REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

85[**A, J, L, M.**].—I. S. Gradshteyn & I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1965, xiv + 1086 pp., 23 cm. Price $10.50. (Translated from the 4th Russian edition by Scripta Technica and Alan Jeffrey. The 4th Russian edition had actually been prepared by Yu. V. Geronimus and M. Yu. Tseytlin after the deaths of both original authors.)

This is the first American edition. It is "translated" from the much enlarged, fourth Russian edition—that is, the text is translated, while the thousands of mathematical formulas are reproduced photographically. For detailed reviews of earlier editions, and for table errata listed in this journal, see *MTAC*, v. 1, 1945, RMT **219,** pp. 442–443; *Math. Comp.*, v. 14, 1960, RMT **69,** pp. 381–382, and MTE **293,** pp. 401–403; *Math. Comp.*, v. 17, 1963, MTE **326,** p. 102; *Math. Comp.*, v. 20, 1966, MTE **392,** p. 468. See also, if you wish, the numerous notices in *Math. Reviews*: MR **14,** p. 643; MR **22,** #3120; MR **28,** #1326; MR **28,** #5198; and MR **30,** #5458.

This fourth edition has "more than twice as many formulas as any of the previous editions" and is advertised as "the most comprehensive table of integrals ever published." The main increases over the third edition have been in the tables of definite integrals of elementary functions (four times as long), and of special functions (ten times as long). But the chapter entitled Indefinite Integrals of Elementary Functions has also been doubled, and new material on special functions, e.g., Mathieu, Struve, Lommel, etc. has been added. On the other hand, some numerical tables in the third edition have been dropped, namely Lobachevskiy's function, values of $\zeta(n)$, and numerical coefficients involving factorials.

Because of its inclusiveness, one is tempted to refer to this volume as the definitive reference book of its type, but unfortunately it is flawed in several directions: mediocre printing, imperfect translation, and persistence in repeating errors that had been pointed out long ago.

Photographic reprinting is not bad if the proper care is taken, but some pages here, e.g., pp. 1065, 1073, are actually shoddy. (As an aside, when so much of the volume is photographic, and no authorization by the Russian authors is indicated, the reviewer is curious about the legal status of the copyright, and particularly the warning printed there: "No part of this book may be reproduced in any form, by photostat, . . . ").

Translation of a table of integrals might seem to make minimal demands upon a translator, but some crude errors are found here. On p. 909 elliptic functions are called "rational" instead of "meromorphic," and are falsely stated to have "no more than two simple and one second-order pole in such a parallelogram." Statement 6 there is not expressed clearly, and in Statement 8 the stipulation of nonconstancy is omitted. On p. 933, $\Gamma(z)$ is a "fractional" analytic function, and on p. 1074 it is alleged that an "uncountable" set of zeros of $\zeta(z)$ have been proven to have real part $\frac{1}{2}$.

Of the errors indicated in the MTE **293** mentioned above, about one-half of them remain, although now on different pages. Specifically, still uncorrected are those errors in MTE **293** previously on pp. 2, 24, 149, 186, 274, 301, 303 together with the erroneous values of Euler's constant and B_{34}.

Particularly charming, and in a way a lesson to us all, is the erroneous

$$\prod_{k=1}^{\infty} \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6}\left(\frac{\pi}{\sqrt{3}}\right)^3$$

on p. 938. The upper limit of the product should be 8, not ∞. The persistence of this error should be an inspiration to everyone. For many years it continued as a misprint in Whittaker and Watson, and though it was finally corrected there, and referred to in MTE **293**, it has managed to elude the combined scrutiny of Ryzhik, Gradshteyn, Geronimus, Tseytlin, Lapko, Scripta Technica, and Jeffrey, and that, in spite of the fact that it is so blatantly false that no mathematician examining it with even casual attention should fail to note that an error is present.

In summary, then, we have a mass of useful information here, but the editing was not of that quality which it deserved.

D. S.

86[F].—Carl Friedrich Gauss, *Disquisitiones Arithmeticae*, Yale University, New Haven, Connecticut, 1966. Translated into English by Arthur A. Clarke, S.J., xx + 472 pp., 24 cm. Price $12.50 (paperback $2.95).

Several years ago [1], the reviewer had occasion to emphasize that Gauss's *Disquisitiones* was still not available in English. At the suggestion of Dr. Herman Goldstine, Professor Arthur A. Clarke, S.J., now offers us a translation, and thus somewhat rectifies this 165-year-old anomaly. For this, English-speaking mathematicians will be somewhat grateful. We say only "somewhat," however, since the translation has unfortunately many defects: peculiar and inaccurate terminology, awkward and undesirable notation, some serious typographical errors, and frequent confusing and inadequate translations. Of course, these are serious charges—which must therefore be documented. Here are some samples.

On p. 168, instead of *convergent fractions*, we find first *approaching fractions*, and two lines later, *approximating fractions*. On p. 342, *trigonal* numbers replace the usual *triangular* numbers, and on p. 360 we find *middle* determinants instead of *mean* determinants. Many similar peculiarities exist.

On p. 240, we find *equation* (I) referring to four equations; on p. 360 *only one class* means *only one genus*; and on pp. 373–374 one finds two examples with inexplicably contradictory terminology: the first, which (correctly) has *four positive genera*, is immediately followed by the second with *eight positive categories*.

For awkward symbolism see f''''''' on p. 162, the undisplayed (I): (1), (3), (5), $\cdots L \cdots$ on p. 170, etc. Unlike Gauss, and (all?) modern writers: Mathews, Dickson, Cohn, etc., Clarke (p. 265) uses + instead of $\times$ to represent the operation on classes called *composition*, and thus, for example, he writes $2K$ instead of K^2 for the *duplication* of a class. This is not only historically wrong, and at variance with customary usage, and in contradiction to earlier symbolism on, say, p. 258, where *F is transformable into ff'*, but it is intrinsically wrong, since, again on p. 258, if a is represented by f and a' by f' then their *product aa'* is represented by the composition class F. Further, this unfortunate symbolism destroys the artistry of Gauss's presentation, since he is clearly suggesting an analogy between the classes of forms under composition with the earlier residue classes under multiplication modulo m—with analogous primitive roots, order of the groups, etc.

Typographical errors that could well cause difficulties are, say, $D = 850/2$, 1550/2, etc. instead of $850\frac{1}{2}$, $1550\frac{1}{2}$, etc. on p. 359, and $m\sqrt{(D - n)}$ instead of $m\sqrt{D} - n$ on p. 364.

For inadequate, false, and/or confusing translations, try these: "unless the congruence" and "But this omission," at the top of p. 4; and "vague computations," at the bottom of p. 5. Again, the statement of the reciprocity law on p. 87: "which, taken positively," etc. is certainly ambiguous. Garbled, and quite misleading, is "That is, there is only a small number . . . " on p. 363.

Beside these many new defects, nothing is done to correct Gauss's own rare notational discrepancies, or his even rarer actual errors. Thus we find $AX^2 + 2BXY + CY^2 \cdots F$, $Ax^2 + 2Bxy + Cy^2 \cdots (F)$, and $F = ax^2 + 2bxy + cy^2$, on pp. 116, 123, and 220. On p. 363, Gauss's error stating that there are 27 determinants with classification IV,1 is still present. Actually, there are 26, and Gauss is presumably classifying 99 as IV, 1 instead of the correct IV, 2.

On the positive side, the volume is very nicely printed on good paper, and it includes many additional notes by the translator, mostly consisting of the exact titles of Gauss's many references. He also includes *Gauss's Handwritten Notes*, which are not given in the French edition [2]. These indicate the dates of Gauss's many discoveries first published in his book. Finally, there is a *List of Special Symbols* and a *Directory of Terms*, but these, again, are not as well done as would be desirable.

For all that, any translation is better than none, and no doubt this volume will introduce many students to Gauss's work. The reviewer must say that he is pleased to own a copy, even with its many defects. It can be best used if the reader also has access to a European translation, say [2], for purpose of comparison.

