less than those given in Table I. Table II gives the value of s corresponding to each value of k.

However this method of producing solutions to (A) with a small number of terms is subject to the following weakness. We had assumed that from any particular solution to (A) solutions of higher degree would be generated containing the least number of terms s, so long as the most frequent difference d was used at each step. After producing the following results this assumption was seen to be false.

When forming Table I the multiplier $(1 - x)$ was used with $\prod_{j=1}^{11} (1 - x^j)$ to produce a solution to the extended problem where $s = 22$ for $k = 11$. This is equivalent to starting with the solution $0, 2 \overset{1}{=} 1, 1$ and using Theorem 2 with $d = 2, 3, \cdots, 11$. Table III compares the lengths of the solutions generated in this manner with those generated from the same initial solution but using the most frequent difference d at each step.

Thus, by a more careful choice of d, the length of solutions can be decreased for $k = 6, 7, 8, 9, 10$. But for $k = 11$ this produces a solution to the extended problem where $s = 24$. This solution is longer than that obtained from a sequence of solutions which was constructed from values for d that did not always represent the most frequent difference.

Finally, although solutions to (A) for $k = 6$ and $s = 7$ exist, we proved that no such solution can be obtained from a sequence generated by any solution for $k = 1$ and $s = 2$ using the most frequent difference d at each step.

Although Theorem 2 was used to generate most solutions for $k \leqq 9$ where $s = k + 1$, it appears that for $k \geqq 10$ it alone will not be sufficient.

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Numerical Solutions of the Diophantine Equation $y^3 - x^2 = k$

By M. Lal, M. F. Jones and W. J. Blundon

Introduction. The distribution of squares and cubes differing by a given integer is very interesting [1] and has attracted many mathematicians over the past few centuries. Probably this is due to the fact that $y^3 - x^2 = k$ is the simplest of all nontrivial Diophantine equations of degree greater than two. The solution of this equation is equivalent to the problem of representation of numbers by binary cubic

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TABLE 1

Values of $|k|$ *with No. of Solutions* $\geqq 6$

K	$N+$	$N-$	K	$N+$	$N-$	K	$N+$	$N-$
17	0	8	1737	0	11	4977	0	8
73	0	6	1792	3	3	5328	0	6
100	3	6	1809	0	6	5400	0	8
113	0	6	1872	0	7	5543	6	0
207	7	0	1900	0	6	5696	3	5
225	0	13	1999	6	0	5841	0	10
252	1	5	2052	0	6	5887	4	2
297	0	9	2071	6	0	6236	6	1
316	1	7	2089	0	14	6400	4	7
343	5	2	2188	5	1	6479	6	0
368	7	1	2241	0	6	6625	0	6
388	0	6	2312	1	5	6908	5	1
431	9	0	2351	10	0	6921	0	6
496	6	1	2600	1	5	7057	0	11
503	8	0	2628	0	9	7100	5	4
512	1	5	2817	0	10	7232	0	8
516	4	3	3025	0	11	7353	0	7
568	0	7	3033	0	8	7568	0	8
648	6	0	3332	6	2	7600	4	2
676	6	1	3356	0	6	7785	0	8
775	7	2	3592	1	5	7804	7	0
828	5	1	3664	0	8	7948	0	6
847	7	0	3807	11	0	8036	3	4
873	0	9	3844	0	8	8225	0	11
892	1	5	3896	11	0	8281	0	9
964	4	3	3969	0	9	8289	0	7
999	6	0	4032	2	4	8433	0	6
1016	0	7	4087	7	0	8532	7	0
1025	0	16	4112	0	10	8623	5	2
1071	6	0	4220	6	0	8673	0	6
1088	0	9	4312	0	10	8676	1	9
1225	1	5	4329	0	7	8712	6	0
1304	0	6	4356	0	8	8900	1	12
1305	0	9	4481	0	12	9052	1	5
1439	8	0	4598	4	2	9297	0	8
1712	6	1	4600	1	5	9559	7	1
1724	5	1	4672	0	7	9748	9	0
1727	7	0	4799	7	0	9936	0	6
1729	1	5	4825	0	7	9967	6	0

forms [2]. Thus the solution of the indeterminate equation of third degree

$$y^3 - x^2 = k \tag{1}$$

is equivalent to solving a finite number of equations $(a, b, c, d) = 1$, where (a, b, c, d) is a binary cubic form.

