ 64611\ 04163$	$2.03468\ 61364\ 40353$
5	$2.16464\ 64674\ 10478$	$2.03468\ 61239\ 92188$
6	$2.16464\ 64674\ 22254$	$2.03468\ 61239\ 68942$
∞	$2.16464\ 64674\ 22276$	$2.03468\ 61239\ 68898$

TABLE 2

c	e_1	e_2 and e_3
0.05	$-6.57973\ 62673\ 89319 \times 10^2$	$3.28986\ 81336\ 94660 \times 10^2 \mp i\ 1.18989\ 31466\ 96521 \times 10^{-3}$
0.1	$-1.64492\ 81173\ 83390 \times 10^2$	$8.22464\ 05869\ 16948 \times 10 \mp i\ 7.66281\ 94807\ 82929 \times 10^{-1}$
0.15	$-7.30584\ 93638\ 31184 \times 10$	$3.65292\ 46819\ 15592 \times 10 \mp i\ 4.66807\ 08317\ 54428$
0.2	$-4.07403\ 58971\ 91193 \times 10$	$2.03701\ 79485\ 95597 \times 10 \mp i\ 9.70788\ 89192\ 14831$
0.25	$-2.51415\ 52474\ 67755 \times 10$	$1.25707\ 76237\ 33877 \times 10 \mp i\ 1.35464\ 73301\ 58955 \times 10$
$\sqrt{3}/6$	$-1.76950\ 31908\ 45431 \times 10$	$8.84751\ 59542\ 27155 \mp i\ 1.53243\ 47153\ 49767 \times 10$
0.3	$-1.59549\ 03411\ 11651 \times 10$	$7.97745\ 17055\ 58256 \mp i\ 1.56616\ 48842\ 85086 \times 10$
0.35	$-9.84355\ 86738\ 94376$	$4.92177\ 93369\ 47188 \mp i\ 1.63303\ 12860\ 75360 \times 10$
0.4	$-5.50761\ 80318\ 03648$	$2.75380\ 90159\ 01824 \mp i\ 1.59915\ 99419\ 38787 \times 10$
0.45	$-2.33992\ 65029\ 49420$	$1.16996\ 32514\ 74710 \mp i\ 1.50346\ 31103\ 67526 \times 10$
0.5	0	$0 \mp i\ 1.37503\ 71636\ 04075 \times 10$
0.6	$3.01540\ 56900\ 68455$	$-1.50770\ 28450\ 34228 \mp i\ 1.09200\ 38547\ 40336 \times 10$
0.7	$4.66025\ 66846\ 27599$	$-2.33012\ 83423\ 13799 \mp i\ 8.33344\ 97665\ 68968$
0.8	$5.55021\ 87470\ 32100$	$-2.77510\ 93735\ 16050 \mp i\ 6.22948\ 64049\ 74943$
$\sqrt{3}/2$	$5.89834\ 39694\ 84770$	$-2.94917\ 19847\ 42385 \mp i\ 5.10811\ 57178\ 32557$
0.9	$6.02887\ 68932\ 08147$	$-3.01443\ 84466\ 04073 \mp i\ 4.60637\ 60054\ 51221$
1	$6.28538\ 81186\ 69387$	$-3.14269\ 40593\ 34693 \mp i\ 3.38661\ 83253\ 97387$
1.25	$6.51845\ 74355\ 05910$	$-3.25922\ 87177\ 52955 \mp i\ 1.55326\ 22270\ 74373$
1.5	$6.56699\ 37384\ 39408$	$-3.28349\ 68692\ 19704 \mp i\ 7.09063\ 29833\ 67643 \times 10^{-1}$
1.75	$6.57708\ 71865\ 96458$	$-3.28854\ 35932\ 98229 \mp i\ 3.23371\ 67067\ 21421 \times 10^{-1}$
2	$6.57918\ 55702\ 81884$	$-3.28959\ 27851\ 40942 \mp i\ 1.47445\ 31124\ 21317 \times 10^{-1}$
2.5	$6.57971\ 24695\ 33558$	$-3.28985\ 62347\ 66779 \mp i\ 3.06512\ 77923\ 13172 \times 10^{-2}$
3	$6.57973\ 52389\ 94009$	$-3.28986\ 76194\ 97004 \mp i\ 6.37177\ 83442\ 38741 \times 10^{-3}$
4	$6.57973\ 62654\ 72430$	$-3.28986\ 81327\ 36215 \mp i\ 2.75349\ 51572\ 21704 \times 10^{-4}$
5	$6.57973\ 62673\ 89319$	$-3.28986\ 81336\ 94660 \mp i\ 1.18989\ 31466\ 96490 \times 10^{-5}$
6	$6.57973\ 62673\ 92899$	$-3.28986\ 81336\ 96450 \mp i\ 5.14199\ 45183\ 96954 \times 10^{-7}$
∞	$6.57973\ 62673\ 92906$	$-3.28986\ 81336\ 96453 \mp i\ 0$

