As we can see, the agreement with the desired value, $u(t) = 1$, is excellent.

7. Discussion. Consider a system of renewal-type equations, given, say, in matrix form:

$$X(t) = F(t) + \int_0^t K(t-s)X(s)\,ds. \tag{7.1}$$

Equations of this type arise naturally in the study of multidimensional branching processes; see [6], [7].

If $X(t)$ is a 5×5 matrix, we are required to store 25 functions (i.e., the elements $x_{ij}(t), i, j = 1, 2, \cdots, 5$) if we proceed in the usual fashion. If high order accuracy were required—say, intervals of 10^{-3} over $0 \leqq t \leqq 5$—we would find that rapid-access storage capacity would be exceeded.

On the other hand, if we use the foregoing technique, differential approximation of order 5 would lead to the task of solving about 250 simultaneous differential equations plus those required to determine $F(t)$. This is a simple matter for a modern computer. Furthermore, it is clear that we could use an approximation of order 10 without coming close to the storage capacity.

The RAND Corporation
Santa Monica, California

1. R. Bellman, J. Jacquez, R. Kalaba & B. Kotkin, *A Mathematical Model of Drug Distribution in the Body: Implications for Cancer Chemotherapy*, The RAND Corporation Report No. RM-3463-NIH, February 1963.
2. R. Bellman & R. Kalaba, *Mathematical Trends in Control Theory*, Dover Publications, New York. (To appear.)
3. R. Bellman, H. Kagiwada & R. Kalaba, *A Computational Procedure for Optimal System Design and Utilization*, RAND Corporation Report No. RM-3174-PR, June 1962; also published in *Proc. Nat. Acad. Sci. USA*, V. 48, 1962, p. 1524–1528.
4. R. Bellman, "Mathematical model-making as an adaptive control process," *Mathematical Optimization Techniques*, University of California Press, Berkeley, 1963, p. 333–339.
5. C. Lanczos, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1956.
6. T. E. Harris, *Branching Processes*, Ergebnisse der Math., Springer, Berlin, 1963.
7. R. Bellman & K. L. Cooke, *Differential-difference Equations*, Academic Press, New York, 1963.

On the Numerical Solution of Equations of the Abel Type

By Henry E. Fettis

The integral equation known as Abel's has the general form

$$f(x) = \int_0^x g(t)(x-t)^{-\alpha}\,dt \tag{1}$$

where α is a real number, and

$$0 < \alpha < 1.$$

Received August 20, 1963.

The function $f(x)$ is supposed known and it is required to find the function $g(t)$. The solution is known to be [1]:

$$g(x) = \frac{\sin(\alpha\pi)}{\alpha\pi}\left[\frac{d}{dx}\int_0^x f(t)(x-t)^{\alpha-1}\,dt\right]. \tag{2}$$

Because of the singularity, the differentiation can not be carried out explicitly. However, if an integration by parts is first made, Equation (2) takes the form

$$g(x) = \frac{\sin\alpha\pi}{\alpha\pi}\frac{d}{dx}\left[x^\alpha f(0) + \int_0^x f'(t)(x-t)^\alpha\,dt\right], \tag{3}$$

and if it is assumed that $f'(t)$ is finite, the differentiation under the integral can be performed and we get

$$g(x) = \frac{\sin\alpha\pi}{\pi}\left[f(0)x^{\alpha-1} + \int_0^x f'(t)(x-t)^{\alpha-1}\,dt\right]. \tag{4}$$

While both Equations (2) and (4) give the theoretically correct solution to Abel's equation, neither are suitable to compute from in problems where no explicit mathematical expression for $f(x)$ is known. First, there is the problem of the singular behavior of the integrand when $t = x$, and while this may be circumvented in several ways, either by an algebraic substitution which removes the singularity or by use of a quadrature formula which inherently takes into account the nature of the singularity (see e.g. [2]), there is an even greater barrier to the numerical problem, namely the fact that both expressions depend not on $f(x)$, but on its derivative. In fact, it is most often the case that $f(x)$ is obtainable only from measured data and, as is well known, the determination of accurate derivatives in such instances is extremely difficult, if not impossible. We, therefore, need a form of the solution in which $f'(x)$ does not appear, and such a solution may readily be obtained if the integration by parts of Equation (4) is carried out in a somewhat different manner; namely, let

$$u = (x-t)^{\alpha-1}, \qquad dv = f'(t)\,dt,$$
$$du = -(\alpha-1)(x-t)^{\alpha-2}\,dt, \qquad v = f(t) - f(x).$$

