# Bounds for the Two-Dimensional Discrete Harmonic Green's Function

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**Abstract.** Estimation of bounds for the two-dimensional discrete harmonic Green's function is obtained. These estimations can then be used to obtain approximate solutions to two-dimensional boundary problems associated with the harmonic difference operator.

**1. Introduction.** In dealing with partial differential operators, it is desirable to obtain the "free-space" Green's function (valid in the entire space under consideration) for the Laplacian operator, $\Delta$, which permits writing solutions for boundary value problems in integral form [4].

A similar situation exists in solving boundary value problems associated with partial difference operators. It is desirable to find the analog free-space discrete harmonic Green's function $g(m, n)$, which permits writing solutions in summation form for the boundary value problems of the difference equations [1]–[3].

Unlike the two-dimensional continuous case, where the Green's function is known to be $\log r$, an exact estimate for $g(m, n)$ (where $m$ and $n$ are integers and where the mesh widths are unity) is not available (see [3] and [6]). This is because the evaluation involves an elliptic integral (see Eqs. (6) and (7)); only asymptotic estimates for $g(m, n)$ are known.

In this paper, we obtain explicit bounds (see Theorem 1) for $g(m, n)$ which yield very reasonable numerical estimates for intermediate values of $m$ and $n$. Then, by making a suitable transformation, similar results are obtained for the discrete harmonic Green's function $g_p(Q)$, associated with mesh widths $h$ in $x$ and $y$ ($x = mh$, $y = nh$).

**2. Known Results.** Let $D$ be the harmonic difference operator; i.e.,

$$\begin{aligned} Du(m, n) &= u(m + 1, n) + u(m - 1, n) + u(m, n + 1) \\ &\quad + u(m, n - 1) - 4u(m, n), \end{aligned} \tag{1}$$

where $m$ and $n$ are integers. Then, $g(m, n)$ is defined [2] as the unique solution of

$$Dg(m, n) = 0, \qquad \text{except at } (0, 0) \tag{2}$$

$$Dg(0, 0) = -1 \tag{3}$$

$$g(0, 0) = 0 \tag{4}$$

(5) the first differences of $g(m, n) \to 0$ as $k = (m^2 + n^2)^{1/2} \to \infty$.

Duffin and Shaffer [3] showed, by means of an operational calculus based on Fourier series, that

$$g(m, n) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1 - \exp[i(mx + ny)]}{4(\sin^2 x/2 + \sin^2 y/2)}. \tag{6}$$

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On the other hand, McCrea and Whipple [6] showed, by considering a two-dimensional random walk problem, that

$$g(m, n) = \frac{1}{2\pi} \int_0^\pi \frac{1 - \exp[-|m|y] \cos nx}{\sinh y} dx. \tag{7}$$

with

$$\cos x + \cosh y = 2. \tag{8}$$

The asymptotic estimates obtained by [3] and [6] are, respectively

$$g(m, n) = \frac{1}{2\pi}\left[\log k + \frac{3}{2}\log 2 + \gamma\right] + O\left(\frac{1}{k^2}\right) \quad \text{as} \quad k = (m^2 + n^2)^{1/2} \to \infty. \tag{9}$$

$$g(m, n) = \frac{1}{2\pi}\left[\log k + \frac{3}{2}\log 2 + \gamma\right] + o\left(\frac{1}{k}\right) \qquad \text{as } k \to \infty. \tag{10}$$

Here $\gamma$ is the Euler's constant.

**3. Statement of Main Results.** The main results of this paper are contained in the following theorem, which will be proved in Section 5.

THEOREM 1. *If $k^2 = m^2 + n^2 \neq 0$ and the mesh widths are unity (i.e., $m, n = 0, \pm 1, \pm 2$, etc.), then*

$$\frac{-53 \cdot 6}{k^2} \leqq 2\pi g(m, n) - \log k - \frac{3}{2}\log 2 - \gamma \leqq \frac{1}{12k^2} + \frac{53 \cdot 6}{k^2}. \tag{11a}$$

*Let $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, and $m = (x_Q - x_P)/h$, $n = (y_Q - y_P)/h$. If $\rho = \overline{PQ} = \sqrt{((x_Q - x_P)^2 + (y_Q - y_P)^2)}$, then the bounds for the Green's functions $g_P(Q)$ associated with mesh widths $h$ are*

$$\frac{53 \cdot 6h^2}{\rho^2} \leqq 2\pi g_P(Q) - \log \rho - \frac{3}{2}\log 2 - \gamma \leqq \frac{53 \cdot 6h^2}{\rho^2} + \frac{h^2}{12\rho^2}, \ \rho \geqq h > 0. \tag{11b}$$

*Remark.* As will be shown later, estimates (11) could be improved to also contain terms of the form $O(1/k^4)$ or $O(1/\rho^4)$, $O(1/k^6)$ or $O(1/\rho^6)$, etc.