σ_4 and σ_6 may be found from the relations

$$\sigma_4 = \tfrac{1}{30}(e_1^2 + e_2^2 + e_3^2), \qquad (15)$$
$$\sigma_6 = \tfrac{1}{35}e_1e_2e_3 .$$

Values of σ_4 and σ_6 are computed by the first method and those of e_1, e_2, e_3 by the second method for $c = 0.05(0.05)0.5(0.1)1(0.25)2(0.5)3(1)6$ and ∞. They are then checked by (15). A separate check is also made on the imaginary part of e_2 or e_3, of which the numerical value is alternately equal to $8\pi^2pC^4$ by virtue of (12), where

$$C = 1 - p^2 - p^6 + p^{12} + p^{20} - p^{30} - \cdots . \qquad (16)$$

Those for $c = \sqrt{3}/6$ and $c = \sqrt{3}/2$ are also computed. In both such cases, the value of σ_4 vanishes identically. The results to 16D are shown in Tables 1–2. In performing the computation, auxiliary tables of $e^{\pi a}$ and $e^{-\pi a}$ with adequate guarding figures are first prepared for $a = 0.1(0.1)10$. Besides, the following values are prepared.

$$e^{\sqrt{3}\pi/2} = 1.51909\ 37703\ 74778\ 01395 \times 10, \qquad (17)$$
$$e^{-\sqrt{3}\pi/2} = 6.58287\ 21011\ 29665\ 13439 \times 10^{-2}.$$

It is thought that the tables of $e^{\pi a}$ and $e^{-\pi a}$ may be of use elsewhere and therefore they are also shown here in Table 3 to 21D.

If σ_{2k} and e_s are expressed as functions of c, the following relations hold.

$$\sigma_4(c') = (2c)^4\sigma_4(c), \qquad (18)$$
$$\sigma_6(c') = -(2c)^6\sigma_6(c)$$

and

$$e_1(c') = -(2c)^2e_1(c),$$
$$e_2(c') = -(2c)^2e_3(c), \qquad (19)$$
$$e_3(c') = -(2c)^2e_2(c)$$

where

$$cc' = \tfrac{1}{4}. \qquad (20)$$

It is noted that here the pair of double periods may alternately be represented by the pair $be^{-i\alpha}$ and $be^{i\alpha}$, in which

$$b = (\tfrac{1}{4} + c^2)^{1/2}, \qquad \alpha = \tan^{-1} 2c. \qquad (21)$$

It appears that the rectangular and rhombic primitive period-parallelograms are the only cases in which σ_4 and σ_6 are both real.