Anyone interested in Gauss's work would do well to examine Mathew's *Theory of Numbers* [3], since, of all textbooks in English, this is the one most in harmony with Gauss's subject matter and treatment. Also of value here is the chapter by G. J. Rieger in the *Gauss Gedenkband* [4]. Free tip to publishers: This last volume should (also) be translated into English. As luck would have it, this may well be a perfect translation, and appear in much less than 165 years.

D. S.

1. DANIEL SHANKS, *Solved and Unsolved Problems in Number Theory*, Vol. 1, Spartan, Washington, D. C., 1962, p. 62.
2. CH. FR. GAUSS, *Recherches Arithmétiques*, reprinted by Blanchard, Paris, 1953.
3. G. B. MATHEWS, *Theory of Numbers*, reprinted by Chelsea, New York, 1961.
4. GEORG JOHANN RIEGER, "Die Zahlentheorie bei C. F. Gauss," *C. F. Gauss Gedenkband Anlässlich des 100. Todestages am 23. Februar* 1955, Teubner, Leipzig, 1957, pp. 37–77.

87[F].—MARVIN WUNDERLICH, *Tables of Fibonacci Entry Points*, edited by Brother U. Alfred, published by The Fibonacci Association, San Jose State College, San Jose, California, January 1965, vii + 54 pp., 28 cm. Spiral bound. Price $1.00.

88[F].—DOUGLAS LIND, ROBERT A. MORRIS & LEONARD D. SHAPIRO, *Tables of Fibonacci Entry Points, Part Two*, edited by Brother U. Alfred, published by

The Fibonacci Association, San Jose State College, San Jose, California, September 1965, 50 pp., 28 cm. Spiral bound. Price \$1.50.

The first of these two publications contains a table of the rank of apparition (here called "entry point") in the Fibonacci sequence for every prime to 48163, inclusive; the second contains a continuation to 99991, inclusive. Inverse tables are also presented, which together permit the identification of all primes less than 10^5 that correspond to ranks of apparition, $Z(p)$, less than 10^5, arranged in numerical order.

The tables in the first book were calculated on an IBM 709 at the Computation Center of the University of Colorado; those in the second, on an IBM 1620 at the Reed College Computing Center.

Previously published tables of such information are quite limited in scope. That of Kraitchik [1] gives (in different notation) the value of $(p - \epsilon)/Z(p)$ for $p < 10^3$, while that of Yarden [2] extends to all $p \leqq 1511$. Here, according to a well-known theorem of Lucas, the quantity ϵ is equivalent to the Legendre symbol $(5/p)$.

The first volume under review contains a discussion of the relationship between the entry points for the Fibonacci sequence and the Lucas sequence, and illustrations of the application of the tables to the factorization of members of both sequences. Also included in the introductory material is a brief discussion of the periods of the Fibonacci and Lucas sequences with respect to a given prime modulus. A list of four references is appended.

The list of typographical corrections appearing on p. vii of the first book should be increased by three additional corrections noted by this reviewer upon making a single comparison of the published tables with the underlying manuscript tables of Mr. Wunderlich, which were lent him by John D. Brillhart. Thus, on p. 16 the value of $Z(p)$ corresponding to $p = 26459$ should read 26458 in place of 26459, on p. 25 the value of $Z(p)$ when $p = 44071$ should read 8814 in place of 8614, and on p. 41 the argument $Z(p) = 8968$ corresponding to the entry $p = 17737$ should be replaced by 8869.

The editor has informed this reviewer that the tables in the first book were typed from the printed computer output, thus accounting for many of the typographical errors present.

On the other hand, the tables in the second book were printed directly from the computer output, so that such imperfections appear to have been obviated. Moreover, the entries in the second set of tables are arranged systematically in blocks of eight lines each; this results in a more pleasing appearance than that of the first set, where no similar separation of the entries occurs.

Finally, it should be emphasized that these two volumes represent a major contribution to the numerical information that is available on both the Fibonacci and the Lucas sequences.

J. W. W.

1. M. Kraitchik, *Recherches sur la Théorie des Nombres*, Gauthier-Villars, Paris, 1924, p. 55.
2. D. Yarden, "Luach Tsiyune-hehofa'a Besidrath Fibonatsi" [Tables of the ranks of apparition in Fibonacci's sequence], *Riveon Lematematika*, v. 1, no. 3, December 1946, p. 54. (See *MTAC*, v. 2, 1947, pp. 343–344, RMT **439**.)

89[F].—M. LAL, M. F. JONES & W. J. BLUNDON, *Tables of Solutions of the Diophantine Equation* $Y^3 - X^2 = K$, Department of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, 1965, iii + 162 pp., 28 cm. Price $10.00.

Nicely printed and bound are the following three tables:

Table 1: The 5370 solutions of

$$y^3 - x^2 = k \quad (1)$$

with $0 \leqq x < 10^{10}$ and $-10^4 < k < 0$.

Table 2: The 3223 solutions of (1) with $0 < k < 10^4$ and the same range of x. (Actually, there are no solutions for $9989 < k < 10^4$.)

Table 3: The number of solutions for each k in the previous two tables. (Although the text indicates that $k = 0$ was not done, it appears that it really was, since we find here that there are 2155 solutions for $k = 0$, which number is, in fact, $[10^{10/3}] + 1$.)

Most previous authors have used the notation $y^2 - k = x^3$, and it may be hoped that the present changes will not cause too much confusion. However, perhaps that is too much to expect, since even the authors write "for $0 < k \leqq -100$, all solutions of (1) have been found." (When taken literally, this would appear to be a very minor accomplishment.)

The tables are prefaced by (essentially) the authors' note previously appearing in this journal [1]. It is known from Hemer [2], (but see the account of his earlier errata in the *MTAC* review of his [3]), that there are no other solutions of (1) for $-10^2 < k < 0$. But for $0 < k < 10^2$ there are 20 cases [4] whose completeness is in doubt. The authors surmise that their table is nonetheless complete here also. They obtain no new solutions for $0 < k < 10^2$ not already listed in [2].

The authors do not mention the earlier work of Robinson [5] concerning $| y^2 - x^3 | < x$ for $0 \leqq y < (\frac{1}{27}) \times 10^9$, which preceded [2], and led to corrections there [3]. The present table contains no other solutions for $-225 < k < 207$ of the type tabulated by Robinson in spite of their substantially greater range of what they call x and he calls y. For a greater range of k, namely, $-1000 \leqq k \leqq +1000$, I find that the present table picks up only 6 new solutions not in Robinson:

k	y	x
−618	421351	273505487
−353	117188	40116655
−225	720144	611085363
207	367806	223063347
307	939787	911054064
847	657547	533200074

It may well be, therefore, that the present table is complete for $-200 \leqq k \leqq 200$.

The new solution for $k = -618$ may be of some special interest, since the only other solution for that k is the very small $7^3 = 31^2 - 618$. Another solution found in the table is a pleasant Fermat Frustrater:

$$11^3 + 37^3 = 228^2.$$

The reviewer notes that when there are many solutions for some k, say 8 or more, then usually there are none at all for $-k$.

The programming for these tables is, of course, quite simple. A real challenge to number-theoretically inclined programmers would be to program the *theory* necessary for showing completeness, at least in the easier cases for negative k.

D. S.

1. M. Lal, M. F. Jones & W. J. Blundon, "Numerical solutions of the Diophantine equation $y^3 - x^2 = k$," *Math. Comp.*, v. 20, 1966, pp. 322–325.
2. O. Hemer, "Notes on the Diophantine equations $y^2 - k = x^3$," *Ark. Mat.*, v. 3, 1954, pp. 67–77. See also RMT **1208**, *MTAC*, v. 8, 1954, pp. 149–150.
3. Ove Hemer, *On the Diophantine Equation $y^2 - k = x^3$*, Uppsala, 1952. See also RMT **1068**, *MTAC*, v. 7, 1953, p. 86.
4. W. Ljunggren, "On the Diophantine equation $y^2 - k = x^3$," *Acta Arith.*, v. 8, 1963, pp. 451–463.
5. R. Robinson, "Table of integral solutions of $|y^2 - x^3| < x$," UMT **125**, *MTAC*, v. 5, 1951, p. 162.

