Zeros of Sections of the Zeta Function. I

By Robert Spira

1. Introduction. The sections of the title are the Dirichlet polynomials:

$$\zeta_M(s) = \sum_{n=1}^{M} n^{-s} \tag{1}$$

We write $s = \sigma + it$, and take $M \geqq 3$. Turán [1], [2] showed that the Riemann hypothesis would be true provided the zeros of $\zeta_M(s)$ all had real parts $\leqq 1 + k/M^{1/2}$, for some positive k. The author verified that these real parts were $\leqq 1$ for $M \leqq 100$ and $|t| \leqq 1000$. Apostol [3] generalized some of Turán's results to L-series. The present author's work also raises interest in these zeros in relation to the Riemann hypothesis for the functions

$$g_M(s) = \zeta_M(s) + \chi(s)\zeta_M(1 - s) \tag{2}$$

where $\chi(s)$ is the functional equation multiplier for the ζ-function, i.e., $\zeta(s) = \chi(s)\zeta(1 - s)$.

In this paper, theorems on zero-free regions of $\zeta_M(s)$ are derived, the methods used for calculating the zeros are given, and the locations of the zeros are described. The numerical values (to $6D$) of the zeros may be found in Spira [7]. The zeros calculated are:

(a) M: 3(1)12, $0 < t \leqq 100$,

(b) $M = 10^k$, k: 2(1)5, $-1 \leqq \sigma$, $0 < t \leqq 100$,

(c) $M = 10^{10}$, a string of zeros, $0 < t \leqq 100$,

(d) Sequences of zeros s_M, where s_{M+1} is obtained by applying Newton's method to $\zeta_{M+1}(s)$ with initial approximant s_M. The following sequences were calculated: $M = 4(1)50$, s_4 = lowest zero; $M = 10(1)35$, s_{10} = next to lowest zero; $M = 10(1)40$, $s_{10} = .35 + 14.50i$; $M = 11(1)50$, $s_{11} = .54 + 37.65i$; six sequences, $M = 11(1)25$, $s_{11} = .60 + 25.00i$, $.60 + 30.43i$, $.57 + 32.86i$, $.57 + 40.86i$, $.53 + 43.25i$, $.54 + 48.10i$.

Figures 1, 2, and 3 give the zeros (a). Two sequences (d) are also given in Figure 3. Figure 4 gives the zeros (b) and (c). Papers by Langer [4] and Wilder [5] estimate the number of zeros of $\zeta_M(s)$ to be within M of $T(\log M)/2\pi$. Table I gives the number of zeros found and the values of $100(\log M)/2\pi$ for comparison.

2. Zero-Free Regions.

THEOREM 1. *If* $\sigma \geqq 1.85$, $\zeta_M(s) \neq 0$.

Proof.

$$|\zeta_M(s)| \geqq 1 - \left[\sum_{n=2}^{M} n^{-\sigma}\right] \geqq 1 - 2^{-\sigma} - \int_2^M x^{-\sigma}\,dx$$

$$\geqq 1 - [2^{-\sigma} + 2^{1-\sigma}/(\sigma - 1)] = 1 - 2^{-\sigma}(\sigma + 1)/(\sigma - 1).$$

