On the Sum of the Series $\sum_{\nu=0}^{\infty} (t^\nu/[u\nu + m]!)$

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An estimation of error in summing the series

$$(1) \qquad \sum_{\nu=0}^{\infty} \frac{t^\nu}{\Gamma(\beta\nu + 2)}, \qquad 0 < \beta < 1,$$

obtained by a solution of Volterra's equation

$$(2) \qquad \phi(x) = ax + b \int_0^x (x - z)^{\beta-1}\phi(z)\, dz, \qquad a = \text{const}, b = \text{const},$$

leads to series of the form

$$(3) \qquad S_{u,m}(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{[u\nu + m]!},$$

where $[x]$ denotes the largest integer no greater than the real number x, m a non-negative integer, and u real, positive and rational.

The series (3) is absolutely and uniformly convergent for all real values of t and therefore not dependent on the sequence of terms. Changing the sequence of terms in (3) we obtain

$$(4) \qquad S_{u,m}(t) = \sum_{j=0}^{q-1} \sigma_{u,m}^j(t),$$

where

$$(5) \qquad \sigma_{u,m}^j(t) = \sum_{i=0}^{\infty} \frac{t^{qi+j}}{[uj + m + pi]!}$$

and

$$(6) \qquad u = \frac{p}{q},$$

where p and q are positive integers with greatest common divisor equal to 1. Substituting in (5)

$$(7) \qquad z^p = t^q,^*$$

we have

$$(8) \qquad \sigma_{u,m}^j(t) = t^{j-[uj+m]/u} s_{u,m}^j(z),$$

where

$$(9) \qquad s_{u,m}^j(z) = \sum_{i=0}^{\infty} \frac{z^{[uj+m+pi]}}{[uj + m + pi]!}.$$

Differentiating (9) $[uj + m]$ times with respect to z we obtain

$$(10) \qquad s(z) = \frac{d^{[uj+m]} s_{u,m}^j(z)}{dz^{[uj+m]}} = \sum_{i=0}^{\infty} \frac{z^{pi}}{(pi)!}.$$

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* z is an imaginary number when q is odd and p even.

But (10) is the solution of the pth order linear differential equation

$$s^{(p)}(z) = s(z) \tag{11}$$

with initial conditions

$$s(0) = 1, \qquad s^{(k)}(0) = 0 \qquad (k = 1, 2, \cdots, p - 1). \tag{12}$$

The characteristic equation of (12) is

$$r^p - 1 = 0. \tag{13}$$

If r denotes a primitive root of pth order of unity, the general solution of (11) has the form

$$s(z) = \sum_{l=1}^{p} A_l \exp r^l z, \tag{14}$$

and taking (12) into consideration we have

$$\begin{aligned} &\sum_{l=1}^{p} A_l = 1, \qquad A_l = \text{const}, \\ &\sum_{l=1}^{p} A_l r^{lk} = 0 \qquad (k = 1, 2, \cdots, p - 1). \end{aligned} \tag{15}$$

For finding A_l we use Cramer's rule, and since there are Vandermonde's determinants in our case

$$A_l = \frac{\prod_{k=2}^{p} \prod_{n=1}^{k-1} (b_k - b_n)}{\prod_{k=2}^{p} \prod_{n=1}^{k-1} (r^k - r^n)}, \qquad \text{where } b_n = \begin{cases} r^n & \text{when } n \neq l, \\ 0 & \text{when } n = l, \end{cases} \tag{16}$$

or

$$A_l = \prod_{k=1;k\neq l}^{p} \frac{1}{1 - r^{l-k}} = \prod_{k=1}^{p-1} \frac{1}{1 - r^k}, \tag{17}$$

it follows that all A_l are equal, and, thus, from (15),

$$A_l = \frac{1}{p}. \tag{18}$$

Putting (18) into (14) we have

$$s(z) = \frac{1}{p} \sum_{l=1}^{p} \exp (r^l z). \tag{19}$$

Integrating both sides of (19) $[uj + m]$ times in the interval $(0, z)$ we obtain

$$s^j_{u,m}(z) = \frac{1}{p} \sum_{l=1}^{p} \exp (r^l z) r^{-l[uj+m]} - B, \tag{20}$$

where

$$B = \frac{1}{p} \sum_{k=1}^{[uj+m]} \frac{z^{[uj+m]-k}}{[uj + m - k]!} \sum_{l=1}^{p} r^{-lk}. \tag{21}$$

But

$$\sum_{l=1}^{p} r^{-lk} = \sum_{l=1}^{p} r^{lk} = \begin{cases} 0 & \text{when } k \text{ is divisible by } p, \\ p & \text{when } k \text{ is not divisible by } p, \end{cases} \tag{22}$$

and therefore

$$B = \sum_{k=1}^{[(uj+m)/p]} \frac{z^{[uj+m]-pk}}{[uj+m-pk]!}. \tag{23}$$

Returning to the variable t we obtain finally

$$S_{u,m}(t) = \sum_{j=0}^{q-1} t^{j-[uj+m]/u} \cdot \left\{ \frac{1}{p} \sum_{l=1}^{p} \exp(r^l t^u) r^{-l[uj+m]} - \sum_{k=1}^{[(uj+m)/p]} \frac{t^{[uj+m-pk]/u}}{[uj+m-pk]!} \right\}. \tag{24}$$

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