Rational Chebyshev Approximations for Fermi-Dirac Integrals of Orders $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$*

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Abstract. Rational Chebyshev approximations are given for the complete Fermi-Dirac integrals of orders $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$. Maximal relative errors vary with the function and interval considered, but generally range down to 10^{-9} or less.

1. Introduction. The complete Fermi-Dirac integrals are usually defined by

$$F_k(x) = \int_0^\infty \frac{t^k\,dt}{e^{t-x}+1}, \qquad k > -1, \tag{1}$$

although Dingle [4] prefers the definition

$$\mathfrak{F}_k(x) = (k!)^{-1} \int_0^\infty \frac{t^k\,dt}{e^{t-x}+1} \tag{2}$$

which places no restriction on k. We will use definition (1) but will employ some formulas, suitably modified, derived by Dingle.

These integrals appear in a variety of applications subject to Fermi-Dirac statistics, for example in the theory of semiconductors. The most frequently used functions are those for which k is either an integer or a half-integer. Function values are quite difficult to compute for k a half-integer and x positive. Consequently a number of useful tables have been published over the last 30 years (e.g., [1], [2], [4], [9]). Recently Werner and Raymann [12] used interpolation in the McDougall and Stoner [9] table to generate a compatible pair of Chebyshev approximations for the case $k = \frac{1}{2}$. Their work allows easy computation of $F_{1/2}(x)$ with a maximal relative error less than 5×10^{-4}. The present work presents portions of the arrays, termed by Rice [11] the L_∞ Walsh arrays, of rational Chebyshev approximations for $k = -\frac{1}{2}$, $\frac{1}{2}$, and $\frac{3}{2}$. Maximal errors range down to 10^{-9} or less.

2. Functional Discussion. The well-known expansion [4], [9]

$$F_k(x) = k! \sum_{r=1}^{\infty} (-1)^{r-1} \frac{e^{rx}}{r^{k+1}} \tag{3}$$

is convergent for $k > -1$ and $x < 0$. The Taylor series

$$F_k(x) = k! \sum_{r=0}^{\infty} \frac{x^r(1-2^{r-k})\zeta(k+1-r)}{r!} \tag{4}$$

where ζ is the Riemann zeta-function, is convergent for $k > -1$ and $|x| < \pi$.

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For positive x, Dingle [4] has shown

$$
(5)\qquad \begin{aligned} \mathfrak{F}_k(x) &= \cos \pi k \mathfrak{F}_k(-x) + 2 \sum_{r=0}^{[(k+1)/2]} \frac{t_{2r} x^{k+1-2r}}{(k+1-2r)!} \\ &+ \frac{2 \sin k\pi}{\pi} \sum_{r=[(k+3)/2]}^{\infty} \frac{t_{2r}(2r-k-2)!}{x^{2r-k-1}}, \end{aligned}
$$

where $[x]$ denotes the integer part of x, and

$$
t_{2r} = \tfrac{1}{2}(2\pi)^{2r}(1 - 2^{1-2r})| B_{2r} |/(2r)!
$$

where the B_r are the Bernoulli numbers. This expansion is finite, hence exact, for k an integer. However, for k half an odd integer the expansion is only asymptotic, equivalent to the well-known Sommerfeld representation [9]

$$
(6)\qquad F_k(x) = \frac{x^{k+1}}{k+1}\left\{1 + \sum_{r=1}^{n} a_{2r} x^{-2r}\right\} + R_{2n}
$$

where

$$
(7)\qquad a_{2r} = \frac{(1-2^{1-2r})(k+1)!(2\pi)^{2r}}{(k+1-2r)!(2r)!} | B_{2r} |,
$$

and R_{2n} is a remainder term. Dingle [4], [5], [6] has transformed (5) into a convergent representation by replacing the final sum with

$$
(8)\qquad \sum_{r=[(k+3)/2]}^{n} \frac{t_{2r}(2r-k-2)!}{x^{2r-k-1}} + \frac{(2n-k)!}{x^{2n-k+1}} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^{2n+2}} A_{2n-k}(jx)
$$

where

$$
(9)\qquad A_s(x) = \frac{-\pi x^{s+1}}{2(s!) \sin \pi s}(e^x - e^{-x} \cos \pi s) - \sum_{m=1}^{\infty} \frac{(s-2m)!}{s!} x^{2m}.
$$

In the Sommerfeld form,

$$
(10)\qquad R_{2n} = \frac{x^{k+1}}{k+1} \frac{2}{\pi} \frac{\sin(k\pi)(k+1)!(2n-k)!}{x^{2n+2}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^{2n+2}} A_{2n-k}(jx).
$$

The reader is referred to Dingle's works for the derivations of (8) and (9) and for some useful asymptotic expressions for the $A_s(x)$.