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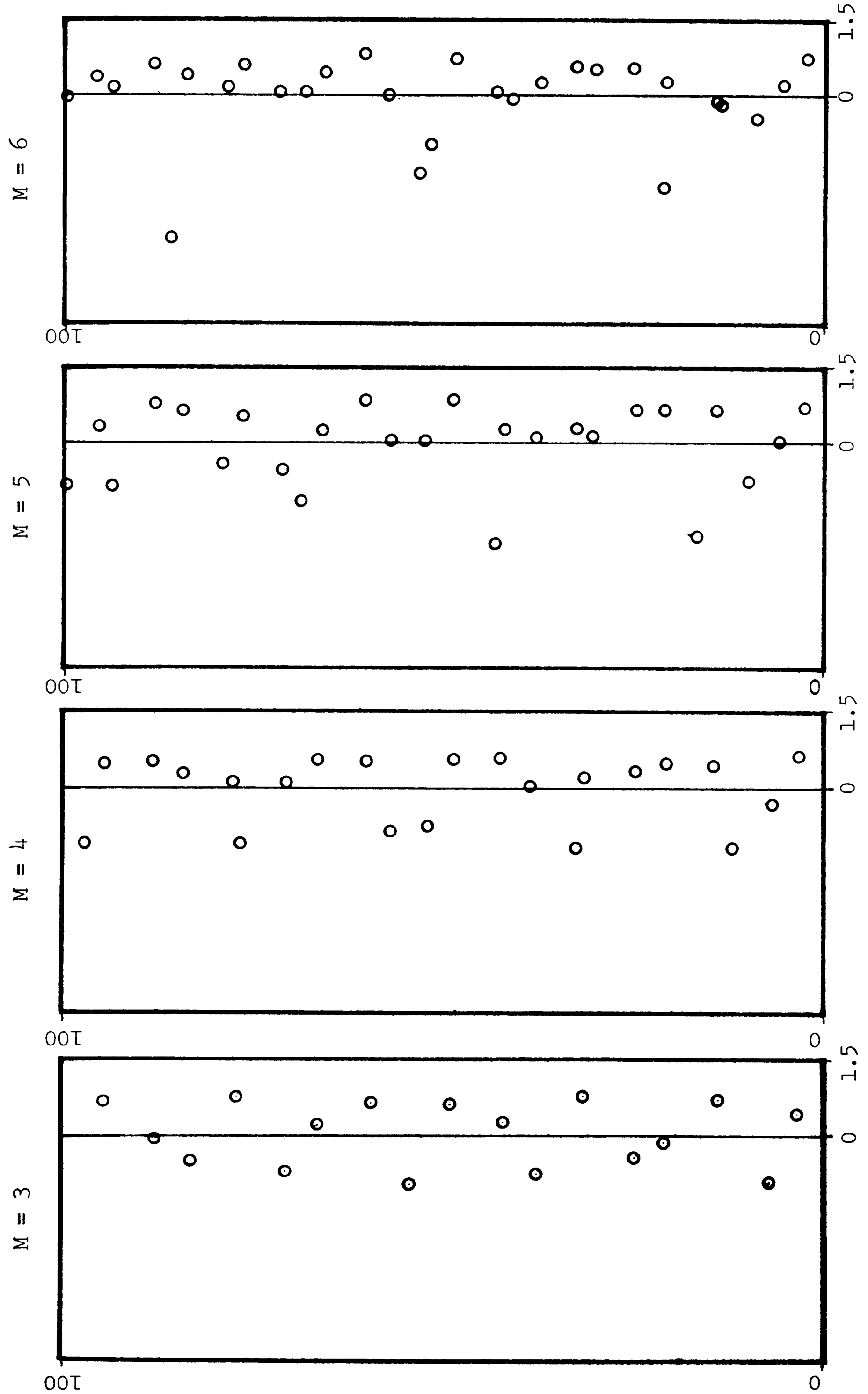