**4. Preliminary Results and Lemmas.** To obtain explicit bounds for $g(m, n)$, we shall use representation (7) with the two properties:

$$g(m, n) = g(n, m) \tag{12}$$

and

$$g(m, m) = \frac{1}{\pi}\sum_{j=1}^{m} \frac{1}{2j - 1} \tag{13}$$

obtained in [6]. Next, one notes from Eq. (7) that $g(m, n) = g(m, -n)$ and, hence

$$g(m, n) = g(m, -n) = g(n, m) = g(n, -m) = g(-m, n) = g(-m, -n). \tag{14}$$

This means that it is sufficient to consider the behavior of $g(m, n)$ for $m \geqq n \geqq 0$ and, without loss of generality, assume that $m \geqq 1$. In addition, for any given $x$, we shall consider only the non-negative values of $y$ determined by relation (8).

From Eq. (7),

$$2\pi[g(m,n) - g(m,m)] = \int_0^\pi \frac{(\cos mx - \cos nx)}{\sinh y} e^{-my}\, dx. \tag{15}$$

When $m$ is large, the important part of the integral above occurs with small values of $x$. Accordingly, we divide the range of integration at $\epsilon$, defined by $\epsilon = m^{-1/3}$. Next, define

$$\begin{aligned} A &= \int_0^\infty \frac{(\cos mx - \cos nx)}{x} e^{-mx}\, dx; \\ B &= \int_\epsilon^\infty \frac{(\cos mx - \cos nx)}{x} e^{-mx}\, dx; \\ C &= \int_\epsilon^\pi \frac{(\cos mx - \cos nx)}{\sinh y} e^{-my}\, dx; \\ D &= \int_0^\epsilon [\cos mx - \cos nx] \left[\frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{x}\right] dx. \end{aligned} \tag{16}$$

Then, by relation (15) and by letting $A^* = A + 2\pi g(m,n)$, there results

$$A^* - |B| - |C| - |D| \leqq 2\pi g(m,n) \leqq A^* + |B| + |C| + |D|. \tag{17}$$

4.1. Bounds for $A^*$ and $|B|$. By using Laplace transforms,

$$A = \tfrac{1}{2} \log (m^2 + n^2)/2n^2.$$

Next, it is well known [5] that

$$\sum_{j=1}^{n} \frac{1}{2j-1} = \frac{1}{2}(\gamma + \log n) + \log 2 + \frac{B_1}{8n^2} - \frac{(2^3 - 1)B_2}{64n^4} + \cdots, \tag{18}$$

where the $B_j$'s are Bernouli-type numbers given by $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, etc. Combining results, therefore, we have the following bounds for $A^*$:

$$0 \leqq A^* - \log k - \frac{3}{2}\log 2 - \gamma \leqq \frac{1}{24m^2}, \qquad m \geqq 1. \tag{19}$$

Finally, by recalling $B$ from Eq. (16), it is easily deduced that

$$|B| \leqq 2m^{-2/3} \exp[-m^{2/3}], \qquad m \geqq 1. \tag{20}$$

4.2. Bounds for $|C|$. To find bounds for $|C|$, the following lemma is needed:

LEMMA 1. *If* $\epsilon \leqq x \leqq \pi$, *where* $0 < \epsilon \leqq 1$, *then* $y > 9\epsilon/10$.

*Proof*. By considering the series for $\cosh y = 2 - \cos x$ and $\cosh 9\epsilon/10$, it can easily be shown that $\cosh y > \cosh 9\epsilon/10$, so that $y > 9\epsilon/10$.

Using Lemma 1 and the fact that $1/\sinh y < 1/y < 10/9\epsilon$ for $0 < \epsilon \leqq 1$, $\epsilon \leqq x \leqq \pi$, one obtains, after neglecting negative valued terms,

$$|C| < \tfrac{20}{9}\pi m^{1/3} \exp[-9m^{2/3}/10], \qquad m \geqq 1. \tag{21}$$

4.3. Bounds for $|D|$. To obtain bounds for $|D|$, the intermediary inequalities contained in the following lemma are needed:

