and the corresponding value y_0,

$$(22) \qquad\qquad F_p(y) = F_p(y_0) - I(x_0) + I(x)$$

is the final integral.

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Doppler Broadening Integrals*

By Van E. Wood, R. P. Kenan and M. L. Glasser

Asymptotic expansions in terms of Chebyshev polynomials of the integrals

$$I_n(x, v) = v^{1/2} \int_{-\infty}^{\infty} y^n (1 + y^2)^{-1} \exp\left[-v(y - x)^2\right] dy, \qquad n = 0, 1, \cdots$$

have recently been given by Thompson [1]. In this note, we wish to point out that these integrals can be expressed in terms of tabulated (and readily calculable) functions for any values of the parameters, a fact which is not mentioned in the literature we have seen on this subject. Specifically, we find

$$I_0(x, v) = \pi v^{1/2} \, \mathrm{Re} \left[\exp\left[v(1 + ix)^2\right] \mathrm{erfc}\,(v^{1/2}(1 + ix))\right],$$

$$I_1(x, v) = \pi v^{1/2} \, \mathrm{Im} \left[\exp\left[v(1 + ix)^2\right] \mathrm{erfc}\,(v^{1/2}(1 + ix))\right].$$

These results may be obtained by introducing the representation $(1 + y^2)^{-1} = \int_0^{\infty} e^{-z} \cos yz \, dz$, or by solving the coupled differential equations $dI_0/dx = 2v(I_1 - xI_0)$, $dI_1/dx = 2v(\pi^{1/2} - I_0 - xI_1)$, or by simply making appropriate changes of variable in some tabulated integrals [2]. Now a method for calculating error functions of complex argument using a rapidly converging infinite series has been described by Salzer [3]; hence the integrals can be obtained easily without using the asymptotic expansion.

By introducing the well-known asymptotic expansion for the co-error function, one may obtain in a very simple way the asymptotic expansions in Chebyshev polynomials

$$I_0 \sim \pi^{1/2} \sum_{m=0}^{\infty} (2m - 1)!!(-2v)^m (1 + x^2)^{-m-1/2} T_{2m+1}((1 + x^2)^{-1/2}),$$

$$I_1 \sim \pi^{1/2} \sum_{m=0}^{\infty} (2m - 1)!!(+2v)^m (1 + x^2)^{-m-1/2} T_{2m+1}(x/(1 + x^2)^{1/2}).$$

where

$$(2m - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2m - 1), \qquad\qquad m > 0,$$

$$= 1 \qquad\qquad m = 0.$$

These expansions agree with those of Thompson [1] except that the factor of $\pi^{1/2}$ has been inadvertently omitted in his paper and there is a sign error in his expansion for I_1. (Apparently the author intended to include a factor $\pi^{-1/2}$ in defining the integrals, as the reciprocal factor is otherwise necessary in all the other expressions for the integrals in the paper. There are also other minor misprints in Eq. (2), the equation preceding it, and in the first, second, third, and fifth equations following Eq. (1).)

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Singular and Invariant Matrices Under the QR Transformation*

By Beresford Parlett

0. Introduction. The above two classes of matrix are usually ignored in discussions of the QR algorithm [1], [3], [4]. Some familiarity with the algorithm is assumed.

There are good reasons for this neglect. Firstly the algorithm is not well defined for singular matrices and thus its behavior is difficult to describe. Secondly the problem of describing all matrices which are left invariant under the QR transformation is in general rather difficult.

The purpose of this note is to point out that with a preliminary reduction to Hessenberg form both difficulties disappear. We show first that singular matrices reveal their zero eigenvalues, one per iteration. Secondly we describe all matrices which are invariant.

A given square matrix A (real or complex) may be reduced to Hessenberg form A_1 ($a_{ij} = 0$, $i > j + 1$) in a variety of ways. If any subdiagonal elements $a_{i+1,i}$ vanish then A_1 is said to be reduced and may be partitioned appropriately as

(1)
$$A_1 = \begin{pmatrix} H_1 & H_{12} & \cdot & H_{1m} \\ 0 & H_2 & \cdot & H_{2m} \\ \cdot & & \cdot & \cdot \\ & & & H_m \end{pmatrix}$$