FIGURE 1. ZEROS OF $\zeta_M(s)$

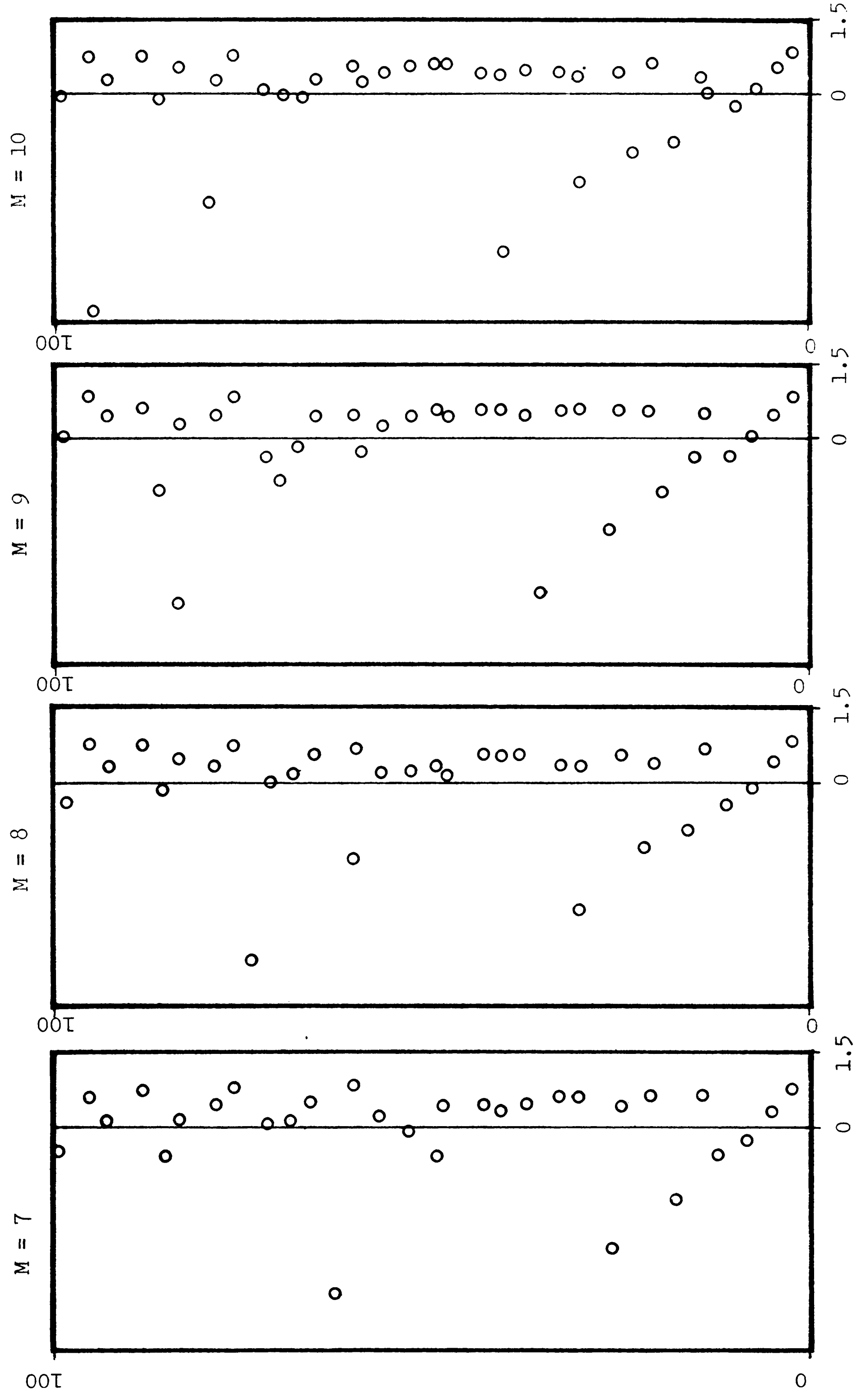

FIGURE 2. ZEROS OF $\zeta_M(s)$

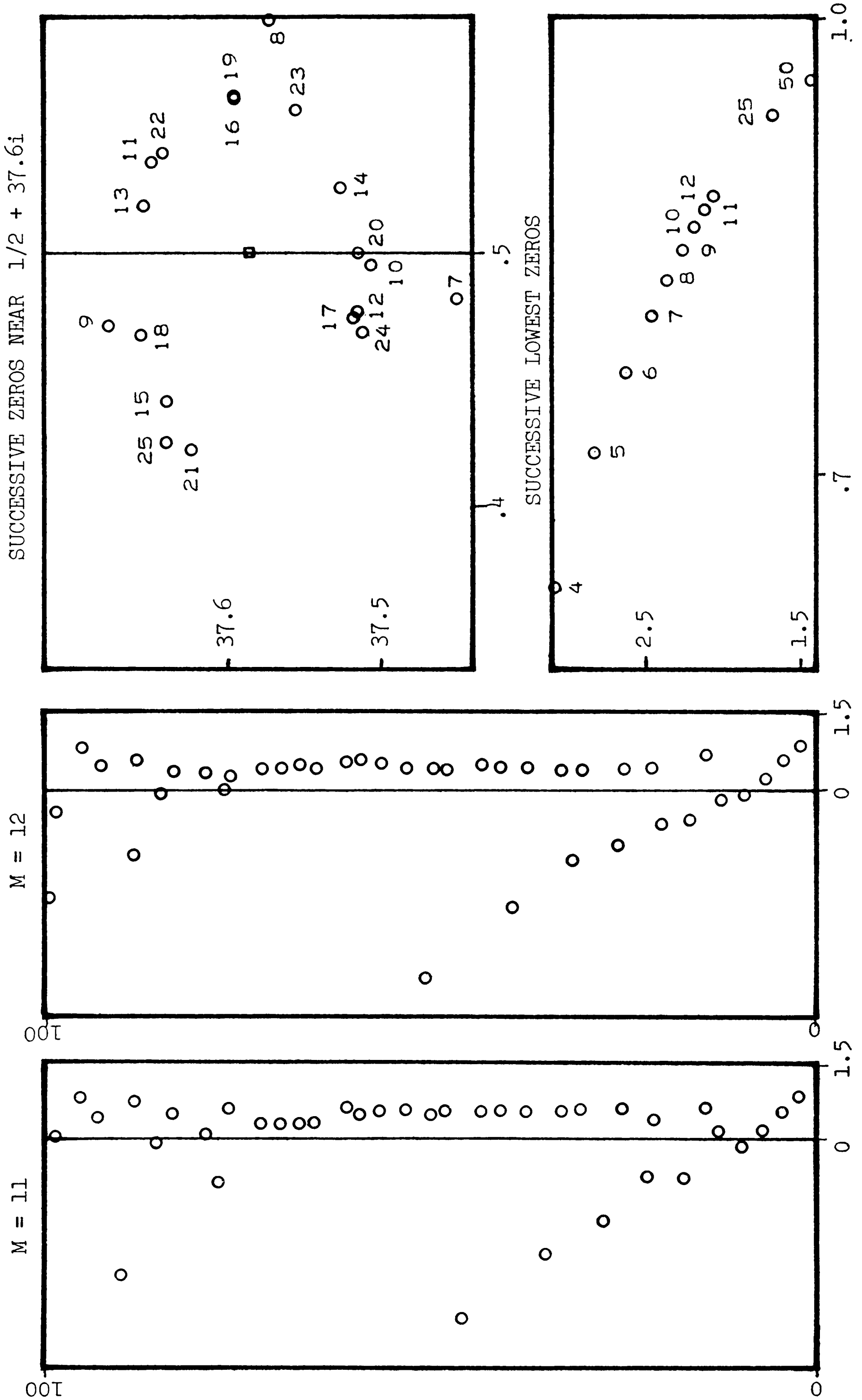

FIGURE 3. ZEROS OF $\zeta_M(s)$

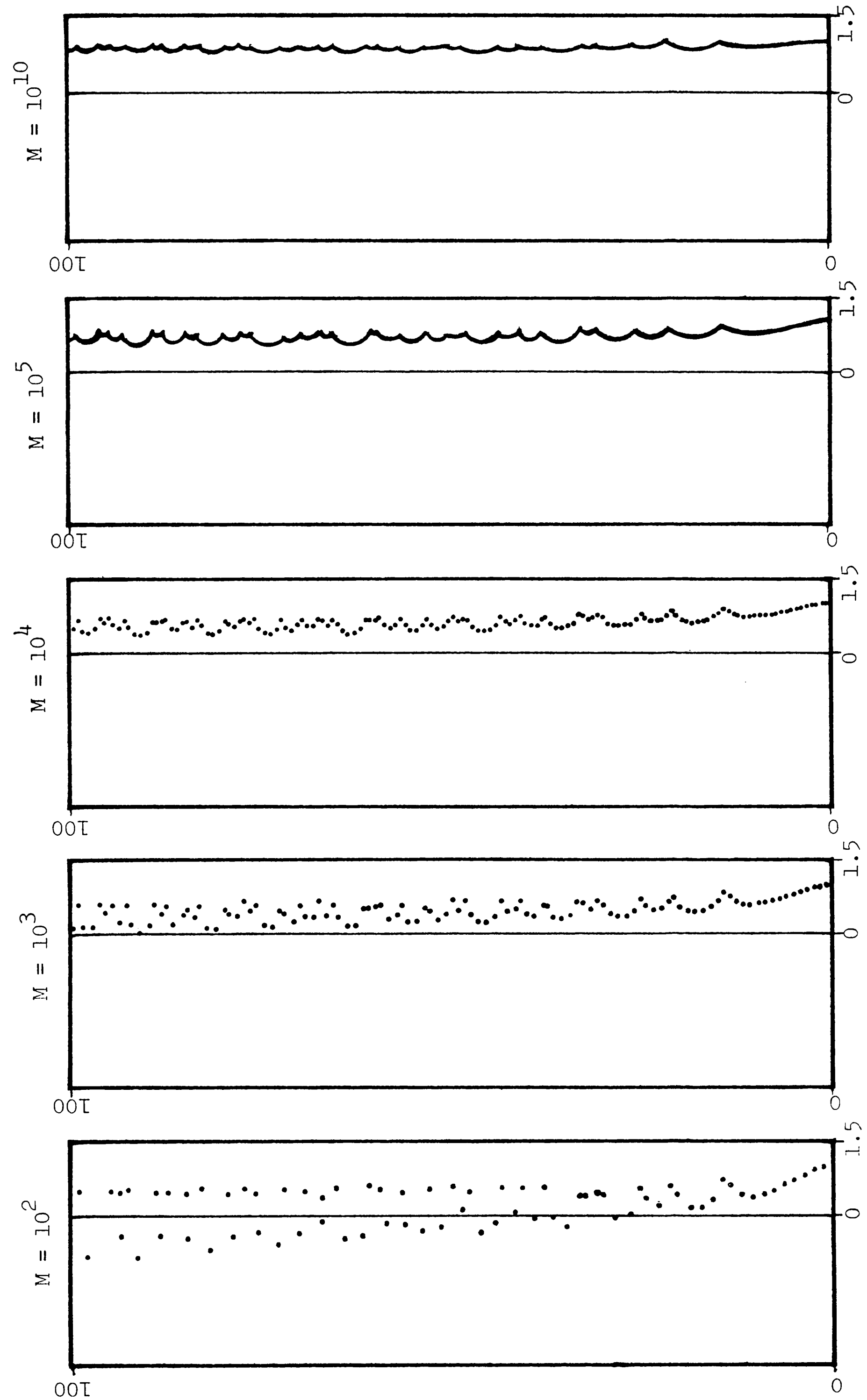

FIGURE 4. ZEROS OF $\zeta_M(s)$

TABLE I

M	Number of zeros < 100	100 (Log M)/2π	Left limit of zeros
3	17	17.485	$-1.$
4	22	22.064	$-1.73050\ 73579$
5	26	25.615	$-2.42601\ 27644$
6	29	28.517	$-3.11889\ 57337$
7	31	30.970	$-3.81199\ 94585$
8	33	33.095	$-4.50517\ 30392$
9	35	34.970	$-5.19834\ 67486$
10	37	36.647	$-5.89151\ 57183$
11	38	38.164	$-6.58468\ 12961$
12	40	39.549	$-7.27784\ 42797$
10^2	73	73.294	
10^3	110	109.940	
10^4	146	146.587	
10^5	183	183.234	
10^{10}	366	366.468	

The last five entries for number of zeros means the number found for $\sigma \geqq -1$. It is probable that there are no others for $t \leqq 100$.

Since $2^{-\sigma}$ and $(\sigma + 1)/(\sigma - 1)$ are decreasing functions, we need only seek the root of $1 = 2^{-\sigma}(\sigma + 1)/(\sigma - 1)$, which is easily seen to lie to the left of 1.85.

THEOREM 2. *If* $\sigma \leqq 1 - M$, $\zeta_M(s) \neq 0$.