Then, Equation (4) becomes

$$g(x) = \frac{\sin\alpha\pi}{\pi}\left\{f(0)x^{\alpha-1} - [f(t)-f(x)][(x-t)^{\alpha-1}]_0^x + (1-\alpha)\int_0^x \frac{f(x)-f(t)}{x-t}(x-t)^{\alpha-1}\,dt\right\} \tag{5}$$

or

$$g(x) = \frac{\sin\alpha\pi}{\pi}\left[x^{\alpha-1}f(x) + (1-\alpha)\int_0^x \frac{f(x)-f(t)}{x-t}(x-t)^{\alpha-1}\,dt\right]. \tag{6}$$

It will be noted that while (6) inherently implies the existence of $f'(x)$ in some sense in order that the $\text{limit}_{t\to x}\,[(f(x)-f(t))/(x-t)]$ be defined, it does not explicitly involve the derivative in any way. Thus, the only difficulty in evaluating Equation (6) numerically by quadrature would arise if it became necessary to evaluate the integrand at the point $t = x$. This would indeed be the case if a "closed" quadrature formula (i.e., one which involves the end points) were used. However, other quadrature formulae are available which do not require knowledge of the integrand except at interior points of the interval of integration, and, of these, perhaps the most suitable is the Gaussian type. Here the approximate value of the integral is expressed as a linear combination of the integrand (not including the singular term $(x - t)^{\alpha-1}$) evaluated at properly selected points and multiplied by appropriate weighting factors. (See e.g. [3].)

For this purpose, it is more convenient always to have a fixed interval of integration, and to this end we set

$$t = x(1 - u)$$

and direct attention to the integral involved in Equation (6) in the form

$$-\int_0^1 \frac{f(x) - f[x(1 - u)]}{u} u^{\alpha-1} \, du. \tag{7}$$

To determine the ordinates and weight factors for the Gaussian quadrature of the above integral we need to find the zeros of that set of polynomials $\mathfrak{F}_n(\alpha, u)$ which are orthogonal on (0, 1) with respect to $u^{\alpha-1}$ as weighting function. These polynomials therefore satisfy the relation

$$\int_0^1 u^{\alpha-1}\mathfrak{F}_m(\alpha, u)\mathfrak{F}_n(\alpha, u) \, du = 0 \tag{8}$$

whenever $m \neq n$. They belong to a more general class of orthogonal polynomials known as the Jacobi Polynomials which satisfy a similar orthogonality relationship with respect to the weighting function

$$u^{\alpha-1}(1 - u)^{\alpha-\gamma}. \tag{9}$$

The first four Jacobi polynomials for the case $\alpha = \gamma$ are given below:

$$\mathfrak{F}_0(\alpha, u) = 1,$$

$$\mathfrak{F}_1(\alpha, u) = 1 - \frac{\alpha + 1}{\alpha} u,$$

$$\mathfrak{F}_2(\alpha, u) = 1 - 2\frac{\alpha + 2}{\alpha} u + \frac{(\alpha + 2)(\alpha + 3)}{\alpha(\alpha + 1)} u^2,$$

$$\mathfrak{F}_3(\alpha, u) = 1 - 3\frac{\alpha + 3}{\alpha} u + 3\frac{(\alpha + 3)(\alpha + 4)}{\alpha(\alpha + 1)} u^2 - \frac{(\alpha + 3)(\alpha + 4)(\alpha + 5)}{\alpha(\alpha + 1)(\alpha + 2)} u^3.$$

The general expression for $\mathfrak{F}_n$ is readily deduced by induction. All zeros of the $\mathfrak{F}_n$ are real and lie in the interval $0 < u < 1$. Further, if u_i is any such zero, and if

$$H_i = \frac{1}{\mathfrak{F}_n'(u_i)} \int_0^1 \frac{u^{\alpha-1}\mathfrak{F}_n(u)}{u - u_i} \, du, \tag{10}$$

then the integral

$$\int_0^1 \phi(u)u^{\alpha-1}\,du \tag{11}$$

is approximated by

$$H_1\phi(u_1) + H_2\phi(u_2) + \cdots + H_n\phi(u_n), \tag{12}$$

and the approximation coincides with the exact value if ϕ is a polynomial of degree $(2n - 1)$ or less.