90[H, X].—J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, New York, 1965, xviii + 662 pp., 25 cm. Price $17.50.

This excellent book is the work of an expert. He has given a unified treatment of the theoretical and practical aspects of the algebraic eigenvalue problem.

The reader will find this presentation to be clear, complete and up to date. Several of the author's recent results appear here for the first time. The simple listing of the chapter headings may indicate the scope of the work:

1. Theoretical Background
2. Perturbation Theory
3. Error Analysis
4. Solution of Linear Algebraic Equations
5. Hermitian Matrices
6. Reduction of a General Matrix to Condensed Form
7. Eigenvalues of Matrices of Condensed Forms
8. The LR and QR Algorithms
9. Iterative Methods

In the preface the author states, "The eigenvalue problem has a deceptively simple formulation and the background theory has been known for many years; yet the determination of accurate solutions presents a wide variety of challenging problems." He then systematically disposes of most of the problems. E. I.

91[I].—K. A. Karpov, *Tables of Lagrange Interpolation Coefficients*, The Macmillan Company, New York, 1965, viii + 75 pp., 25 cm. Price $5.75.

This book, which is Volume 28 of the Pergamon Press Mathematical Tables Series, is an attractively printed and bound English translation by D. E. Brown of the Russian *Tablitsy Koéffitsientov interpoliatsionnoĭ Formuly Lagranzha*, published in 1954 by the Academy of Sciences, U.S.S.R. and reviewed in this journal [1].

An appropriate reference that has appeared since the original edition of this book is the tables of Karmazina & Kurochkina [2].

J. W. W.

1. *MTAC*, v. 11, 1957, pp. 209–210, RMT **85**.
2. L. N. Karmazina & L. V. Kurochkina, *Tablitsy interpoliatsionnykh koéffitsientov*, Academy of Sciences of the USSR, Moscow, 1956. See *MTAC*, v. 12, 1958, p. 149, RMT **66**.

92[K].—B. M. BENNETT & C. HORST, *Supplement to Tables for Testing Significance in a* 2×2 *Contingency Table* (*Five and One Percent Significance Points for* $41 \leqq A \leqq 50$, $B \leqq A$), Cambridge University Press, New York, 1966, 28 pp., 26 cm. Price $1.00 (paperbound).

These tables are an extension of Table 2 in D. J. Finney, R. Latscha, B. M. Bennett & P. Hsu, *Tables for Testing Significance in a* 2×2 *Contingency Table*, Cambridge University Press, 1963, previously reviewed here (*Math. Comp.*, v. 18, 1964, pp. 514–515). The notation is that of the earlier publication. The present table gives the 5% and 1% one-tail significant values, $b_{.05}$ and $b_{.01}$, for $41 \leqq A \leqq 50$, $B \leqq A$; the exact probabilities are not given.

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93[K].—F. ZABRANSKY, MASAAKI SIBUYA & A. K. MD. EHSANES SALEH, *Tables for Estimation of the Exponential Distribution by the Linear Combinations of the Optimal Subset of Order Statistics*, The University of Western Ontario, London, Canada, undated. 184 computer sheets. Copy deposited in UMT file.

If $Y(1) > \cdots > Y(N)$ is a decreasingly ordered sample from the exponential distribution

$$f(x; \sigma) = \sigma^{-1} \exp(-x/\sigma), \qquad x \geqq 0, \qquad \sigma \geqq 0,$$

then σ can be estimated by linear forms

$$\sigma = B(1)Y(n_1) + \cdots + B(k)Y(n_k),$$

where

$$1 \leqq \nu + 1 \leqq n_1 < \cdots < n_k \leqq N,$$

which implies that ν upper observations are censored.

The first three tables give the ranks n_i, the coefficients $B(i)$, and the inverse of the minimized variance K^* of the minimum-variance unbiased estimator. These tables can be used also for lower and doubly censored samples and for the estimation of both σ and θ for the distribution $f(x - \theta; \sigma)$.

Table 1 (35 pp.) gives K^* and $B(i)$, each to 5D, for $N = 1(1)15$, $k = 1(1)10$, $N - \nu = k(1)N$.

Table 2 (90 pp.) gives 5D values of K^* and $B(i)$ for $N = 16(1)45$, $k = 1(1)10$, $(N - \nu)/N = 0.5(0.1)1.0$.

Table 3 (40 pp.) gives 4D values of K^* and $B(i)$ for $\nu = 0(1)9$, $k = 1(1)8$, $N = (k + \nu)(1)(k + \nu + 23)$.

Table 4 and 5 relate to the asymptotic case ($N \to \infty$). For given k and $p = \nu/N$, they give $p_i = n_i/N$, $B(i)$, and $V(p, k)$, which is determined by the relation: the minimum variance $= N^{-1}V(p, k) + O(N^{-2})$.

Specifically, Table 4 (16 pp.) gives p_i to 4D, $B(i)$ to 4D and $V(p, k)$ to 5D for $k = 1(1)8$, $p = 0.02(0.02)0.98$; while Table 5 (3 pp.) gives 8D values of p_i, $B(i)$, and $V(p, k)$ for $p = 0$ (uncensored) and $k = 1(1)15$.

The data in Tables 1, 2, 3, and 5 are accurately rounded; however, Table 4 contains errors attributable to the discrete approximation in the maximizing process. For $p > 0.8$ the tabulated values of p_i are accurate to 2D; for $0.7 < p \leqq 0.8$, p_i is accurate to 3D; and for smaller values of p, the maximum error in p_i is 5×10^{-4}.

The coefficients $B(i)$ are exact in the sense that they give the minimum-variance unbiased linear combination of the order statistics which are chosen according to the table.

The underlying theory, the algorithms used in the tabulation, and a list of previous publications may be found in a paper by Sibuya [1].

AUTHORS' SUMMARY

1. M. SIBUYA, "Maximization with respect to partition of an interval and its application to the best systematic estimators of the exponential distribution," *Ann. Math. Statist.* (To appear.)

94[L].—V. A. DITKIN, Editor, *Tablit͡sy Logarifmichesko͡ĭ Proizvadnoĭ Gamma-funkt͡sii i ee Proizvodnykh v Kompleksnoĭ Oblasti*, Akad. Nauk SSSR, Moscow, 1965, xiv + 363 pp., 27 cm. Price 3.15 roubles.

This volume contains two tables: the first, occupying 320 pages, consists of 7S decimal approximations to the real and imaginary parts of $\psi(x + iy)$, the logarithmic derivative of the gamma function, for $x = 1(0.01)2$ and $y = 0(0.01)4$; the second, occupying 40 pages, consists of 7S values (in floating-point form for positive exponent) of the real and imaginary parts of the derivatives $\psi^{(n)}(x + iy)$ for $n = 1(1)10$, $x = 1(0.1)2$, and $y = 0(0.1)4$. No tabular differences are provided; however, interpolation with second differences is explained and illustrated in the introduction with the aid of a nomogram.

The real and imaginary parts for any argument appear on facing pages, with six tabular columns of 51 lines each on a page, the last column being repeated as the first column on the following page. This format was adopted from that in the tables of Abramov [1] for $\ln \Gamma(x + iy)$, to which the present tables are related, as noted in the preface.

The numerical evaluation of the tabulated functions outside the range of the tabular arguments is discussed, and a number of relevant formulas and series are included.

This reviewer has compared the tabular values herein for $\psi(x + iy)$ when $x = 1(0.01)2$, $y = 0$ with the corresponding entries in the 10D tables of Davis [2], to which reference is made in the bibliographic list of 10 titles at the end of the introduction. It was thereby discovered that, with very few exceptions, there exists a consistent lack of conventional rounding-up of the final digit in the main table under review. On the other hand, this source of error was not observed in the values of the derivatives of $\psi(z)$ for real argument, which occupy the first line throughout the second table.