3. Approximation Forms. Three different approximation forms and associated intervals were chosen for each function, reflecting the basically different functional behaviours displayed in Eqs. (3), (4), and (6). The forms and intervals are:

$$
(11)\qquad (F_k)_{l,m}^{1*}(x) = e^x[\Gamma(k+1) + e^x R_{k,l,m}^{1*}(e^x)], \qquad -\infty < x \leqq 1;
$$

$$
(12)\qquad (F_k)_{l,m}^{2*}(x) = R_{k,l,m}^{2*}(x), \qquad 1 \leqq x \leqq 4;
$$

and

$$
(13)\qquad (F_k)_{l,m}^{3*}(x) = x^{k+1}\left[\frac{1}{k+1} + \frac{1}{x^2} R_{k,l,m}^{3*}(1/x^2)\right], \qquad 4 \leqq x < \infty;
$$

where the $R_{k,l,m}^{i*}$ are rational Chebyshev approximations of degree l in the numerator and m in the denominator. The first and third forms were also used by Werner and

Raymann [12], although our choice of interval for the third form is different. The choice of intervals used here is the result of experimentation. Reasonable choices of l and m give reasonable accuracy on each interval and, although not optimal in this sense, a given choice of k, l and m results in about the same accuracy for each interval.

4. Computations. All computations to be described were carried out on a CDC-3600 computer in 25-decimal floating point arithmetic.

The basic tools for obtaining the approximations were two versions of the second algorithm of Remes [3], [7]. Functional values were computed as needed in a number of ways. For $x < -1$, Eq. (3) gave at least 20S results. The series in Eq. (4) was transformed by the QD algorithm [8] into a continued fraction, the first 40 terms of which gave about 11S for $|x| < 4$ (higher accuracy for smaller x and less accuracy for larger x). Finally the Sommerfeld-Dingle expansion, Eqs. (6)–(10), was the basis for a computation that gave maximal relative errors of 3×10^{-9} for $k = -\frac{1}{2}$, 3×10^{-11} for $k = \frac{1}{2}$ and 5×10^{-13} for $k = \frac{3}{2}$ and $x \geqq 4$. Because of large subtraction errors in the Dingle method, these last accuracies appear nearly maximal using 25-decimal arithmetic. All three methods of computation were cross-checked in regions where they overlapped, and were checked for gross errors against existing tables in the literature, although none of the tables contained as many significant figures as the computations. Additional detailed numerical checking was made in the case of the computations based on Dingle's work because of the large subtraction error involved in Eq. (9) for certain values of s and x. As a final

TABLE IA

$$E^{i*}_{-1/2,l,m} = -100 \log \left\| \frac{F_{-1/2}(x) - (F_{-1/2})^{i*}_{l,m}(x)}{F_{-1/2}(x)} \right\|_\infty$$

$m \backslash l$	0	1	2	3	4	5	6	7	8
				$i = 1,\quad -\infty < x \leqq 1$					
0	65	129	187	243	298	351	404	457	510
1	265	345	418	486	552				
2		492	580	661	738				
3		580	714	807	893				
4			806	934	1030				
				$i = 2,\quad 1 \leqq x \leqq 4$					
0	44	173	285†	316	397	534†	558	621	727
1		266	312	362	484	557		701	
2			413	561	752	795	863		
3				744	794	822			
4					879				
				$i = 3,\quad 4 \leqq x < \infty$					
0	317†	348	454†	465	503	566	607	617	644
1		381	465		560	600			
2			524	616†	633				
3				630					
4					762				

† Nonstandard error curve.