Since the most frequently encountered value for α in Abel's equation is $\frac{1}{2}$, the ordinates and weight factors for this case are listed in Table A1, Appendix 1, to ten places, for $n = 1$ up to $n = 8$. The following example illustrates their use and also demonstrates the accuracy obtainable with relatively few ordinates. The equation considered is

$$\int_0^x (x - t)^{-1/2} g(t)\,dt = e^x, \tag{13}$$

for which the explicit solution is

$$\pi g(x) = x^{-1/2}\left[1 + 2x^{1/2}e^x \int_0^{x^{1/2}} e^{-t^2}\,dt\right]. \tag{14}$$

For this example, Equation (4) takes the form

$$\pi g(x) = x^{-1/2}\left[e^x + \tfrac{1}{2}\int_0^1 \left(\frac{e^x - e^{x(1-t)}}{t}\right) t^{-1/2}\,dt\right]. \tag{15}$$

Table 1 gives the results using the Gaussian coefficients for selected values of n

TABLE 1

$\pi g(x)$ as found from Equation (14)

$x^{1/2}$	$\pi g(x)$
.5	3.184593
1.0	5.060157
1.5	16.9132453

as well as the more exact values calculated from Equation (14). This table illustrates the high degree of precision which is attainable by the use of the present method.

Applied Mathematics Research Laboratory
Aerospace Research Laboratories
Office of Aerospace Research
Wright-Patterson Air Force Base, Ohio

1. F. G. TRICOMI, *Integral Equations*, Interscience, New York, 1957.
2. E. L. KAPLAN, "Numerical integration near a singularity," *J. Math. and Phys.*, v. 31, 1952, p. 1–28.
3. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.

TABLE 1—*Continued*

Calculation of $g(x)$ *from Equation* (15)

	t_i	$\phi(t_i)$	H_i
$x = .25$			
one point	.3333333	.3079950	2
			$\pi g(x) = 3.184040$
two points	.1155871 .7415557	.3164127 .2930075	1.3042903 .6257097
			$\pi g(x) = 3.1845929$
$x = 1.0$			
one point	.3333333	2.311644	2
			$\pi g(x) = 5.029920$
two points	.1155871 .7415557	2.567064 1.919435	1.3042903 .6957097
			$\pi g(x) = 5.060161$
three points	.0569391 .4371978 .8694994	2.6423418 2.2019707 1.8158532	.9358279 .7215231 .3426490
			$\pi g(x) = 5.060157$
$x = 2.25$			
two points	.1155871 .7415557	18.797264 10.382250	1.3042903 .6957097
			$\pi g(x) = 16.90520$
three points	.0569391 .4371978 .8694994	20.03654 13.58656 9.36913	.9358279 .7215231 .3426490
			$\pi g(x) = 16.91319$
four points	.0336483 .2761843 .6346775 .9221560	20.55933 15.89908 11.36443 8.99664	.7253676 .6274133 .4947621 .2024571
			$\pi g(x) = 16.91325$

APPENDIX 1

TABLE A1

Ordinates and weights for Gaussian quadrature with weight function $x^{-1/2}$

$$\int_0^1 x^{-1/2} f(x)dx \cong \sum_{i=1}^{n} H_i f(x_i)$$

		x_i	H_i
$n = 1$	1	.33333 33333	2.00000 00000
$n = 2$	1	.11558 71100	1.30429 03097
	2	.74155 57471	.69570 96903
$n = 3$	1	.05693 91160	.93582 78691
	2	.43719 78528	.72152 31461
	3	.86949 93949	.34264 89848
$n = 4$	1	.03364 82681	.72536 75668
	2	.27618 43139	.62741 32917
	3	.63467 74762	.44476 20689
	4	.92215 66085	.20245 70726
$n = 5$	1	.02216 35688	.59104 84494
	2	.18783 15677	.53853 34386
	3	.46159 73615	.43817 27251
	4	.74833 46284	.29890 26983
	5	.94849 39263	.13334 26886
$n = 6$	1	.01568 34066	.49829 40916
	2	.13530 00117	.46698 50730
	3	.34494 23794	.40633 48535
	4	.59275 01277	.32015 66571
	5	.81742 80133	.21387 86520
	6	.96346 12787	.94350 67278
$n = 7$	1	.01167 58719	.43052 77068
	2	.10183 27040	.41039 62274
	3	.26548 11513	.37107 67950
	4	.47237 15370	.31440 63344
	5	.68426 20157	.24303 71414
	6	.86199 13332	.16031 61744
	7	.97275 57513	.70238 92066
$n = 8$	1	.00902 73770	.37890 12208
	2	.07939 05598	.36520 68301
	3	.20977 93686	.33831 30388
	4	.38177 10534	.29919 19776
	5	.57063 58202	.24925 79425
	6	.74931 73785	.19031 70234
	7	.89222 19743	.12450 70479
	8	.97891 42102	.05430 49188