As a further check, this reviewer also compared the tabulated values of the real part of $\psi(1 + iy)$ for $y = 0(0.01)4$ with the corresponding 10D values in Table II of the NBS tables of Coulomb wave functions [3], and the same general lack of conventional rounding was again observed in the Russian table.

We are informed in the preface that these tables were computed on the electronic computer Strela. Presumably the observed rounding errors are attributable to a deficiency in the computer program. It is interesting to note that no other tabular discrepancies were observed.

The present attractively printed tables are by far the most extensive of their kind, and accordingly constitute an important accession to the growing store of mathematical tables. It is to be hoped that an emended edition eventually will be forthcoming.

J. W. W.

1. A. A. Abramov, *Tablitsy* ln Γ(z) *v kompleksnoĭ oblasti*, Izdat. Akad. Nauk SSSR, Moscow, 1953. (See *MTAC*, v. 12, 1958, pp. 150–151, RMT **70**.)

2. H. T. Davis, *Tables of the Higher Mathematical Functions*, Vols. 1, 2, Principia Press, Bloomington, Indiana, 1933 and 1935. Revised edition, entitled *Tables of the Mathematical Functions*, published by The Principia Press of Trinity University, San Antonio, Texas, 1963. (See *Math. Comp.*, v. 19, 1965, pp. 696–698, RMT **131**.)

3. NBS Applied Mathematics Series, No. 17: *Tables of Coulomb Wave Functions*, U. S. Government Printing Office, Washington, D. C., 1952. (See *MTAC*, v. 7, 1953, pp. 101–102, RMT **1091**.)

95[L].—Roddam Narasimha, *On the Incomplete Gamma-function with One Negative Argument*, Report AE 123A, Department of Aeronautical Engineering, Indian Institute of Science, Bangalore, India, 16 pp. + 2 figs., 29 cm. Copy deposited in UMT file.

Let $g(\alpha, x) = \alpha e^{-x} \int_0^1 t^{\alpha-1} e^{xt}\, dt$ and $G(\alpha, x) = -\alpha e^{x} \int_1^\infty t^{\alpha-1} e^{-xt}\, dt$; then this report presents 5D tables of $g(\alpha, x)$ and $G(\alpha, x)$, the first for $\alpha = 0(0.2)2(0.5)5$, $x = 0(0.1)2(0.25)3(0.5)5(1)10$, and the second for $-\alpha = 0(0.2)2(0.5)5$ and for x as above.

In an introduction the author discusses the properties of these functions and the procedures followed in the calculation of these tables on an IBM 7090 system. Methods for extending the range of the tables are also described.

The author alludes to the application of the incomplete gamma function to the solution of problems in statistics, radiative transfer, and the kinetic theory of gases. A list of nine references is appended to the introduction.

Additional information concerning these functions, including related tabular data, is presented in a treatise [1] by this reviewer and in the NBS *Handbook* [2].

Y. L. L.

1. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill Book Co., New York, 1962. (See *Math. Comp.*, v. 17, 1963, pp. 318–320, RMT **51**.)

2. M. Abramowitz & I. A. Stegun, Editors, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, D. C., 1964. (See *Math. Comp.*, v. 19, 1965, pp. 147–149, RMT **1**.)

96[L].—E. Wai-Kwok Ng, *Lommel Functions of Two Imaginary Arguments*, Department of Astronomy, Columbia University, New York, undated ms. of 13 pp., deposited in UMT file.

This manuscript contains tables to 6S in floating-point form of

$$Y_n(w, z) = \sum_{m=0}^{\infty} (w/z)^{n+2m} I_{n+2m}(z)$$

and

$$\Theta_n(w, z) = \sum_{m=0}^{\infty} (z/w)^{n+2m} I_{n+2m}(z),$$

where $I_n(z)$ is the modified Bessel function of the first kind of order n, for the following ranges: $w = 0.1(0.1)1$, $z = 0.1(0.1)1$ for Y_1, Y_2, Θ_0, Θ_1; $w = 1(1)z$, $z = 2(1)20$ for Y_1, Y_2; $w = 2(1)20$, $z = 1(1)w$ for Θ_0, Θ_1.

Lommel's functions of two variables are usually represented by the symbols $U_n(w, z)$ and $V_n(w, z)$; these are related to the above functions by the formulas $Y_n(w, z) = i^{-n}U_n(iw, iz)$ and $\Theta_n(w, z) = i^{-n}V_n(iw, iz)$.

Tables of U_n and V_n have been calculated by Dekanosidze [1] and Boersma [2].

Y. L. L.

1. E. N. Dekanosidze, *Tablit͡sy t͡silindricheskikh funkt͡siĭ ot dvukh peremennykh* (*Tables of cylinder functions*), Acad. Sci. USSR, Moscow, 1956. (See *MTAC*, v. 12, 1958, pp. 239–240, RMT **107**.) English translation published by Pergamon Press, New York, 1960. (See *Math. Comp.*, v. 16, 1962, p. 383, RMT **36**.)

2. J. Boersma, "On the computation of Lommel's functions of two variables," *Math. Comp.*, v. 16, 1962, pp. 232–238.

97[L, M].—Rory Thompson, *Table of* $I_n(b) = (2/\pi)\int_0^\infty ((\sin x)/x)^n \cos bx\, dx$, ms. of 26 computer sheets deposited in the UMT file.

The integral in the title is tabulated to 8D for $n = 3(1)100$, $b = 0(0.1)9$. Previous tables [1], [2] have been limited to the case $b = 0$. The method used in computing the present tables has been described by the author in [3].

In a marginal handwritten note the author notes 12 rounding errors detected by a comparison with the earlier tables, which extended to 10D. The presence of other rounding errors in this table is alluded to by the author; some of these are obvious among the early entries.

Apparently no attempt was made to edit the computer output constituting this table; for example, the fact that $I_n(b) = 0$ for $b \geqq n$ could have been used to reduce the number of entries shown for $n \leqq 8$. Furthermore, the obvious rounding errors referred to could have been removed in an improved copy.

Despite these flaws, this table is a valuable extension of the earlier, related tables.

A FORTRAN listing of the program used in the calculations is included.

J. W. W.

1. K. Harumi, S. Katsura & J. W. Wrench, Jr., "Values of $(2/\pi)\int_0^\infty ((\sin t)/t)^n\, dt$," *Math. Comp.*, v. 14, 1960, p. 379.

2. R. G. Medhurst & J. H. Roberts, "Evaluation of the integral $I_n(b) = (2/\pi)\int_0^\infty ((\sin x)/x)^n \cos (bx)\, dx$," *Math. Comp.*, v. 19, 1965, pp. 113–117.

3. Rory Thompson, "Evaluation of $I_n(b) = (2/\pi)\int_0^\infty ((\sin x)/x)^n \cos (bx)\, dx$ and of similar integrals", *Math. Comp.*, v. 20, 1966, pp. 330–332.

98[L, M].—Shigetoshi Katsura, Yuji Inoue, Seiji Hamashita & J. E. Kilpatrick, *Tables of Integrals of Threefold and Fourfold Products of Associated Legendre Functions*, The Technology Reports of the Tôhoku University, v. 30, 1965, pp. 93–164.

These extensive tables list the values, to accuracies varying from 11 to 15 significant figures, of the integrals of the product of three and four normalized associated Legendre functions. The integral of the product of three functions, $\Theta(l_1, m_1) \cdot \Theta(l_2, m_2)\Theta(l_3, m_3)$, is computed for all values of the l's from 0 to 8, and all permitted values of the m's satisfying the condition $m_1 + m_2 + m_3 = 0$, which is the condition for the nonvanishing of the integral of the product of three spherical harmonics. Similarly, the integrals of the product of four functions are tabulated for all values of the l's from 0 to 4 and for m's satisfying the condition $m_1 + m_2 + m_3 + m_4 = 0$.