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Table 3

a	$e^{\pi a}$	$e^{-\pi a}$
0.1	1.36910 77706 24846 88484	7.30402 69104 86456 10873 (−1)*
0.2	1.87445 60875 85338 35057	5.33488 09109 11032 51176 (−1)
0.3	2.56633 23952 08135 32094	3.89661 13737 53467 94902 (−1)
0.4	3.51358 56242 85733 63770	2.84609 54333 60292 80116 (−1)
0.5	4.81047 73809 65351 65547	2.07879 57635 07619 08547 (−1)
0.6	6.58606 19626 94724 85836	1.51835 80198 06488 86883 (−1)
0.7	9.01702 86109 42078 24233	1.10901 27836 41952 22245 (−1)
0.8	1.23452 83939 18736 85795 (1)	8.10025 92157 94312 88095 (−2)
0.9	1.69020 24171 71154 60199 (1)	5.91645 11294 07757 88852 (−2)
1.0	2.31406 92632 77926 90057 (1)	4.32139 18263 77224 97744 (−2)
1.1	3.16821 02101 17924 35952 (1)	3.15635 62190 61546 64985 (−2)
1.2	4.33762 12176 45429 13719 (1)	2.30541 10763 10682 04408 (−2)
1.3	5.93867 09151 05567 24216 (1)	1.68387 84541 10676 64716 (−2)
1.4	8.13068 04970 54802 50287 (1)	1.22990 93542 81271 53095 (−2)
1.5	1.11317 77848 98562 26027 (2)	8.98329 10211 29427 88966 (−3)
1.6	1.52406 03553 91575 92353 (2)	6.56141 99363 06069 66927 (−3)
1.7	2.08660 28754 67872 35481 (2)	4.79247 87785 78186 16706 (−3)
1.8	2.85678 42110 11213 73297 (2)	3.50043 93966 67033 38810 (−3)
1.9	3.91124 54622 93824 99384 (2)	2.55673 03551 78298 63043 (−3)
2.0	5.35491 65552 47647 36503 (2)	1.86744 27317 07988 81443 (−3)
2.1	7.33145 78668 37191 20979 (2)	1.36398 51966 18748 94871 (−3)
2.2	1.00375 55935 49546 24197 (3)	9.96258 45816 08502 26303 (−4)
2.3	1.37424 95829 36839 19562 (3)	7.27669 85882 06595 17424 (−4)
2.4	1.88149 57827 76781 53285 (3)	5.31492 02307 75977 42717 (−4)
2.5	2.57597 04965 97570 55092 (3)	3.88203 20392 67662 47233 (−4)
2.6	3.52678 12237 92079 54462 (3)	2.83544 66482 18162 15904 (−4)
2.7	4.82854 35787 87543 23018 (3)	2.07101 78621 83408 03034 (−4)
2.8	6.61079 65345 18733 02916 (3)	1.51267 70197 48574 28994 (−4)
2.9	9.05089 29054 29406 22166 (3)	1.10486 33659 11803 90247 (−4)
3.0	1.23916 47807 91669 74815 (4)	8.06995 17570 30459 92392 (−5)
3.1	1.69655 01304 66510 05654 (4)	5.89431 44799 67793 83033 (−5)
3.2	2.32275 99668 76296 70719 (4)	4.30522 31580 55474 73556 (−5)
3.3	3.18010 87199 46649 78016 (4)	3.14454 65802 08667 28460 (−5)
3.4	4.35391 15599 10793 22659 (4)	2.29678 52843 12226 31551 (−5)
3.5	5.96097 41492 87215 58845 (4)	1.67757 81524 22578 70825 (−5)
3.6	8.16121 60282 82962 95159 (4)	1.22530 75969 73866 47167 (−5)
3.7	1.11735 84282 07025 47474 (5)	8.94967 96619 20613 64147 (−6)
3.8	1.52978 41066 31403 67916 (5)	6.53687 01090 90149 06637 (−6)
3.9	2.09443 93077 67444 13665 (5)	4.77454 75187 14898 47932 (−6)
4.0	2.86751 31313 66532 99747 (5)	3.48734 23562 08995 49178 (−6)
4.1	3.92593 45105 22707 69216 (5)	2.54716 42415 82974 76484 (−6)
4.2	5.37502 74453 20893 81174 (5)	1.86045 56165 95087 22855 (−6)
4.3	7.35899 18427 10655 01612 (5)	1.35888 17889 37618 96908 (−6)
4.4	1.00752 52915 82001 87728 (6)	9.92530 91545 70345 60877 (−7)
4.5	1.37941 07058 05983 40218 (6)	7.24947 25159 87938 10837 (−7)