TABLE IB

$$E^{i*}_{1/2,l,m} = -100 \log \left\| \frac{F_{1/2}(x) - (F_{1/2})^{i*}_{l,m}(x)}{F_{1/2}(x)} \right\|_{\infty}$$

$m \backslash l$	0	1	2	3	4	5	6	7	8
				$i = 1, \quad -\infty < x \leqq 1$					
0	109	181	247	309	369	428	485	542	598
1	271	359	439	513	584				
2		495	588	674	755				
3		586	715	812	902				
4			811	935	1033				
5		746							
				$i = 2, \quad 1 \leqq x \leqq 4$					
0	21	118	265	393†	431	519	666†	692	760
1		207	374	427		614	691		
2		289	436	531	667	802	920		
3		371	516	648	846	904			
4		456			905				
5		544							
				$i = 3, \quad 4 \leqq x < \infty$					
0	312	407	480	516	586	640	652	687	737
1		464							
2			647†						
3				654					
4					817				

† Nonstandard error curve.

TABLE IC

$$E^{i*}_{3/2,l,m} = -100 \log \left\| \frac{F_{3/2}(x) - (F_{3/2})^{i*}_{l,m}(x)}{F_{3/2}(x)} \right\|_{\infty}$$

$m \backslash l$	0	1	2	3	4	5	6	7	8
				$i = 1, \quad -\infty < x \leqq 1$					
0	150	232	305	374	439	503	565	625	685
1	301	397	483	563	639				
2		528	626	716	801				
3		622	750	850	944				
4			848	971	1072				
				$i = 2, \quad 1 \leqq x \leqq 4$					
0	13	79	199	358	499	542	635	789	821
1	59	171	304	479	538		737	819	
2	122		397	546	645	781	908		
3	195			630	766	942			
4					858				
				$i = 3, \quad 4 \leqq x < \infty$					
0	291	485†	488	567	643	657	699	761	811†
1		488							
2			637						
3				784					
4					944				

† Nonstandard error curve.

check, the relative error functions

$$\delta^{i*}_{k,l,m}(x) = \frac{F_k(x) - (F_k)^{i*}_{l,m}(x)}{F_k(x)} \tag{14}$$

were plotted on a cathode-ray tube, photographed and examined for smoothness.

Because the approximation forms (11) and (13) correctly emulate the asymptotic behaviour of $F_k(x)$ as $x \to \pm\infty$, the errors (14) vanish asymptotically. Thus computations in the Remes algorithm could be restricted to large finite intervals, generally $[-10, 1]$ in the first case and $[4, 60]$ in the second.

In the original computations all error curves were levelled to at least 3S. The rounded coefficients presented in this paper were separately tested for 2000 random arguments against the original function routines. In each case the maximal error agreed within 2S in magnitude and position with one of the extremal points found in the Remes algorithm.

Out of about 200 different approximations generated for the intervals and approximation forms described above, almost a dozen gave nonstandard error curves on the interval considered, or a slightly larger interval, or computational difficulty because of near-degeneracy. The nonstandard error curves were typified by an extra extremal point of magnitude different from the others, while the near-degeneracy was frequently typified by a pole just outside the approximation interval, and a near-common factor in the numerator and denominator. As expected, if

TABLE IIA

$$F_{-1/2}(x) \cong e^x \left[\Gamma(1/2) + e^x \sum_{s=0}^{n} p_s e^{sx} \Big/ \sum_{s=0}^{n} q_s e^{sx} \right], \qquad -\infty < x \leqq 1$$

s	p_s		q_s	
		$n = 1$		
0	−1.24470	(00)	1.00000	(00)
1	−1.52654	(−02)	7.98207	(−01)
		$n = 2$		
0	−1.25322 15	(00)	1.00000 00	(00)
1	−6.01723 59	(−01)	1.29585 46	(00)
2	−1.22715 51	(−03)	3.54694 31	(−01)
		$n = 3$		
0	−1.25331 32212	(00)	1.00000 00000	(00)
1	−1.17174 61092	(00)	1.75140 13572	(00)
2	−2.11467 70891	(−01)	8.91719 38220	(−01)
3	−1.26856 62408	(−04)	1.21919 85358	(−01)
		$n = 4$		
0	−1.25331 41288 20	(00)	1.00000 00000 00	(00)
1	−1.72366 35577 01	(00)	2.19178 09259 80	(00)
2	−6.55904 57292 58	(−01)	1.60581 29554 06	(00)
3	−6.34228 31976 82	(−02)	4.44366 95274 81	(−01)
4	−1.48838 31061 16	(−05)	3.62423 22881 12	(−02)