Proof.

$$| \zeta_M(s) | \geqq M^{-\sigma} - \left[\sum_{n=1}^{M-1} n^{-\sigma} \right] \geqq M^{-\sigma} - \left[1 + 2^{-\sigma} + \int_2^{M-1} x^{-\sigma}\, dx \right]$$
$$\geqq M^{-\sigma} - [1 + 2^{-\sigma} + (M - 1)^{1-\sigma}/(1 - \sigma)]$$
$$\geqq M^{-\sigma} - [1 + 2^{-\sigma} + (M - 1)^{-\sigma}],$$

since $M - 1 \leqq 1 - \sigma$. Thus $\zeta_M(s) \neq 0$, provided $M^{-\sigma} > 1 + 2^{-\sigma} + (M - 1)^{-\sigma}$. Dividing this last by $(M - 1)^{-\sigma}$, and noting that the resulting right hand side is $\leqq 2.25$ for $M \geqq 3$, our theorem will be true provided $[M/(M - 1)]^{M-1} \geqq 2.25$. Putting $m = M - 1$, we know by the binomial theorem that $(1 + 1/m)^m$ is an increasing function, so that $[M/(M - 1)]^{M-1}$ is always $\geqq 2.25$ as it is so for $M = 3$.

One can obtain closer bounds in particular cases.

PROPOSITION. *If* $M^{-\sigma} > \sum_{n=1}^{M-1} n^{-\sigma}$ *and* $\sigma_1 < \sigma$, *then* $M^{-\sigma_1} > \sum_{n=1}^{M-1} n^{-\sigma_1}$.

Proof. Let $\sigma = \sigma_1 + k$, $k > 0$. Then $M^{-\sigma_1} = M^{-\sigma+k} = M^{-\sigma}M^k > M^k \sum_{n=1}^{M-1} n^{-\sigma} = \sum_{n=1}^{M-1} M^k n^{-\sigma} \geqq \sum_{n=1}^{M-1} n^k n^{-\sigma} = \sum_{n=1}^{M-1} n^{-\sigma_1}$, Q.E.D. Thus, the root of $M^{-\sigma} = \sum_{n=1}^{M-1} n^{-\sigma}$ gives a right hand bound for a zero-free half plane of $\zeta_M(s)$. Column 4 of Table I gives this root, rounded to ten places, for $3 \leqq M \leqq 12$.

A crude upper bound for this root is $-M^{1/2}$, for $M \geqq 18$. To show this, let $\alpha = M^{1/2}$, then

$$\sum_{n=1}^{M-1} n^{\alpha} > \int_0^{M-2} x^{\alpha}\, dx + (M - 1)^{\alpha} \tag{3}$$

and

$$\int_0^{M-2} x^\alpha\,dx = (M-2)^{\alpha+1}/(\alpha+1) = (\alpha-1)((M-2)/(M-1))(M-2)^\alpha$$

$$> ((M-2)^{1/2}-1)(6/7)(M-2)^\alpha = (6/7)(M-2)^{\alpha+1/2} - (6/7)(M-2)^\alpha.$$

We can drop the second term as it is overpowered by the second term of (3), so we need only show $\frac{6}{7}(M-2)^{\alpha+1/2} > M^\alpha$. After dividing by $(M-2)^\alpha$ and setting $m = M-2$, this transforms to $\frac{6}{7}m^{1/2} > (1+2/m)^{(m+2)^{1/2}}$. Now $(m+2)^{1/2} < m/2$ for $m \geqq 6$, so the right hand side of the last inequality is bounded by e, which the left hand side will surely exceed if $m \geqq 16$, or $M \geqq 18$.

As shown in Turán [2], the 1.85 bound of Theorem 1 can be reduced to 1 for $M \leqq 5$. For $M = 3$, one can do slightly better. If $\zeta_3(s) = 0$, then

$$(4)\qquad \begin{aligned} 2^{-\sigma}\cos(t\log 2) + 3^{-\sigma}\cos(t\log 3) &= -1,\\ 2^{-\sigma}\sin(t\log 2) + 3^{-\sigma}\sin(t\log 3) &= 0. \end{aligned}$$

Squaring and adding, we obtain

$$(5)\qquad \cos(t\log\tfrac{3}{2}) = \{6^\sigma - [(\tfrac{2}{3})^\sigma + (\tfrac{3}{2})^\sigma]\}/2 = g(\sigma)$$

and

$$(6)\qquad g'(\sigma) = \{6^\sigma[\log 6 + (\log\tfrac{3}{2})(9^{-\sigma} - 4^{-\sigma})]\}/2.$$

For $\sigma \leqq 0$, $9^{-\sigma} \geqq 4^{-\sigma}$, so $g'(\sigma) > 0$, and for $\sigma > 0$, $\log 6 > 4^{-\sigma}\log\frac{3}{2}$, so $g'(\sigma) > 0$ for all σ and $g(\sigma)$ is strictly increasing. From (5), $|g(\sigma)| \leqq 1$, and a detailed calculation gives -1 and $.78788\,49110\cdots$ as the limits on σ.