LEMMA 2. *If* $0 \leqq x \leqq 1$, *then*

$$x \geqq y \geqq x - x^3/10 \tag{22}$$

$$\sinh y \geqq 9x/10 \tag{23}$$

$$\sinh x \leqq 6x/5 \tag{24}$$

$$\cosh x < 8/5. \tag{25}$$

*Proof.* $\cosh y = 1 + y^2/2! + y^4/4! + \cdots \geqq 1 + y^2/2$ and $\cosh y = 2 - \cos x \leqq 1 + x^2/2$. Therefore, $y \leqq x$. Next,

$$\cosh y = 2 - \cos x \geqq 1 + \frac{x^2}{2} - \frac{x^4}{24}$$

and

$$\cosh\left(x - \frac{x^3}{10}\right) - 1 \equiv \cosh t - 1 = \frac{t^2}{2} + \frac{t^4}{24}\left[1 + \frac{t^2}{5\cdot 6} + \frac{t^4}{5\cdot 6\cdot 7\cdot 8} + \cdots\right]$$

$$\leqq \frac{t^2}{2} + \frac{t^4}{24}\left[1 + \frac{t^2}{5^2} + \frac{t^4}{5^4} + \cdots\right] = \frac{t^2}{2} + \frac{25t^4}{24(25 - t^2)} = \cdots$$

$$= \frac{x^2}{2} + \frac{x^6 - 20x^4}{200} + \frac{10^4x^4 - 4000x^6 + 600x^8 - 40x^{10} + x^{12}}{96[2500 - 100x^2 + 20x^4 - x^6} \leqq \cdots \leqq \frac{x^2}{2} - \frac{9x^4}{200}.$$

Hence

$$\cosh\left(x - \frac{x^3}{10}\right) \leqq 1 + \frac{x^2}{2} - \frac{9x^4}{200} \leqq 1 + \frac{x^2}{2} - \frac{x^4}{24} \leqq \cosh y$$

and, therefore

$$y \geqq x - \frac{x^3}{10}.$$

Now

$$\sinh y \geqq y \geqq x - \frac{x^3}{10} \geqq x - \frac{x}{10} = \frac{9x}{10}.$$

Therefore,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \leqq x + \frac{x}{3!} + \frac{x}{5!} + \cdots$$

$$= x \sinh 1 = \frac{x(e^2 - 1)}{2\epsilon} \leqq \frac{6x}{5}.$$

Finally

$$\cosh^2 x = 1 + \sinh^2 x \leqq 1 + \frac{36\,x^2}{25} \leqq 1 + \frac{36}{25} = \frac{61}{25}$$

so that

$$\cosh x < \tfrac{8}{5}.$$

This completes the proof of Lemma 2.

Now, from Eq. (16)

$$|D| \leqq 2\int_0^{\epsilon} \left| \frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{x} \right| dx.$$

However

$$\begin{aligned} \left| \frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{x} \right| &= \left| \frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{\sinh x}\frac{\sinh x}{x} \right| \\ &= \left| \frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{\sinh x}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) \right| \\ &\leqq \left| \frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{\sinh x} \right| + \left| \frac{e^{-mx}}{\sinh x}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right) \right|. \end{aligned}$$

Therefore,

$$|D| \leqq 2(d_1 + d_2), \quad \text{where} \quad d_1 = \int_0^{\epsilon} \left| \frac{e^{-my}}{\sinh y} - \frac{e^{-mx}}{\sinh x} \right| dx$$

and

$$d_2 = \int_0^{\epsilon} \frac{e^{-mx}}{\sinh x}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right) dx.$$

We shall first consider $d_1$. Let $f(x) = e^{-mx}/\sinh x$, so that $d_1 = \int_0^{\epsilon} | f(x) - f(y) | \, dx$. Since $m \geqq 1$ is assumed in all of these estimates of $g(m, n)$, we need only to consider $x$ in the range $0 \leqq x \leqq 1$ (as in Lemma 2). By the Mean Value Theorem, we have

$$d_1 = \int_0^{\epsilon} (x - y) \, | f'(\xi) | \, dx$$

where

$$y \leqq \xi \leqq x.$$

Now

$$| f'(\xi) | = \left| \frac{e^{-m\xi}}{\sinh^2 \xi} (m \sinh \xi + \cosh \xi) \right|$$

and, since

$$y \leqq \xi \leqq x,$$

we have

$$\sinh y \leqq \sinh \xi \leqq \sinh x \quad \text{and} \quad \cosh \xi \leqq \cosh x.$$

Therefore

$$|f'(\xi)| \leqq \frac{e^{-my}}{\sinh^2 y} (m \sinh x + \cosh x).$$

But from Eqs. (22) through (25), we have

$$e^{-my} \leqq e^{-m(x-x^3/10)} \leqq e^{-m(x-x/10)} = e^{-9mx/10}, \quad \sinh^2 y \geqq \frac{81x^2}{100} \geqq \frac{5x^2}{5},$$

and

$$m \sinh x + \cosh x \leqq \frac{6mx}{5} + \frac{8}{5}.$$

Therefore

$$| f'(\xi) | \leqq \frac{e^{-9mx/10}(6mx + 8)}{4x^2}$$

and, hence

$$d_1 \leqq \int_0^\epsilon \frac{x^3}{10} \frac{e^{-9mx/10}(6mx + 8)}{4x^2} \, dx = \frac{3m}{20} \int_0^\epsilon x^2 e^{-9mx/10} \, dx + \frac{1}{5} \int_0^\epsilon x e^{-9mx/10} \, dx.$$

