

# On Convergence Rates for Line Overrelaxation

By John Gary

This note is devoted to the solution of Poisson's equation by finite difference methods. We will empirically determine the convergence rate for successive line overrelaxation (Liebmann method) to show that there is a qualitative difference between Neumann and Dirichlet boundary conditions. Keller has shown that the convergence rate for Dirichlet boundary conditions on a rectangular mesh depends on the ratio  $\alpha = (\Delta y / \Delta x)^2$  [2]. Convergence is much faster if this ratio is small, provided the implicit direction is parallel to the  $y$ -axis.

The computation described here indicates that in certain cases the convergence rate is independent of  $\alpha$  for Neumann boundary conditions. These seem to be the cases in which the solution of Poisson's equation contains a Fourier component which is independent of  $y$ , for example,  $u = \cos x + \cos x \cos y$ . Such cases do arise in applications. One such application is the numerical solution of the Navier-Stokes equations for incompressible viscous fluid flow between two plates. In this case, a Poisson equation with Neumann boundary conditions and  $\alpha \sim .01$  must be solved to obtain the pressure. Under certain conditions the pressure tends to resemble the function  $\cos x(1 - e^{-y})$ .

The computation also shows that convergence for the Liebmann method can be very slow with Neumann boundary conditions. On the other hand, the computations indicate that convergence in the Dirichlet case is quite rapid for small values of  $\alpha$  independent of the form of the solution  $u$ . This is in agreement with the theoretical results by Keller.

For comparison we will describe convergence rates obtained with an ADI (alternating direction implicit) method applied to Dirichlet and Neumann boundary conditions. The results indicate that convergence is slower in the Neumann case. However, the difference between Dirichlet and Neumann conditions with ADI is much less than with the Liebmann method.

To our knowledge there is no published work which yields an expression for the convergence rate of the Liebmann method with Neumann boundary conditions. There are a few papers concerned with the convergence rate of iterative methods for singular systems [1], [3], [4].

We wish to solve the equations:

$$\alpha[2u_{ij} - \theta_i^+ u_{i+1,j} - \theta_i^- u_{i-1,j}] + 2u_{ij} - \Psi_j^+ u_{i,j+1} - \Psi_j^- u_{i,j-1} = \rho_{ij}$$

or

$$\alpha[(\theta_i^+ + \theta_i^-)u_{ij} - \theta_i^+ u_{i+1,j} - \theta_i^- u_{i-1,j}] + (\Psi_j^+ + \Psi_j^-)u_{ij} - \Psi_j^+ u_{i,j+1} - \Psi_j^- u_{i,j-1} = \rho_{ij}$$

where  $1 \leq i \leq N_x$ ,  $1 \leq j \leq N_y$ ,  $\alpha = (\Delta y / \Delta x)^2$ ,

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$$\begin{aligned} \theta_i^+ = 1, \quad i < N_x, \quad \theta_i^- = 0, \quad i = 1, \quad \Psi_j^+ = 1, \quad j < N_y, \quad \Psi_j^- = 0, \quad j = 1, \\ = 0, \quad i = N_x, \quad = 1, \quad i > 1, \quad = 0, \quad j = N_y, \quad = 1, \quad j > 1. \end{aligned}$$

The first equation corresponds to Dirichlet boundary conditions, the second to Neumann boundary conditions. The convergence rate is determined as follows. We first select a function  $u_{ij}$ ,  $1 \leq i \leq N_x$ ,  $1 \leq j \leq N_y$ . Then the right-hand side  $\rho_{ij}$  is computed from the equations above. Therefore, we can easily determine the error at any stage in the numerical solution of these equations since the exact solution  $u_{ij}$  is known.