TABLE 3—Continued

a	$e^{\pi a}$	$e^{-\pi a}$
4.6	1.88856 19162 02076 47085 (6)	5.29503 42343 60785 53694 (−7)
4.7	2.58564 47947 78413 81653 (6)	3.86750 72539 71822 59597 (−7)
4.8	3.54002 63806 06813 87980 (6)	2.82483 77059 51176 91520 (−7)
4.9	4.84667 76259 05740 65394 (6)	2.06326 90622 02422 28930 (−7)
5.0	6.63562 39993 41134 23327 (6)	1.50701 72753 90064 61075 (−7)
5.1	9.08488 43804 42670 74431 (6)	1.10072 94734 01701 04231 (−7)
5.2	1.24381 85800 49235 82598 (7)	8.03975 76948 91610 22921 (−8)
5.3	1.70292 16831 92971 91696 (7)	5.87226 06557 27887 98907 (−8)
5.4	2.33148 33092 25041 56976 (7)	4.28911 49854 82733 65777 (−8)
5.5	3.19205 19157 42137 17455 (7)	3.13278 11276 13661 21624 (−8)
5.6	4.37026 30820 80489 01787 (7)	2.28819 17660 75428 61239 (−8)
5.7	5.98336 11453 51290 55253 (7)	1.67130 14235 76846 05372 (−8)
5.8	8.19186 62385 55235 84751 (7)	1.22072 30573 33960 68163 (−8)
5.9	1.12155 47723 12530 90732 (8)	8.91619 40610 18549 86382 (−9)
6.0	1.53552 93539 54466 93923 (8)	6.51241 21360 79900 72821 (−9)
6.1	2.10230 51705 21611 64610 (8)	4.75668 33494 10617 94951 (−9)
6.2	2.87828 23451 85932 29437 (8)	3.47429 43188 75800 38156 (−9)
6.3	3.94067 87248 46367 75598 (8)	2.53763 39200 01905 86296 (−9)
6.4	5.39521 38637 23174 97636 (8)	1.85349 46440 65715 51964 (−9)
6.5	7.38662 92250 06302 56497 (8)	1.35379 74758 69850 17560 (−9)
6.6	1.01130 91470 68071 94014 (9)	9.88817 31951 02024 39235 (−10)
6.7	1.38459 12117 54883 38231 (9)	7.22234 83112 57602 86121 (−10)
6.8	1.89565 45871 52483 67960 (9)	5.27522 26422 33194 26958 (−10)
6.9	2.59535 54256 91101 44405 (9)	3.85303 68137 67871 77194 (−10)
7.0	3.55332 12808 47044 35970 (9)	2.81426 84574 85552 72109 (−10)
7.1	4.86487 97771 34322 34756 (9)	2.05554 92546 80768 60938 (−10)
7.2	6.66054 47060 30274 03357 (9)	1.50137 87072 01871 18741 (−10)
7.3	9.11900 35136 20234 64547 (9)	1.09661 10480 23383 27988 (−10)
7.4	1.24848 98570 85274 50206 (10)	8.00967 66050 99546 92873 (−11)
7.5	1.70931 71648 81753 87131 (10)	5.85028 93467 94089 01097 (−11)
7.6	2.34023 94129 02041 86211 (10)	4.27306 70823 11625 73556 (−11)
7.7	3.20403 99653 26715 06920 (10)	3.12105 96959 51795 89636 (−11)
7.8	4.38667 60139 21370 58104 (10)	2.27963 04008 46659 38446 (−11)
7.9	6.00583 22178 73377 47364 (10)	1.66504 81793 74702 70619 (−11)
8.0	8.22263 15585 59499 52750 (10)	1.21615 56709 40930 83974 (−11)
8.1	1.12576 68761 80890 65249 (11)	8.88283 37478 93270 22920 (−12)
8.2	1.54129 61780 91317 24348 (11)	6.48804 56735 98971 03015 (−12)
8.3	2.11020 05742 59200 32308 (11)	4.73888 60196 43211 04041 (−12)
8.4	2.88909 20037 95285 41152 (11)	3.46129 51013 20206 20908 (−12)
8.5	3.95547 83124 46234 88485 (11)	2.52813 92565 17773 08240 (−12)
8.6	5.41547 60941 08196 18967 (11)	1.84655 97163 06303 62722 (−12)
8.7	7.41437 04020 76625 99127 (11)	1.34873 21859 72147 77499 (−12)
8.8	1.01510 72131 77397 90288 (12)	9.85117 61813 79790 86996 (−13)
8.9	1.38979 11735 78508 43714 (12)	7.19532 55928 74119 82759 (−13)
9.0	1.90277 38952 92161 29169 (12)	5.25548 51760 06448 55522 (−13)

TABLE 3—Continued

a	$e^{\pi a}$	$e^{-\pi a}$
9.1	2.60510 25257 86606 78556 (12)	3.83862 05153 21374 94491 (−13)
9.2	3.56666 61113 28858 90997 (12)	2.80373 87543 05271 02946 (−13)
9.3	4.88315 02882 44645 96778 (12)	2.04785 83311 41947 37983 (−13)
9.4	6.68555 90047 64705 70119 (12)	1.49576 12359 52466 78871 (−13)
9.5	9.15325 07843 94276 31478 (12)	1.09250 80319 05929 90960 (−13)
9.6	1.25317 86775 39217 86849 (13)	7.97970 80649 63507 85426 (−14)
9.7	1.71573 66654 00312 45627 (13)	5.82840 02444 31926 72695 (−14)
9.8	2.34902 84009 45530 65488 (13)	4.25707 92230 41663 13798 (−14)
9.9	3.21607 30371 52984 44509 (13)	3.10938 21205 16908 18044 (−14)
10.0	4.40315 05860 63202 90114 (13)	2.27110 10683 24093 83868 (−14)

* (n) at the end of the number stands for the factor 10^n.

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