A curious consequence of the above analysis is the following:

PROPOSITION. *If* $\sigma \neq \frac{1}{2}$, $\zeta_3(s)$ *and* $\zeta_3(1-s)$ *cannot both be zero.*

Proof. Let $\zeta_3(s) = 0$. We can take $t \geqq 0$. From the second equation of (4), it follows that $\sin(t\log 2)$ and $\sin(t\log 3)$ are both zero or both nonzero. If they both vanished, we would have $t\log 2 = k\pi$, $t\log 3 = j\pi$, k and j nonnegative integers. If $t \neq 0$, (and hence $k, j \neq 0$), we can divide these last two equations, obtaining $\log 2/\log 3 = k/j$, or $3^k = 2^j$, which is impossible for $k, j > 0$. If $t = 0$, we could deduce from the first equation of (4) that $2^{-\sigma} + 3^{-\sigma} = -1$, which is impossible. Hence $\sin(t\log 2)$ and $\sin(t\log 3)$ are nonzero and we can write

$$(\tfrac{3}{2})^\sigma = -(\sin(t\log 3))/(\sin(t\log 2)).$$

Thus, σ is determined as a function of t, so that two distinct values of σ are impossible for a given t.

This shows that if $g_3(s)$ is zero off $\sigma = \frac{1}{2}$, it cannot happen for the reason $\zeta_3(s) = \zeta_3(1-s) = 0$. In Spira [6], it was shown that for t sufficiently large, $g_1(s)$ and $g_2(s)$ satisfy the Riemann hypothesis. In Spira [10], the calculations are described indicating zeros off the critical line of $g_M(s)$ for $M \geqq 3$.

3. Method of Calculation. The zeros for $M \neq 10^{10}$ were found by locating a zero within a square, searching the square by absolute value tests for small functional values with the four possibilities of signs for the real and imaginary parts, closing in with linear interpolation, and then a final tightening with a high precision Newton's method. For the zeros with $M \leqq 12$, it was possible to search the entire strip within which the zeros were known to be located. For larger $M \neq 10^{10}$, only the region to the right of $\sigma = -1$ was searched. From the general appearance of the zeros, and the fact that the numbers of zeros found are so close to $100(\log M)/2\pi$, it seems likely that all the zeros for $0 \leqq t \leqq 100$ have been found for the M considered. An integration was performed over the boundaries of regions searched, verifying the number of zeros obtained.

The zeros for $M = 10^{10}$ were calculated in a sequence using Newton's method, with initial approximants:

$$s_0 = 1 + 2\pi i/\log 10^{10},$$
$$s_1 = 1 + 4\pi i/\log 10^{10},$$
$$s_{m+1} = s_m + (s_m - s_{m-1}).$$

For large M, the direct use of series (1) would involve an impractical amount of time. Thus, it was necessary to use the asymptotic expansion derived from the Euler-McLaurin formula:

$$\begin{aligned}\zeta_M(s) = {} & \sum_{n=1}^{N-1} n^{-s} + \frac{N^{-s}}{2} - \frac{N^{1-s}}{1-s} + \frac{M^{-s}}{2} + \frac{M^{1-s}}{1-s} \\ & + \sum_{v=1}^{m} \frac{B_{2v}}{(2v)!}\left(\prod_{j=0}^{2v-2} (s+j)\right) N^{1-s-2v} \\ & - \sum_{v=1}^{m} \frac{B_{2v}}{(2v)!}\left(\prod_{j=0}^{2v-2}\right)(s+j) M^{1-s-2v} \\ & + \text{error}.\end{aligned}$$