Since the convergence rate depends on  $u_{ij}$ , we used several different functions  $u_{ij}$  which are displayed in Table 1. In all cases the initial guess was taken equal to zero, that is,  $u_{ij}^{(0)} = 0$ . Then the successive iterates  $u_{ij}^{(k)}$  were computed along with the relative error.

$$\epsilon^{(k)} = \text{Max}_{ij} |u_{ij}^{(k)} - u_{ij}| / \text{Max}_{ij} |u_{ij}|.$$

The Liebmann algorithm for  $u_{ij}^{(k)}$  (Dirichlet case) is

$$\begin{aligned} & \gamma[(2 + 2\alpha)u_{ij}^{(k+1)} - \Psi_j^+ u_{i,j+1}^{(k+1)} - \Psi_j^- u_{i,j-1}^{(k+1)}] \\ &= (\gamma - 1)[(2 + 2\alpha)u_{ij}^{(k)} - \Psi_j^+ u_{i,j+1}^{(k)} - \Psi_j^- u_{i,j-1}^{(k)}] \\ & \quad + \alpha[\theta_i^+ u_{i+1,j}^{(k)} + \theta_i^- u_{i-1,j}^{(k+1)}] + \rho_{ij}. \end{aligned}$$

TABLE 1  
Log error after 48 iterations

$$\begin{aligned} \epsilon &= \text{relative error} \\ \alpha &= (\Delta y / \Delta x)^2, \quad N_x = N_y = 40, \quad 0 \leq x \leq 4\pi, \quad 0 \leq y \leq 2\pi \end{aligned}$$

Solution $u =$	$\alpha$	$-\log_{10}(\epsilon)$			
		Line relaxation		ADI	
		Dirichlet	Neumann	Dirichlet	Neumann
$\cos x$	1.0	3.4	2.2	11.9	9.7
	0.01	13.0	2.2	11.3	7.4
$(1 - e^{-y/\pi}) \cos x$	1.0	3.6	2.2	11.9	9.9
	0.01	12.4	2.2	11.6	7.6
$.01x/4\pi + \cos x(1 + .05 \cos y)$	1.0	3.3	2.1	11.2	9.0
	0.01	12.0	2.2	11.3	6.0
$x(x - 4\pi)y(y - 2\pi)$	1.0	3.4	2.3	12.0	9.7
	0.01	12.4	2.2	11.5	7.4
$\cos x \cos y$	1.0	3.5	3.4	12.1	11.9
	0.01	12.6	12.5	11.8	11.5
$\cos(x + y)$	1.0	3.5	2.1	12.0	11.8
	0.01	12.6	12.4	11.8	10.5

For values of  $\gamma$  in the range  $\frac{1}{2} \leq \gamma < 1$  we have overrelaxation. The optimum value of  $\gamma$  was found experimentally by observing the convergence rate at various values of  $\gamma$ .

The ADI method is defined in terms of the operators  $H$  and  $V$  given below (for the Dirichlet case):

$$\begin{aligned}Hu &= \alpha[2u_{ij} - \theta_i^+ u_{i+1,j} - \theta_j^- u_{i,j-1}], \\Vu &= 2u_{ij} - \Psi_j^+ u_{i,j+1} - \Psi_j^- u_{i,j-1}.\end{aligned}$$

One ADI iteration consists of the two steps:

$$\begin{aligned}(r_k I + H)u^{(k+1/2)} &= (r_k I - V)u^{(k)} + \rho, \\(r_k I + V)u^{(k+1)} &= (r_k I - H)u^{(k+1/2)} + \rho.\end{aligned}$$

The sequence of positive numbers  $r_k$  is defined as follows:

$$r_k = \beta \delta^{(17-2k)} \qquad \text{for } k = 1, 2, \dots, 8,$$

where  $\beta = \mu(\pi/N_x)^2$ ,  $\delta = (4/\beta)^{1/16}$  and  $\mu$  is adjusted to obtain the most rapid convergence [5]. The remaining values of  $r_k$  are defined modulo 8, that is  $r_9 = r_1$ ,  $r_{10} = r_2$ , etc. This probably does not provide an optimal choice of these eight parameters, so that the ADI method may be somewhat better than the results below indicate.