The computations were carried out on a large-scale digital computer. It is stated that the Gaunt formula, which gives the integral of the threefold product in closed form, was not suitable for programming. Rather, the integrals were obtained in the following straightforward manner: The Legendre polynomials $P_l(x)$ are generated by means of the recursion formula. The associated polynomials $P_l^m(x)$ are obtained by m-fold differentiation. Then three or four of these polynomials, $(1 - x^2)$ to the appropriate power, and the normalizing factors are multiplied to obtain the integrand; the indefinite integral is taken and the limits $x = -1$ and $+1$ are substituted.

To the reviewer's knowledge, these are the first extensive tables of their kind. Previous computations involving the integral of three associated Legendre functions have tabulated either the Clebsch-Gordan coefficients or the $3j$-symbols, which are these integrals multiplied by certain other factors.

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99[L, M].—HENRY E. FETTIS & JAMES C. CASLIN, *Elliptic Integral of the First Kind* and *Elliptic Integral of the Second Kind*, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, February 1966. Two mss., each of 36 computer sheets, deposited in the UMT File.

These companion tables consist of 10D values (without differences) of the elliptic integrals of the first and second kinds, respectively, in Legendre's form, namely, $F(\theta, k)$ and $E(\theta, k)$, for $\theta = 0°(1°)90°$ and arcsin $k = 0°(1°)90°$.

The authors have informed this reviewer that essentially the same subroutine was used in computing these tables on an IBM 1620 with 16-digit arithmetic as was employed in the computation of their published tables [1] of elliptic integrals, wherein the modulus serves as one argument, rather than the modular angle as in the present tables.

It is interesting to note that these new tables in range and precision resemble rather closely the celebrated 9–10D tables of Legendre [2], [3], which may now be considered superseded after more than a century.

J. W. W.

1. HENRY E. FETTIS & JAMES C. CASLIN, *Tables of Elliptic Integrals of the First, Second and Third Kind*, Report ARL 64-232, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, December 1964. (See *Math. Comp.*, v. 19, 1965, p. 509, RMT **81**.)
2. A. M. LEGENDRE, *Exercises de Calcul Intégral*, v. 3, Paris, 1816.
3. A. M. LEGENDRE, *Traité des Fonctions Elliptiques et des Intégrales Eulériennes*, v. 2, Paris, 1826. A facsimile reproduction of Tables II and VIII therein appears in K. PEARSON, *Tables of the Complete and Incomplete Elliptic Integrals*, reissued from Tome II of Legendre's *Traité des Fonctions Elliptiques*, London, 1934. (See also ALAN FLETCHER, "Guide to tables of elliptic functions," *MTAC*, v. 3, 1948, pp. 229–281.)

100[L, M].—HENRY E. FETTIS & JAMES C. CASLIN, *Table of the Jacobian Zeta Function*, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio. Ms. of 36 computer sheets deposited in the UMT File.

Using the notation of Byrd & Friedman [1], the authors tabulate 10D values of the product $K(k)Z(\beta, k)$ of the complete elliptic integral of the first kind and the Jacobian zeta function, for $\beta = 0°(1°)90°$ and $\arcsin k = 1°(1°)89°$. No provision is made for interpolation; indeed, interpolation to the full precision of the table is not generally feasible because of the large number of successive differences that would be required in both arguments.

By means of the well-known relation $K(k)Z(\beta, k) = K(k)E(\beta, k) - E(k)F(\beta, k)$, the tabular entries before rounding to 10D were derived from values of both complete and incomplete elliptic integrals of the first and second kinds that were initially calculated to about 16S by use of double-precision arithmetic. (See the preceding review.)

Errors detected in the corresponding 6D table in [1], as the result of the present authors' comparison thereof with their table, are listed separately in this issue.

J. W. W.

1. P. F. BYRD & M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, Berlin, 1954.

101[L, M].—HENRY E. FETTIS & JAMES C. CASLIN, *Heuman Lambda Function*, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio. Ms. of 36 computer sheets deposited in UMT File.

The Heuman lambda function, usually designated $\Lambda_0(\alpha, \beta)$, is the product of $2/\pi$ and the complete elliptic integral of the third kind, namely

$$\int_0^{\pi/2} \frac{1}{1 - p \sin^2 \phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

in the circular case, $k^2 < p < 1$. The variable α is the modular angle (so that $k = \sin \alpha$) and the variable β is defined implicitly by the relation

$$p = \sin^2 \alpha/(\sin^2 \alpha + \cos^2 \alpha \cos^2 \beta).$$

The authors adopt the notation $\Lambda_0(\theta, k)$ in the present 10D table; thus, the range of parameters can be expressed as $\theta = 0°(1°)90°$ and $\arcsin k = 0°(1°)90°$. No differences are provided.

These tabulated values of the lambda function were derived from the computer data underlying the authors' 10D manuscript tables of the elliptic integrals of the first and second kinds by means of the relation

$$\Lambda_0(\theta, k) = (2/\pi)\{E(k)\cdot F(\theta, k') + K(k)\cdot[E(\theta, k') - F(\theta, k')]\}$$

where k' represents the complementary modulus.

This table may be considered a valuable extension of the original 6D table of Heuman [1], which has been abridged in Byrd & Friedman [2].

J. W. W.

1. C. HEUMAN, "Tables of complete elliptic integrals," *J. Math. and Phys.*, v. 20, 1941, pp. 127–206, 336.
2. P. F. BYRD & M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954, p. 344.

102[L, S].—OTTO EMERSLEBEN, "Über lineare Beziehungen zwischen Madelung-konstanten", *Izvestii͡a na Matematicheskii͡a Institut B''lgarska Akademii͡a na Naukite*, Sofia, v. 2, 1957, pp. 87–120.

The electrostatic energy of a crystal structure, assumed to consist of positive and negative point charges $\pm e$, is proportional to e^2/a, where a is a characteristic length, e.g., for cubic crystals the edge of the smallest cube of periodicity (the "cell"), or the smallest distance between charges. In the expression of this energy, referred to the cell or to a neutral group of charges, the "molecule", the factor of e^2/a is called the Madelung Constant (here denoted by ψ). Its value is given by a lattice sum over the reciprocal distances of atoms in the structure; it depends on the structure, and, in a trivial way, on the normalizations mentioned above.

Beginning with his thesis with M. Born (1922) the author has used a specialized brand of Epstein's zeta functions for expressing the constant, namely

$$Z \left| \begin{matrix} 000 \\ xyz \end{matrix} \right| (1)_3 = \Pi(x, y, z),$$

a function which Born interpreted as the "Grundpotential." Thus

$$\psi(\text{NaCl}) = \frac{e^2}{a} \Pi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

and

$$\psi(\text{CsCl}) = \frac{e^2}{a} [\tfrac{3}{4}\Pi(0, 0, \tfrac{1}{2}) + \tfrac{1}{4}\Pi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})], \quad \text{etc.}$$

Since, in a number of simple cubic structures the atomic positions are fixed by the intersections of symmetry elements, only multiples of $a/8$ occur as the atomic coordinates, and only a small number of arguments of Π occur in the expressions of ψ for these structures. This leads to linear relations between the ψ, as has been discussed in an exhaustive way by Pinhas Naor in *Zs. f. Krist.*, v. 110, 1958, pp. 112–126. In the present paper the author deduces the relations

$$2\psi(\text{ZnS}) = 2\psi(\text{CaF}_2) + \psi(\text{NaCl}) = 2[2\psi(\text{CsCl}) + \psi(\text{NaCl})]$$

and

$$2\psi(\text{CaF}_2) = 4\psi(\text{CsCl}) + \psi(\text{NaCl}) = \psi(\text{ZnS}) + 2\psi(\text{CsCl}).$$

Numerical calculations have previously been carried out by Emersleben for ψ(NaCl) to 15 decimals, for a value of order 1; now ψ(CsCl) is added to the same accuracy. The calculation is done in two parts in the way first used by this reviewer. The exponentials were first taken from the tables of Hayashi [1] and later from 17D tables of exp $(-n\pi/20)$, $0 \leqq n \leqq 200$, prepared in the Department of Applied Mathematics of the University of Greifswald. The error function was taken from the author's table in [2].

