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# Primes of the Form $n^4 + 1$

**By M. Lal**

In this note we report 172 new primes of the form $n^4 + 1$ and tabulate all such primes for $1 \leqq n \leqq 4004$.

Factorization of the numbers of the form $n^4 + 1$ has been extensively studied by Cunningham [1] and Gloden [2], [3]. They used a sieve method based on the four solutions of the congruence equation

$$x^4 + 1 \equiv 0 \pmod{p} \tag{1}$$

for all primes of the form $8k + 1$. With primes less than $4 \times 10^6$, numbers $n^4 + 1$ for $n \leqq 2000$ have been completely factorized.

For $p > 4 \times 10^6$, it becomes rather difficult and time consuming to solve (1). Consequently, it renders such a sieve less practical. An analysis of the growing inefficiency of such a sieve with increasing $p$ is given in [5, p. 188]. However, the range for $n$ can be extended by using Alway's method [4] of factorization modified for odd divisors of the form $8k + 1$ and testing each number $n^4 + 1$ individually.

With the modified Alway's method, we found all primes for $1 \leqq n \leqq 1000$. This was done to check the program and to provide independent data which is not readily accessible for this interval. The search was then extended to $2000 \leqq n \leqq 4004$; 172 primes and one prime factor for other composite numbers were indentified. The time required to establish the primality of $n = 4002$ is 2.0 hours on the IBM 1620 computer Model II. As the project required several hundred hours of machine time, the search was made on three IBM 1620 computers—one at Memorial and two at Kingston. All primes of the form $n^4 + 1$, complemented by those given in [3] for $1000 < n < 2040$, are presented in Table 1.

**Discussion of Results.** Shanks [5] has made a conjecture regarding the number of primes $Q(N)$ of the form $n^4 + 1$ for $1 \leqq n \leqq N$ and has given the following expression:

$$Q(N) \sim .66974 \int_2^N \frac{dn}{\log n}. \tag{2}$$

The observed count and those computed by using (2), rounded to the first decimal, are given below in Table 2.