Substituting $\epsilon = m^{-1/3}$ then, after dropping negative valued terms, there results

$$d_1 \leqq \frac{160}{243m^2}, \qquad m \geqq 1. \tag{26}$$

In considering $d_2$, let

$$\frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots = \frac{x^2}{3!} S,$$

where

$$S = 1 + \frac{x^2}{4 \cdot 5} + \frac{x^4}{4 \cdot 5 \cdot 6 \cdot 7} + \cdots \leqq 1 + \frac{x^2}{4^2} + \frac{x^4}{4^4} + \cdots = \frac{16}{16 - x^2} \leqq \frac{16}{15}.$$

Hence,

$$\frac{x^2 S}{3!} \leqq \frac{8x^2}{45} \leqq \frac{x^2}{5}.$$

Furthermore, by Eqs. (22) and (23), $\sinh x \geqq 9x/10$ and, therefore, $d_2 \leqq 2/9 \int_0^\epsilon x e^{-mx} \, dx \leqq 2/9m^2$, $m \geqq 1$ (after negative terms have been dropped and $\epsilon = m^{-1/3}$ has been substituted). Combining the above results, we obtain

$$| D | \leqq 2(d_1 + d_2) \leqq \frac{2}{m^2} \left( \frac{160}{243} + \frac{2}{9} \right) \leqq \frac{9}{5m^2}, \qquad m \geqq 1. \tag{27}$$

**5. Proof of Theorem 1.** Letting

$$F(m) = 2m^{-2/3} \exp [-m^{2/3}] + \frac{20\pi}{9} m^{1/3} \exp \left[ -\frac{9m^{2/3}}{10} \right] + \frac{9}{5m^2}$$

and

$$G(m) = m^2 F(m)$$

one notes that

$$\frac{dG}{dm} < 0 \quad \text{for} \quad m \geqq 8.$$

Therefore, $G(m)$ is a monotonic decreasing function for $m \geqq 8$ i.e.,

$$G(8) > G(m), \qquad m = 9, 10, 11, \text{etc.}$$

One may also verify that

$$\max [G(1), G(2), \cdots, G(8)] = G(8) = 26 \cdot 80.$$

Hence

$$|B| + |C| + |D| \leqq \frac{G(8)}{m^2} = \frac{26 \cdot 80}{m^2}. \tag{28}$$

Combining this with Eqs. (17) and (19), one obtains

$$-\frac{G(8)}{m^2} \leqq 2\pi g(m, n) - \log k - \frac{3}{2} \log 2 - \gamma \leqq \frac{1}{24m^2} + \frac{G(8)}{m^2}, \tag{29}$$

where

$$m \geqq n \quad \text{and} \quad m = 1, 2, 3, \cdots.$$

Finally, since $m^2 + n^2 \geqq 2n^2$, we are led to

$$-\frac{53 \cdot 60}{k^2} \leqq 2\pi g(m, n) - \log k - \frac{3}{2} \log 2 - \gamma \leqq \frac{53 \cdot 60}{k^2} + \frac{1}{12k^2}. \tag{30}$$

This completes the verification of estimate (11a). As for estimate (11b), it follows from the definition

$$Dg_P(Q) = \frac{\delta_{P,Q}}{h^2} = \frac{1}{h^2} \quad \text{if } P = Q; \qquad 0 \text{ if } P \neq Q$$

and from the fact that

$$g_P(Q) = g(m, n) + \frac{1}{2\pi} \log h.$$

To verify the remark following estimates (11), we shall make use of Eq. (18) by taking more terms in that series. The estimates for $|B|$, $|C|$, and $|D|$ remain unchanged, while only $A^*$ changes. For example, by taking an additional term in Eq. (18), one is led to

$$\frac{1}{24m^2} - \frac{7}{960m^4} \leqq A^* - \log k - \frac{3}{2} \log 2 - \gamma \leqq \frac{1}{24m^2} \tag{31}$$

and, hence

$$\begin{aligned} \frac{1}{24k^2} - \frac{53 \cdot 60}{k^2} - \frac{7}{190k^4} &\leqq 2\pi g(a, b) - \log k - \frac{3}{2} \log 2 - \gamma \\ &\leqq \frac{53 \cdot 60}{k^2} + \frac{1}{12k^2}. \end{aligned} \tag{32}$$

This process of taking more terms in relation (18) may be continued indefinitely.

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