In Table 1 the Madelung constants for NaCl, CsCl, ZnS, CaF_2 are listed to better than 12 decimals with different normalizations.

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1. K. Hayashi, *Tafeln der Besselschen, Theta-, Kugel-, und anderer Funktionen*, Springer, Berlin, 1930.

2. O. Emersleben, "Numerische Werte des Fehlerintegrals für $\sqrt{n\pi}$," *Z. Angew. Math. Mech.*, v. 31, 1951, pp. 393–394.

Editorial note: For reviews of earlier, related papers by the same author, see *MTAC*, v. 5, 1951, pp. 77–78, RMT **871**; v. 11, 1957, pp. 109–110, RMT **56**; *ibid.*, p. 113–114, RMT **65**.

103[L, X].—Lucy Joan Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, New York, 1966, xiii + 273 pp., 24 cm. Price $13.50.

The author of this valuable book remarks in the preface that the volume should really be attributed to both the late W. N. Bailey and Miss Slater. It was Professor Bailey's intention to write a comprehensive book on hypergeometric functions with the assistance of Miss Slater. The present work is dedicated to the memory of W. N. Bailey and is based in part on notes for a series of lectures which he gave during the years 1947–1950.

The ordinary hypergeometric or Gauss series is

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}, \qquad |z| < 1, \qquad (a)_k = \Gamma(a+k)/\Gamma(a),$$

and is usually represented by the symbol ${}_2F_1(a, b, c; z)$. A natural generalization is a series like the above but with an arbitrary number of numerator and denominator parameters. Thus (formally at least)

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} .$$

The latter is called a generalized Gauss function or hypergeometric function, and where no confusion can result, is simply called a ${}_pF_q$. If in the above ${}_pF_q$, a numerator parameter is set to unity and the summation index sums from $-\infty$ to ∞, then such a series is called a bilateral series. It obviously can be expressed as a combination of two generalized hypergeometric functions.

The ${}_pF_q$ may be generalized by considering an obvious generalization of the double series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_m}{(c)_{m+n} m! n!} x^m y^m$$

to an arbitrary number of numerator and denominator parameters. These are known as Appell series or functions. We can also have triple, quadruple, or multiple sums which are known as Lauricella functions.

The theory of the ${}_pF_q$ is fundamental in the applications since many of the special functions (Bessel functions, Legendre functions, etc.) are special cases. Appell and Lauricella functions also appear in the applications.

A different view of the problem of generalizing the Gauss function was adopted by E. Heine. He defined a basic number as $a_q \equiv (1 - q^a)/(1 - q)$. Observe that $a_q \rightarrow a$ as $q \rightarrow 1$. Heine defined the basic analogue of the Gauss series as

$$ {}_2\Phi_1(a, b; c; q, z) = 1 + \frac{(1 - q^a)(1 - q^b)z}{(1 - q^c)(1 - q)} + \frac{(1 - q^a)(1 - q^{a+1})(1 - q^b)(1 - q^{b+1})z^2}{(1 - q^c)(1 - q^{c+1})(1 - q)(1 - q^2)} + \cdots, $$

$|q| < 1$, so when $q \rightarrow 1$, the latter approaches ${}_2F_1$. The Heine series and its natural extension to a similar type series with an arbitrary number of numerator parameters and denominator parameters are basic hypergeometric series. We also have basic bilateral series. Most of the applications of the q concept have occurred in the field of pure mathematics, particularly in number theory, modular equations and elliptic integrals.

About two-thirds of the volume is devoted to the Gauss function and its generalizations as previously noted, and the remainder to basic hypergeometric series. A survey of the contents follows. Chapter 1 treats the Gauss function. A historical development of the function is traced. Some of the topics discussed are Kummer's 24 solutions, integral representations and analytic continuation.

The generalized Gauss function is taken up in Chapter 2. Here there is considerable material on the ${}_3F_2$ of unit argument and other special summation formulas. Products of hypergeometric series are also considered. A generalization of the ${}_pF_q$ known as Meijer's G-function is referenced but not discussed.

Barnes-type integral representations for the ${}_pF_q$ and other contour integrals for the ${}_pF_q$ are studied in Chapter 4. Integral transforms of the ${}_pF_q$ such as those of Euler and Mellin are also presented. Chapter 6 deals with bilateral series. Appell series and Lauricella functions are analyzed in Chapter 8.

Basic hypergeometric functions, basic hypergeometric integrals, basic bilateral series and basic Appell series are the subjects of Chapters 3, 5, 7 and 9, respectively. There are five appendices. The first gives properties of the gamma function expressed as a Pochammer symbol. Properties of the analog of this which appear in basic series are delineated in Appendix 2. Appendix 3 contains a list of formulas where the ordinary hypergeometric series for special arguments can be expressed as the product of gamma functions. These are called summation theorems. Summation theorems for basic series are listed in Appendix 4. Appendix 5 gives a table of $[\prod_{n=0}^{\infty} (1 - aq^n)^{-1}]$ for $q = 0(0.05)1.0$, $a = -0.9(0.05)0.9$ mostly to 9S. There is also a table of the above function for $a = 1$ and $q = 0(0.005)0.890$, mostly to 9S.

The bibliography is excellent, though nearly all the entries are dated before 1956. An index of symbols and a general index are provided.

Our casual reading has revealed several typographical errors. In formula (1.1.1.3) on p. 3, $(a)_n$ should be replaced by $(a)_n/n^a\Gamma(n)$. On the same page in (1.1.8), ∞ in the integral limits should read $i\infty$. On p. 17, the right-hand sides of (1.5.6) and (1.5.8) should read $(\sin az)/a \sin z$ and $(\sinh az)/a \sinh z$, respectively. These and possibly other imperfections aside, the volume contains a wealth of information and should be most useful to pure and applied workers.

Y. L. L.

104[L, X].—G. E. ROBERTS & H. KAUFMAN, *Table of Laplace Transforms*, W. B. Saunders Company, Philadelphia, Pennsylvania, 1966, xxx + 367 pp., 27 cm. Price $6.75.

This is a rather comprehensive reference of Laplace transforms and their inverses and should prove useful to pure and applied workers. The volume is in two parts. The first concerns direct transforms and the second inverse transforms. In the presentation of a large list of transform pairs (there are over 3,100), it is of utmost importance to arrange the material so that a result (if given) can be located quickly. Obviously no list is complete, since a table of integral transforms is by its nature infinite in character. To provide quick access, the authors have developed an index system which is not too much different from that used by Erdélyi et al., that is, A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, v. 1, v. 2, McGraw-Hill Book Co., New York, 1954, see *MTAC*, v. 10, 1956, pp. 252–254. The volume under review gives the definitions of the most common special functions. Code numbers are assigned to various types of functions and it is this reference system which permits location of a desired transform. Thus, for example, code numbers 1, 2, 4, and 10 refer to algebraic rational functions, irrational algebraic functions, exponential functions e^x, a^x and Bessel functions $J_\nu(x)$, $Y_\nu(x)$, respectively. Thus the Laplace transforms of $J_\nu(at)$, $J_0[a(t^2 + bt)^{1/2}]$, and $(1 - e^{-t})^{\nu/2}J_\nu[a(1 - e^{-t})^{1/2}]$ are found in Sections 10, 10.2 and 10.3, respectively.

Since the date of the Erdélyi et al. volumes referred to, numerous papers and some books have appeared in the literature on transforms and related topics. However, only six of the eleven items given in the present volume date beyond 1954. A notable omission is the absence of Meijer's G-function, though MacRobert's E-function is listed in the function definitions.