TABLE IIB

$$F_{1/2}(x) \cong e^x \left[\Gamma(3/2) + e^x \sum_{s=0}^{n} pe_s{}^{sx} \Big/ \sum_{s=0}^{n} q_s e^{sx} \right], \qquad -\infty < x \leqq 1$$

s	p_s		q_s	
		$n = 1$		
0	−3.10391	(−01)	1.00000	(00)
1	−1.00423	(−02)	5.38275	(−01)
		$n = 2$		
0	−3.13291 80	(−01)	1.00000 00	(00)
1	−1.42756 95	(−01)	9.98828 53	(−01)
2	−1.00908 90	(−03)	1.97169 67	(−01)
		$n = 3$		
0	−3.13328 14419	(−01)	1.00000 00000	(00)
1	−2.80582 65535	(−01)	1.43979 96246	(00)
2	−4.71780 05580	(−02)	5.80877 04412	(−01)
3	−1.18443 08954	(−04)	6.00800 57319	(−02)
		$n = 4$		
0	−3.13328 53055 70	(−01)	1.00000 00000 00	(00)
1	−4.16187 38522 93	(−01)	1.87260 86759 02	(00)
2	−1.50220 84005 88	(−01)	1.14520 44465 78	(00)
3	−1.33957 93751 73	(−02)	2.57022 55875 73	(−01)
4	−1.51335 07001 38	(−05)	1.63990 25435 68	(−02)

TABLE IIC

$$F_{3/2}(x) \cong e^x \left[\Gamma(5/2) + e^x \sum_{s=0}^{n} p_s e^{sx} \Big/ \sum_{s=0}^{n} q_s e^{sx} \right], \qquad -\infty < x \leqq 1$$

s	p_s		q_s	
		$n = 1$		
0	−2.33268	(−01)	1.00000	(00)
1	−1.06386	(−02)	3.80655	(−01)
		$n = 2$		
0	−2.34974 804	(−01)	1.00000 000	(00)
1	−9.90504 038	(−02)	7.83564 382	(−01)
2	−1.14559 597	(−03)	1.15033 976	(−01)
		$n = 3$		
0	−2.34996 17182	(−01)	1.00000 00000	(00)
1	−1.95392 64014	(−01)	1.19434 20572	(00)
2	−3.06557 11516	(−02)	3.87179 30021	(−01)
3	−1.41222 30260	(−04)	3.08879 90780	(−02)
		$n = 4$		
0	−2.34996 39854 06	(−01)	1.05000 00000 00	(00)
1	−2.92737 36375 47	(−01)	1.60859 71091 46	(00)
2	−9.88309 75887 38	(−02)	8.27528 95308 80	(−01)
3	−8.25138 63795 51	(−03)	1.52232 23828 50	(−01)
4	−1.87438 41532 23	(−05)	7.69512 04750 64	(−03)

$(F_k)^{i*}_{l,m}$ had a nonstandard error curve, $(F_k)^{i*}_{l+1,m+1}$ was nearly degenerate. Behaviours of this type have been noted and commented upon before, particularly by Ralston [10] and Rice [11].

The behaviour of the two versions of the Remes algorithm used in these difficult cases points up a basic difference in the numerical stability of the two approaches. For example, the program based on the Fraser-Hart technique [7] failed to converge to $(F_{1/2})^{3*}_{3,3}(x)$ even when 10S initial guesses at the critical points and a 5S initial guess at the maximal error, based on the approximation obtained by the Cody-Stoer technique [3], were used. The difficulty in this case is that the denominator of $R^{3*}_{1/2,3,3}$ vanishes for $x^2 \approx 15.93994663$, while the numerator vanishes for $x^2 \approx 15.93994749$ and the interval of approximation is $16 \leqq x^2 < \infty$. This approximation is not very stable numerically.

The techniques devised to handle such nearly-degenerate cases are still being revised, and will be the subject of a future paper.