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TABLE 1*

*Primes of the form $n^4 + 1$*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 48 | 132 | 204 | 276 | 364 | 492 | 566 | 702 | 772 |
| 2 | 54 | 140 | 210 | 278 | 374 | 494 | 568 | 710 | 778 |
| 4 | 56 | 142 | 220 | 288 | 414 | 498 | 582 | 730 | 786 |
| 6 | 74 | 154 | 228 | 296 | 430 | 504 | 584 | 732 | 788 |
| 16 | 80 | 160 | 238 | 312 | 436 | 516 | 600 | 738 | 798 |
| 20 | 82 | 164 | 242 | 320 | 442 | 526 | 616 | 742 | 800 |
| 24 | 88 | 174 | 248 | 328 | 466 | 540 | 624 | 748 | 810 |
| 28 | 90 | 180 | 254 | 334 | 472 | 550 | 628 | 758 | 856 |
| 34 | 106 | 194 | 266 | 340 | 476 | 554 | 656 | 760 | 874 |
| 46 | 118 | 198 | 272 | 352 | 488 | 556 | 690 | 768 | 894 |
| 912 | 996 | 1144 | 1246 | 1404 | 1536 | 1610 | 1680 | 1788 | 1910 |
| 914 | 1038 | 1150 | 1252 | 1406 | 1540 | 1612 | 1688 | 1806 | 1916 |
| 928 | 1042 | 1152 | 1270 | 1428 | 1542 | 1618 | 1700 | 1820 | 1926 |
| 930 | 1072 | 1170 | 1280 | 1434 | 1552 | 1622 | 1706 | 1824 | 1932 |
| 936 | 1076 | 1180 | 1302 | 1442 | 1554 | 1638 | 1710 | 1836 | 1934 |
| 952 | 1088 | 1200 | 1322 | 1446 | 1558 | 1644 | 1718 | 1850 | 1942 |
| 962 | 1126 | 1202 | 1330 | 1458 | 1568 | 1646 | 1722 | 1854 | 1944 |
| 966 | 1132 | 1218 | 1344 | 1472 | 1586 | 1650 | 1738 | 1864 | 1948 |
| 986 | 1136 | 1236 | 1382 | 1486 | 1594 | 1652 | 1754 | 1870 | 1952 |
| 992 | 1142 | 1238 | 1388 | 1496 | 1598 | 1666 | 1772 | 1892 | 1956 |
| 1962 | 2074 | 2204 | 2260 | 2374 | 2478 | 2560 | 2674 | 2770 | 2964 |
| 1972 | 2102 | 2206 | 2266 | 2376 | 2482 | 2578 | 2676 | 2798 | 2998 |
| 1978 | 2104 | 2222 | 2292 | 2384 | 2486 | 2590 | 2690 | 2804 | 3006 |
| 1986 | 2108 | 2224 | 2296 | 2400 | 2488 | 2598 | 2698 | 2834 | 3012 |
| 1994 | 2126 | 2226 | 2312 | 2408 | 2510 | 2604 | 2700 | 2866 | 3022 |
| 2040 | 2142 | 2238 | 2322 | 2414 | 2512 | 2612 | 2724 | 2872 | 3030 |
| 2044 | 2152 | 2240 | 2336 | 2428 | 2522 | 2622 | 2732 | 2876 | 3046 |
| 2046 | 2158 | 2250 | 2350 | 2432 | 2536 | 2640 | 2734 | 2902 | 3070 |
| 2058 | 2162 | 2254 | 2360 | 2436 | 2546 | 2642 | 2736 | 2936 | 3084 |
| 2068 | 2192 | 2256 | 2368 | 2438 | 2554 | 2646 | 2740 | 2958 | 3090 |
| 3094 | 3246 | 3410 | 3502 | 3626 | 3720 | 3862 | 3972 | | |
| 3100 | 3254 | 3416 | 3516 | 3632 | 3752 | 3870 | 3982 | | |
| 3104 | 3268 | 3422 | 3522 | 3642 | 3756 | 3872 | 3988 | | |
| 3108 | 3286 | 3450 | 3530 | 3644 | 3764 | 3882 | 3992 | | |
| 3124 | 3288 | 3456 | 3550 | 3666 | 3780 | 3896 | 3998 | | |
| 3128 | 3322 | 3464 | 3574 | 3688 | 3796 | 3910 | 4000 | | |
| 3132 | 3326 | 3468 | 3576 | 3692 | 3802 | 3926 | 4002 | | |
| 3162 | 3378 | 3472 | 3586 | 3700 | 3842 | 3954 | | | |
| 3200 | 3386 | 3480 | 3618 | 3702 | 3854 | 3958 | | | |
| 3244 | 3390 | 3492 | 3620 | 3718 | 3856 | 3960 | | | |

* I am indebted to the late Professor Albert Gloden for his kind permission to include his results for $1000 < n < 2040$.

The agreement between the actual and computed counts is remarkably good. From Table 1, one observes that "twin" primes (those where $n^4 + 1$ and $(n + 2)^4 + 1$ are both primes) occur quite frequently. The number of such twins $P(N, N + 2)$ for $1 \leqq n \leqq N$ is given in Table 3.

On the basis of heuristic arguments [6], one would expect the number

TABLE 2

| $N$ | Counts | |
|---|---|---|
| | Observed | Computed |
| 500 | 63 | 67.5 |
| 1000 | 111 | 118.3 |
| 1500 | 150 | 165.3 |
| 2000 | 205 | 210.1 |
| 2500 | 254 | 253.5 |
| 3000 | 292 | 295.8 |
| 3500 | 330 | 337.3 |
| 4000 | 376 | 377.9 |

TABLE 3
*Twin primes*

| $N$ | $P(N, N + 2)$ | $R = N/(\log N)^2$ | $P(N, N + 2)/R$ |
|---|---|---|---|
| 1000 | 18 | 20.96 | .86 |
| 2000 | 30 | 34.62 | .87 |
| 3000 | 46 | 46.80 | .98 |
| 4000 | 58 | 58.15 | 1.00 |

$P(N, N + 2)$ to be proportional to $N/(\log N)^2$ and it is gratifying to note that this is indicated in column 4 above. These twins are analogous to the Gaussian twin primes examined in [7].

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