In a work of this kind, it is inevitable that typographical errors should occur. A casual reading has revealed several blemishes. On p. xxi, there is given $\int_x^\infty u^{-1}I_\nu(u)\,du$ where $I_\nu(u)$ is the modified Bessel function. This integral does not exist unless, of course, ∞ is interpreted as $i\infty$. See also p. 104 formulas (6), (10). In the list of function definitions, there is given the so-called Schlömilch function, which is better known as the incomplete gamma function. On p. 115, for formulas (3) and (4) and p. 220 for formulas 125, 126, the condition given for the validity of an integral reads $R(s) > 0$, whereas it should read $R(s) > 2^{-2/3}$. Actually, the result is valid for $|\arg(s - 2^{-2/3})| < \pi$.

A natural question is whether the present compendium is more detailed than that of Erdélyi et al. I do not think so. It must be remembered that the latter reference gives, in addition to Laplace and inverse Laplace transforms, Fourier sine and cosine transforms which are essentially Laplace transforms, and exponential Fourier transforms which are in a sense two-sided Laplace transforms. They also give Mellin and inverse Mellin transforms which are virtually two-sided Laplace transforms. We do not find in the volume under review references to the two-sided Laplace transforms. Further, the Erdélyi et al. volumes give Hankel, J, Y, K and H transforms and, of course, many of these are Laplace transforms.

In any event, we find the present volume very usable. Certainly for a tome of this size, the price of $6.75 is quite reasonable.

Y. L. L.

105[P, X].—SOLOMON LEFSCHETZ, *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965, xi + 150 pp. Price $7.50.

This monograph gives an up-to-date, essentially self-contained, treatment of the problem of absolute stability of closed-loop control systems. It discusses in detail the sufficient conditions based on the construction of suitable Lyapunov functions as well as Popov's sufficient conditions utilizing Fourier-transform techniques. It is the first such monograph written outside the Soviet Union, by an author who has been for more than two decades the main force in making Lyapunov's approach to stability theory widely known and used in this country.

The book consists of nine chapters and two appendices. The first chapter is introductory, the last contains supplements on Jordan canonical forms, Lyapunov's matrix equation and the basic stability theorems. The construction of a Lyapunov function is discussed in Chapters 2, 3 for the case of indirect controls, in Chapter 4 for direct controls and in Chapter 5 for systems represented by a single equation. Discontinuous characteristics are briefly taken up in Chapter 6. The theorems of Popov are stated and proved in Chapter 7, and compared with the preceding results. Chapter 8 concerns essentially a strengthened version of a lemma of Yacubovich from which a somewhat less general necessary and sufficient condition than Kalman's is deduced very simply. The appendices give an application of multiple feedback control and an example from the theory of nuclear power reactors. The book closes with a bibliography of the most important papers and texts on the subject.

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106[P, X].—C. T. LEONDES, Editor, *Advances in Control Systems*, Vol. 2, Academic Press, New York, 1965, x + 313 pp., 24 cm. Price $13.00.

The present volume consists of five contributions.

The first article, "The Generation of Lyapunov Functions" by D. G. Schultz summarizes the methods of constructing Lyapunov functions for autonomous systems as proposed by Aizerman, Szegö, the author and Gibson. The usefulness and relative merits of these methods are discussed in detail and illustrated by numerous examples. A brief discussion of generating Lyapunov functions for nonautonomous systems is also included.

The paper by F. T. Smith, entitled "The Application of Dynamic Programming to Satellite Intercept and Rendezvous Problems," discusses the use of dynamic programming in synthesizing the optimal control and optimal estimation problem. A wealth of numerical data is presented for comparison purposes.

The article "Synthesis of Adaptive Control Systems by Function Space Methods" by the late H. C. Hsieh describes various control problems such as final value and minimum effort problems in the setting of functional analysis. It includes a general discussion of the minimization problem in Hilbert space, the steepest-descent method and its variants, and the least square estimation problem.

The paper "Singular Solutions in Problems of Optimal Control" by C. D. Johnson is essentially concerned with the solution of two-point boundary value problems for systems of ordinary differential equations containing a discontinuity of relay type, as they arise in control theory. A good number of examples is discussed, illustrating the construction of singular trajectories.

The final article "Several Applications of the Direct Method of Liapunov" by R. A. Nesbit describes in terms of Lyapunov functions and linear bounds various sufficient conditions for the stability of an equilibrium point of a nonlinear system of ordinary differential equations. Numerical examples are presented to illustrate the usefulness of these bounds.

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107[S, V, X, Z].—Lewis F. Richardson, *Weather Prediction by Numerical Process*, Dover Publications, Inc., New York, 1965, xvi + 236 pp., 24 cm. Price $2.00.

The author is the English scientist (1881–1953) known most widely for his iterative finite-difference method of solving elliptic differential equations and for the criterion ("Richardson number") concerning the onset of turbulence in stratified shear flow. This remarkable book, published originally by Cambridge University Press in 1922, describes in detail a visionary plan for the numerical forecasting of weather. The numerical process is the solution of an initial-value problem by finite-differences, the initial data for which are to come from an international meteorological network. Richardson also works out a sample test case, calculating the value, at $t = 0$, of the time rates of change of wind, pressure, and temperature at a limited grid network in Europe. The results differ greatly from the observed values (especially that for surface pressure), and he concludes that the initial data then available are too inaccurate.

Although the book made considerable impression when first published, it then appears to have been almost completely ignored until the late 1940's, when J. Charney and J. von Neumann began the modern era of numerical weather prediction at Princeton. (None of the books on dynamic meteorology published before 1948 discuss Richardson's book, and only two even mention its existence.) *Weather Prediction by Numerical Process* was thus 25 years ahead of time in 1922. The technical developments of electronic computers and radiosondes, and the theoretical developments of atmospheric hydrodynamics and the concept of computational stability, were all necessary before Richardson's basic idea could be put into successful operation. Although the book's attraction today is primarily one of historical interest, it still makes stimulating reading for meteorologists and, I should think, applied mathematicians. Its reissue now in an inexpensive form is therefore very welcome. Sydney Chapman has written an *Introduction* to this edition which conveys much of the dedication and passion which seem to have characterized Richardson.

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108[X].—Wiktor Eckhaus, *Studies in Non-Linear Stability Theory*, Springer-Verlag, New York, 1965, viii + 117 pp., 24 cm. Price $5.50.

This monograph is concerned with the problem of stability of solutions of nonlinear partial differential equations of the form $u_t = L(u) + F(u)$, with boundary conditions, where $L(u)$ and $F(u)$ are linear and nonlinear operators, respectively, involving u and the derivatives of u with respect to the space variables, and the order of the derivatives in F is lower than of those in L. Both cases, of one and two space variables, are discussed. The basic idea throughout is to illustrate how expansions in series of eigenfunctions (the eigenfunctions being obtained from the linear problem) with time-varying coefficients can be used to discuss stability (or instability) and oscillatory phenomena in the solution of such boundary-value problems, as well as to obtain asymptotic forms for the solutions. The discussion centers around the case where the instability in the linear equation is determined by only one eigenvalue. There are also applications of the results to some particular problems that arise in fluid dynamics and a comparison of his results with those of other contributors. The author has certainly systematized this method to a point where in principle it can be applied to many problems, especially the scaling techniques necessary to determine relative magnitudes of the Fourier coefficients.

The book is fairly easy to read and should serve as a guide to further research in this area.

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109[X].—Richard Courant & Fritz John, *Introduction to Calculus and Analysis*, Vol. I, John Wiley & Sons, Inc., New York, 1965, xxiii + 661 pp., 24 cm. Price $10.50.