5. Results. Table I lists the values of

$$E^{i*}_{k,l,m} = -100 \log \max | \delta^{i*}_{k,l,m}(x) | ,$$

where the maximum is taken over the appropriate interval, for the initial segments of the L_∞ Walsh arrays. An examination of the tables indicates that $E^{i*}_{k,l,m}$ is generally close to maximal for fixed k and $l + m$ along the line $l = m$. Tables II, III and IV present the coefficients for cases $l = m$, $l = 0, 1, \cdots, 4$ for each interval. All coefficients are given to an accuracy greater than that justified by the maximal

TABLE IIIA

$$F_{-1/2}(x) \cong \sum_{s=0}^{n} p_s x^s \Big/ \sum_{s=0}^{n} q_s x^s, \qquad 1 \leqq x \leqq 4$$

s	p_s		q_s	
		$n = 1$		
0	9.2012	(−01)	1.0000	(00)
1	1.0331	(00)	7.5323	(−02)
		$n = 2$		
0	− 1.17909 1	(00)	1.00000 0	(00)
1	1.33436 7	(00)	8.97500 7	(−01)
2	1.15108 8	(00)	1.15382 4	(−01)
		$n = 3$		
0	1.07161 9310	(00)	1.00000 0000	(00)
1	7.59564 5943	(−01)	7.79454 5888	(−02)
2	2.52371 0602	(−01)	9.21173 5007	(−02)
3	5.09743 3764	(−02)	2.49051 0221	(−03)
		$n = 4$		
0	1.07381 27694	(00)	1.00000 00000	(00)
1	5.60033 03660	(00)	4.60318 40667	(00)
2	3.68822 11270	(00)	4.30759 10674	(−01)
3	1.17433 92816	(00)	4.21511 32145	(−01)
4	2.36419 35527	(−01)	1.18326 01601	(−02)

TABLE IIIB

$$F_{1/2}(x) \cong \sum_{s=0}^{n} p_s x^s \Big/ \sum_{s=0}^{n} q_s x^s, \qquad 1 \leqq x \leqq 4$$

s	p_s		q_s	
		$n = 1$		
0	4.7314	(−01)	1.0000	(00)
1	7.7863	(−01)	−9.5883	(−02)
		$n = 2$		
0	6.94327 4	(−01)	1.00000 0	(00)
1	4.91885 5	(−01)	−5.45621 4	(−04)
2	2.14556 1	(−01)	3.64878 9	(−03)
		$n = 3$		
0	6.76208 535	(−01)	1.00000 000	(00)
1	6.51664 310	(−01)	1.59803 695	(−01)
2	2.63424 203	(−01)	3.05417 676	(−02)
3	6.96443 154	(−02)	−8.78750 815	(−04)
		$n = 4$		
0	6.78176 62666 0	(−01)	1.00000 00000 0	(00)
1	6.33124 01791 0	(−01)	1.43740 40039 7	(−01)
2	2.94479 65177 2	(−01)	7.08662 14845 0	(−02)
3	8.01320 71141 9	(−02)	2.34579 49473 5	(−03)
4	1.33918 21294 0	(−02)	−1.29449 92883 5	(−05)

TABLE IIIC

$$F_{3/2}(x) \cong \sum_{s=0}^{n} p_s x^s \Big/ \sum_{s=0}^{n} q_s x^s, \qquad 1 \leqq x \leqq 4$$

s	p_s		q_s	
		$n = 1$		
0	4.986	(−01)	1.000	(00)
1	1.729	(00)	−1.468	(−01)
		$n = 2$		
0	1.19607	(00)	1.00000	(00)
1	7.33852	(−01)	−1.53064	(−01)
2	3.52295	(−01)	1.04035	(−02)
		$n = 3$		
0	1.15000 145	(00)	1.00000 000	(00)
1	9.43296 764	(−01)	−7.28698 650	(−02)
2	3.26281 283	(−01)	1.15139 877	(−02)
3	7.72617 906	(−02)	−5.74907 929	(−04)
		$n = 4$		
0	1.15302 13402	(00)	1.00000 00000	(00)
1	1.05915 58972	(00)	3.73489 53841	(−02)
2	4.68988 03095	(−01)	2.32484 58137	(−02)
3	1.18829 08784	(−01)	−1.37667 70874	(−03)
4	1.94387 55787	(−02)	4.64663 92781	(−05)