At present, very little is known about the theory of binary cubic forms. In this respect, the theory for negative discriminants is better developed and for $0 < -k \leqq 100$, all solutions of (1) have been found [3]. However for positive discriminants, progress has been rather slow and for the equivalent positive range,

TABLE 2

Range of y	Number of solutions	
	Positive K	Negative K
$-21 \leqq y < 0$	0	1774
$0 \leqq y < 10^2$	2140	2669
$10^2 \leqq y < 10^3$	712	649
$10^3 \leqq y < 10^4$	294	185
$10^4 \leqq y < 10^5$	64	66
$10^5 \leqq y < 10^6$	13	23
$10^6 \leqq y < 4.64 \times 10^6$	0	4
Total	3223	5370

20 cases remain to be resolved [3], [4]. For $|k| > 100$, it appears that complete solutions are lacking. It was therefore felt desirable to obtain solutions of (1) by means of a numerical search.

This search was conducted with

$$|k| \leqq 9999, k \neq 0 \text{ and}$$

$$0 \leqq x < 10^{10}.$$

These parameters fix the range of y to be

$$-21 \leqq y < y_{\max}; \qquad y_{\max} = 4{,}641{,}589.$$

We anticipate that the results of such an extensive search will be useful for checking some of the conjectures concerning this equation and also provide further insight into the theory of binary cubic forms.

Method. By rewriting (1) as

$$x^2 = y^3 - k,$$

then, for a given y, x is bounded by

$$x_{\min} < x < (y^3 + 9999)^{1/2}; \qquad x_{\min} = (y^3 - 9999)^{1/2}.$$

Thus, if y is large, the possible values of x in this search are severely limited.

There are two methods of finding the starting values of x for a given y.

(1) To compute the square-root directly.

(2) To compute it by a search routine.

A routine was programmed, using the fact for $y > 21$, $x_{\min}$ is a monotonic increasing function of y. This was found to be considerably quicker than (1) and was used throughout.

Results. The final output is rather large and it is intended to deposit a copy in the UMT file. A limited number of copies have been retained by the authors for distribution to interested mathematicians.

The tables contain all solutions found in ascending order of k as well as a summary giving the total number of solutions for each value.

The present paper contains in Table 1 a somewhat shortened version of that summary, and lists all values of $|k|$ for which six or more solutions were found.

We summarize some of our results as follows:

(1) For positive $k \leqq 100$, no solution could be appended to the Table in [3].

(2) For negative $k \geqq -9999$, the last solution found was

$$(1{,}775{,}104)^3 - (2{,}365{,}024{,}826)^2 = -5412;$$

whilst, for positive $k \leqq 9999$, the last solution was

$$(939{,}787)^3 - (911{,}054{,}064)^2 = 307.$$

(3) In addition to solutions for $|k| \leqq 9999$, we have solutions, for $y \geqq 10^4$ and $|k| \leqq 99999$; there are 1221 for positive k and 799 for negative k.

The vast majority of solutions are with $y < 100$ and Table 2 gives the number of solutions for various ranges of y.

The fact that the number of solutions is a rapidly decreasing function of y suggests that for at least some k the solution set may be complete.

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Experiments on Gram-Schmidt Orthogonalization

By John R. Rice*

1. Orthogonalization Procedures. In this note we present a brief resumé of some experiments made on orthogonalization methods. We have a set $\{u_i \mid i = 1, 2, \cdots, n\}$ of m-vectors and wish to obtain an equivalent orthonormal set $\{v_i \mid i = 1, 2, \cdots, n\}$ of m-vectors. We consider the following methods:

(a) *Gram-Schmidt* (GS). $v_1 = u_1/\| u_1 \|$.

$$v_k' = u_k - \sum_{j=1}^{k-1} (v_j, u_k)v_j, \quad v_k = v_k'/\| v_k' \|; \qquad k = 2, \cdots, n.$$

(b) *Modified Gram-Schmidt* (MGS). $v_1 = u_1/\| u_1 \|$,

$$u_j^{(1)} = u_j - (u_j, v_1)v_1, \qquad j = 2, \cdots, n.$$

$$\left.\begin{aligned} v_k &= u_k^{(k-1)}/\|u_k^{(k-1)}\| \\ u_j^{(k)} &= u_j^{(k-1)} - (u_j^{(k-1)}, v_k)v_k \quad j = k+1, \cdots, n \end{aligned}\right\} k = 2, \cdots, n.$$

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