We briefly report on this masterfully written volume on "calculus" for functions of a single variable, because of its treatment of selected topics in the field of numerical methods. The following subjects are presented maturely and in a clear, mathematically sound and practically useful manner:

i. interpolation by polynomials (including the case of coincident points of interpolation),
ii. approximation by algebraic and trigonometric polynomials,
iii. computation of integrals (Simpson's rule),
iv. calculus of errors,
v. solution of simultaneous (nonlinear) equations (Newton's method, false position method, iteration method),
vi. Bernoulli polynomials (numbers),
vii. Euler-Maclaurin summation formula.

Only topics iii–v are treated in the short separate chapter on numerical methods. The rest are interwoven through the main body of the book. Many other useful numerical methods are developed throughout the text, for example, the summation of series, and the evaluation of asymptotic series (not formally introduced).

I quote from the preface: "(The intention is) to lead the student directly to the heart of the subject and to prepare him for active application of his knowledge. It avoids the dogmatic style which conceals the motivation and the roots of the calculus in intuitive reality. To exhibit the interaction between mathematical analysis and its various applications and to emphasize the role of intuition remains an important aim of this new book. Somewhat strengthened precision does not, as we hope, interefere with this aim."

The authors succeed admirably in achieving this aim and we look forward eagerly to reading the second volume on the calculus for functions of several variables.

E. I.

110[X, Z].—J. A. ZONNEVELD, *Automatic Numerical Integration*, Mathematical Centre Tracts 8, Mathematisch Centrum Amsterdam, 1964, 110 pp., 24 cm.

This tract is concerned with the automatic integration of systems of ordinary differential equations with initial conditions. First order and second order equations are considered, including second order with first derivatives appearing, and without.

Equations which must be satisfied by the parameters in Runge-Kutta formulas are developed in a standard way, and formulas are obtained for all orders up to and including fifth order. Additional equations are developed for parameters which can be used to determine the accuracy of the method, and this leads to formulas for approximating the last term retained in the Taylor expansion of the true solution. (The increment in the Runge-Kutta formula approximates the *sum* of a certain number of terms in this expansion. The new formula approximates the last of these terms, and can be used to keep the error below a prescribed tolerance.)

The formula for the last term can be evaluated only at the cost of a slight increase in the number of function evaluations per step.

Formulas are given in each case for differential equations in which the independent variable appears explicitly, and also for equations in which it does not appear.

There is an interesting chapter on the choice of step-size, and on changing the variable of integration, including the use of the arc length for this variable.

Nine ALGOL 60 procedures are given, some for first order and some for second order equations. Two of them change the variable of integration automatically, and one uses the arc length.

Five numerical examples are presented to illustrate various possibilities. Two involve van der Pol's equation, one consists of 15 second order equations, and another contains a singularity. One is used to show how a "virtually foolproof" strategy can fail in special circumstances.

A bibliography of 24 items is included.

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111[Z].—S. H. HOLLINGDALE & G. C. TOOTILL, *Electronic Computers*, Penguin Books, Inc., Baltimore, Maryland, 1965, 335 pp., 19 cm. Price $1.65 (paperbound).

This delightful little book is within reach of everyone, both in price and in content, although some thought and patience will be required of the layman to realize the full rewards of a careful reading. It is an honest and apparently successful attempt at popularization of the "black arts" of computers.

Because the book was written in 1963 and 1964 the latest fashions in computing now sweeping the field, namely, time-sharing and its corollaries, are only mentioned in passing. We can hope for an early revision to bring the laity up to date on "personal computing-1966" as this book already promises to do for computing and the basics of programming with respect to the older techniques.

One of the strong points of this volume is its pleasant pedagogical approach. When words alone do not suffice, a concrete example is used "to fix ideas." These examples illustrate the points being made quite adequately without exhausting the intelligent reader's patience or endurance. The authors manage quite nicely to increase the reader's cultural background through the use of anecdotes and historical sidelights that go well with the lesson. This is often characteristic of gifted authors, perhaps a little more frequently with British writers than with others. One wonders why this is so.

For style and lucidity of exposition and for its skill in pleasing, useful communication, as well as its content, this book is highly recommended.

The plea of G. H. Stearman [1] for the improvement of technical writing, with which this review strongly concurs, would be unnecessary if more of our colleagues wrote like the present authors.

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1. G. H. STEARMAN, "Is switching theory mathematics or engineering?," *IEEE Trans. on Electronic Computers*, v. EC-15, 1966, p. 124.

112[Z].—SEYMOUR V. POLLACK, *A Guide to Fortran* IV, Columbia University Press, New York, 1965, 260 pp., 24 cm. Price $5.00.

The text is written in a clear and lucid manner and contains a glossary of terms (the lack of which sometimes occurs in the best of texts of this type). The problem exercises are interesting but rather limited in scope (many having evolved from medical applications). The illustrations and flow charts are clear and well coordinated with the text, and a complete index is included. However, there is no discussion of the use of disc storage appropriate to some machine configurations and the material is scanty on the use and advantages of binary tape, an important medium for handling large amounts of data conveniently. The most serious criticism of the book concerns the complete lack of material concerned with the basic numerical problems associated with computing hardware. For example, when discussing the arithmetic IF statement, no material is presented on the use of a tolerance when comparing two floating point numbers. This presentation of the Fortran IV language appears to be geared toward researchers in the life sciences.

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113[Z].—CHARLES PHILIP LECHT, *The Programmer's Fortran* II *and* IV, McGraw-Hill Book Co., New York, 1966, xx + 162 pp., 28 cm. Price $7.95.

This concise, compact book is a catalogue of the main features of and contrasts between Fortran II and Fortran IV. Other forms of Fortran such as that for the CDC 3600, for instance, are not included. It is characterized by a simple and uniform arrangement, clear type, sharp explanations, and freedom from extraneous material. Although an index would enhance its value as a reference, a very well organized table of contents serves almost the same purpose. Its compactness is achieved in part by relegating definitions to a glossary, with some words in the glossary referred to other definitions in the glossary. However, a great deal of the substance of the book is thus partially buried.

Although the author carefully points out that this book is intended for programmers and is not a "self-teaching device," it is not clear for whom the book is written. The omission or scanty coverage of a number of fine, technical details hardly supports the author's claim to a complete reference for programmers. For instance, in some systems, such as the Fortran II compiler for the IBM 7094 used until recently at New York University, the value of the index of a *DO* loop after a normal exit is equal to one plus the upper limit. In other systems it is equal to the upper limit. This author disposes of this point by stating that, upon a normal exit, the index of a *DO* loop is not available, which is not even correct. A number of other technical details, especially concerning formats and *DO* loops, are left unexplained. If it is indeed intended for programmers, these details are the only real essentials. One uses a reference to look up the obscure, not the elementary and obvious.

This book is probably most useful for a beginning student of Fortran, to be used with other texts and manuals, much as a student of French uses a bilingual dictionary.

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114[Z].—J. ARSAC, A. LENTIN, M. NIVAT & L. NOLIN, *Algol, Théorie et Pratique*, Gauthier-Villars, Paris, 1965, 204 pp., 27 cm. Price 45 francs.

The title of the book (Algol—Theory and Practice) shows clearly that it is directed both to those interested in the theoretical aspects of algorithmic languages, and to application programmers who mainly want a reference manual.

The beginning (Chapters 1 and 2) gives the basic definitions and concepts concerning the structure of programs and the "ways of thought" responsible for the formulation of the Algol language. Although the material is very abstract, the authors have made a valuable effort to make the reading attractive by giving numerous examples.

The following chapters give a thorough description of the Algol language including a special chapter for Boolean expressions.

Finally, a whole chapter is devoted to examples and at the end a summary of the Algol syntax is given. This will probably be very helpful to those who need a reference manual.

The book should appeal to many people of different interests. Those interested in mechanical languages will appreciate the systematic presentation of the material. Advanced programmers will be interested in the detailed description of the Algol system and in the examples given. However, this is not an elementary book; far from that, beginning programmers should stay away from it lest they be confused by an abstract description which may at times obscure the relative simplicity of a concept. If an English version were available, this book could be an excellent textbook for an advanced programming course.

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