TABLE IVA

$$F_{-1/2}(x) \cong \sqrt{x}\left[2 + x^{-2}\sum_{s=0}^{n} p_s x^{-2s} \Big/ \sum_{s=0}^{n} q_s x^{-2s}\right], \quad 4 \leqq x < \infty$$

s	p_s		q_s	
		$n = 0$		
0	−9.84535	(−01)	1.00000	(00)
		$n = 1$		
0	−5.86246	(−01)	1.00000	(00)
1	−1.58903	(02)	1.50627	(02)
		$n = 2$		
0	−8.14958 47	(−01)	1.00000 00	(00)
1	4.05212 66	(00)	−1.08676 28	(01)
2	−3.25435 65	(02)	3.84615 01	(02)
		$n = 3$		
0	−8.24391 144	(−01)	1.00000 000	(00)
1	−2.04495 807	(00)	−4.88152 379	(−01)
2	−8.96893 377	(02)	8.05727 048	(02)
3	4.88655 638	(03)	−3.56730 597	(03)
		$n = 4$		
0	−8.22255 9330	(−01)	1.00000 0000	(00)
1	−3.62036 9345	(01)	3.93568 9841	(01)
2	−3.01538 5410	(03)	3.56875 6266	(03)
3	−7.04987 1579	(04)	4.18189 3625	(04)
4	−5.69814 5924	(04)	3.38513 8907	(05)

TABLE IVB

$$F_{1/2}(x) \cong x\sqrt{x}\left[2/3 + x^{-2}\sum_{s=0}^{n} p_s x^{-2s} \Big/ \sum_{s=0}^{n} q_s x^{-2s}\right], \quad 4 \leqq x < \infty$$

s	p_s		q_s	
		$n = 0$		
0	8.66045	(−01)	1.00000	(00)
		$n = 1$		
0	8.16118 1	(−01)	1.00000 0	(00)
1	8.76882 9	(00)	8.94339 7	(00)
		$n = 2$		
0	8.22713 535	(−10)	1.00000 000	(00)
1	5.27498 049	(00)	5.69335 697	(00)
2	2.90433 403	(02)	3.22149 800	(02)
		$n = 3$		
0	8.22752 754	(−10)	1.00000 000	(00)
1	−7.55890 283	(00)	−9.89594 310	(00)
2	2.07024 852	(02)	2.31227 330	(02)
3	−4.71158 007	(03)	−5.22142 719	(03)
		$n = 4$		
0	8.22449 97626	(−01)	1.00000 00000	(00)
1	2.00463 03393	(01)	2.34862 07659	(01)
2	1.82680 93446	(03)	2.20134 83743	(03)
3	1.22265 30374	(04)	1.14426 73596	(04)
4	1.40407 50092	(05)	1.65847 15900	(05)

TABLE IVC

$$F_{3/2}(x) \cong x^2\sqrt{x}\left[2/5 + x^{-2}\sum_{s=0}^{n} p_s x^{-2s} \Big/ \sum_{s=0}^{n} q_s x^{-2s}\right], \quad 4 \leqq x < \infty$$

s	p_s		q_s	
		$n = 0$		
0	2.4247	(00)	1.0000	(00)
		$n = 1$		
0	2.46929 3	(00)	1.00000 0	(00)
1	−6.56036 0	(−01)	9.64792 7	(−02)
		$n = 2$		
0	2.46721 347	(00)	1.00000 000	(00)
1	1.99900 983	(01)	8.36525 114	(00)
2	1.56338 125	(02)	6.90842 636	(01)
		$n = 3$		
0	2.46741 6637	(00)	1.00000 0000	(00)
1	9.77546 3043	(01)	3.99113 6128	(01)
2	2.29398 3407	(03)	9.41059 1778	(02)
3	8.00686 7097	(03)	3.70648 4478	(03)
		$n = 4$		
0	2.46740 02368 4	(00)	1.00000 00000 0	(00)
1	2.19167 58236 8	(02)	8.91125 14061 9	(01)
2	1.23829 37907 5	(04)	5.04575 66966 7	(03)
3	2.20667 72496 8	(05)	9.09075 94530 4	(04)
4	8.49442 92003 4	(05)	3.89960 91564 1	(05)

errors. Reasonable rounding of the coefficients should not affect the overall accuracy.

Coefficients for all approximations indicated in Table I will be published in an Argonne National Laboratory report.

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