One also needs, for Newton's method, a similar asymptotic expansion for $\zeta_M{}'(s)$. The programs for these were obtained as modifications of programs for $\zeta(s)$ and $\zeta'(s)$, available as a separate report [8]. Note that the term $M^{1-s}/(1 - s)$ is very large near $\sigma = 0$, and also changes argument very rapidly as t varies. The direct and asymptotic series were checked against each other for $M = 100$, as were most of the other programs. The set of zeros finally obtained was differenced and also resubstituted to verify that they were good approximations to the true zeros. The computations were carried out at the University Computing Center, University of Tennessee (NSF-G13581).

4. The Zeros. For $M \leqq 12$, the zeros appear to have a pattern which has two parts. One part is a line of $M - 1$ zeros stretching upward in the left half plane with a negative slope which increases negatively with M. The other part consists of zeros which at first form a line near $\sigma = \frac{1}{2}$, the zeros of $\zeta_M(s)$ lying quite close to the zeros of $\zeta(s)$, and then this line disappearing in a general scattering or blossoming. For $M = 3$, this scattering appears almost immediately, while for increasing M, the start of scattering moves progressively upward, being around 70 or 80 units up at $M = 12$. Since

$$\zeta(s) = \zeta_{M-1}(s) + M^{-s}/2 + M^{1-s}/(s-1) + \sum_{v=1}^{m} \frac{B_{2v}}{(2v)!}\left(\prod_{j=0}^{2v-2} (s+j)\right) M^{1-s-2v} + R$$

we can expect a reasonable proximity of the zeros of $\zeta(s)$ and $\zeta_{M-1}(s)$ whenever the remaining terms are small. For fixed s, the B_{2v} terms become small when M exceeds $t/2\pi$ (Lehmer [9]). Near $\sigma = \frac{1}{2}$, $M^{1-s}/(s-1) \sim M^{1/2}/t$ which will certainly not be small when $M \sim t^2$. Turning the argument about, we can see that $\zeta(s)$ is roughly approximated by $\zeta_M(s)$ near the critical line for $t < 2\pi M$, but t also large enough so that $M^{1/2}/t$ is small.

In Figure 3, one can observe the zeros of $\zeta_M(s)$, for successive M, circling first in one direction and then in the other around the zero $s_o = .5 + 37.586i$ of $\zeta(s)$ (denoted by a small square). This circling can be explained by setting s_M to be such a zero, then $\zeta_{M+1}(s_M) = (M+1)^{-s_M}$, and s_M can be moved slightly to s_M' so that the vectors Ms_M' have overcome the disturbance $(M+1)^{-s_M}$. The disturbance for each M will be approximately M^{-s_0}, and as this rotates depending on M, the zeros s_M will be rotating one way or another depending on the quadrant of M^{-s_0}.

The successive lowest zeros, also on Figure 3, appear to have imaginary part slightly less than $2\pi/\log M$. Thus, the vectors n^{-s} have consecutive arguments spread between 0 and $2\pi - \epsilon$, for this lowest zero. This appears to be true for every M. The real parts appear to be strictly increasing with upper limit 1.

As noted in Turán [2], every point on the line $\sigma = 1$ is an accumulation point of the zeros of $\zeta_M(s)$, but to see the approach one must proceed to very large M, as in Figure 4. For such large M, the zeros lie on a line which sags to the left between t locations of zeros of $\zeta(s)$, and at such t locations there is a forcing to the right as well as a shortening of the intervals between zeros.

Thus, we have described the empirical behavior of the zeros of $\zeta_M(s)$. In Turán [2] there is given a proof by Jessen that $\zeta_M(s) \neq 0$ for $M \leqq 5$, by showing that $\operatorname{Re} \zeta_M(s) > 0$ for $\sigma \geqq 1$. The present author was able to extend this result to $M = 6$ and $M = 8$. However, one cannot show $\operatorname{Re} \zeta_7(s) > 0$ for $\sigma \geqq 1$, but it appears that a different method can settle this case. These matters require extensive calculation, and further study, and will be taken up in part II of this paper.

University of Tennessee
Knoxville, Tennessee 37916

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