The results of the computation are given in the two tables below. The error  $\epsilon^{(k)}$  is given for various values of the iteration count “ $k$ ” and the parameters  $\alpha$ ,  $\mu$ , and  $\gamma$ . The values of  $\mu$  and  $\gamma$  are chosen to minimize the error on the 48th iteration. The

TABLE 2  
*Log error after  $k$  iterations ( $\epsilon$  = relative error)*  
 $\alpha = (\Delta y/\Delta x)^2$ ,  $N_x = N_y = 40$ ,  $u = \cos x$ ,  $0 \leq x \leq 4\pi$ ,  $0 \leq y \leq 2\pi$   
 $\gamma$  = relaxation parameter,  $\mu$  = convergence rate parameter for ADI

$\alpha$	$\mu$	$k$				$\alpha$	$\mu$	$k$			
		8	24	48	96			8	24	48	96
$-\log_{10}(\epsilon)$						$-\log_{10}(\epsilon)$					
1.0	1.0	2.9	7.4	11.9	11.9	1.0	0.51	1.9	6.0	9.7	11.2
0.1	0.032	2.9	8.0	11.4	11.4	0.1	0.064	1.4	4.7	9.5	10.3
0.01	0.004	2.1	6.4	11.3	11.2	0.01	0.004	1.4	3.9	7.4	9.4
ADI—Dirichlet						ADI—Neumann					
$\alpha$	$\gamma$	$k$				$\alpha$	$\gamma$	$k$			
		8	24	48	96			8	24	48	96
$-\log_{10}(\epsilon)$						$-\log_{10}(\epsilon)$					
1.0	0.56	0.3	0.8	3.4	6.6	1.0	0.53	0.04	0.7	2.2	4.0
0.1	0.62	0.7	2.4	7.0	12.1	0.1	0.53	0.04	0.7	2.2	4.0
0.01	0.88	2.1	6.7	13.0	13.0	0.01	0.53	0.04	0.7	2.1	4.0
Line Relaxation—Dirichlet						Line Relaxation—Neumann					

values of the error shown in the tables correspond to these (approximately) optimal values of  $\mu$  and  $\gamma$ . These computations were performed on the CDC 6600 at the National Center for Atmospheric Research.

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## Canonical Decomposition of Hessenberg Matrices†

By Beresford Parlett

**1. Introduction.** A square matrix  $A$  is said to be in (upper) Hessenberg form if  $a_{ij} = 0$  for  $i > j + 1$ . Such matrices occur frequently in connection with the eigenvalue problem. In practical work it is an important fact that any square matrix may be transformed in a stable manner into a similar Hessenberg matrix, see [5]. Apart from possible economies in computing the eigenvalues we may ask whether a preliminary reduction of a full matrix to this form offers any other advantages.

We show here that this reduction replaces an arbitrary independent set of eigenvectors by one which has some useful theoretical properties. In other words if  $J$  is the (lower) Jordan canonical form of  $A$ , say

$$(1.1) \quad A = Y^{-1}JY,$$

then the rows of  $Y$  are the row eigenvectors of  $A$ . When  $A$  is defective we must interpret eigenvectors in the generalized sense (as principal vectors). For general  $A$  we can say nothing about  $Y$  other than  $\det(Y) \neq 0$ . If  $A$  is a Hessenberg matrix then  $Y$  has the properties summarized in Theorem 1.

We should remark here that our results are fairly straightforward deductions from Lemma 1 which is well known, but not in the form used here. The purpose of this note is just to extract the properties which are latent in that lemma: essentially the triangular factorization of Vandermonde matrices.

As we show in [7] the existence of this factorization helps explain the remarkable convergence properties of the  $QR$  algorithm of J. G. F. Francis [1]. The result is also useful in discussing other problems involving Hessenberg matrices. We note that "the" Jordan form is unique only to within the order of the submatrices of which it is a direct sum. The factors in our decomposition depend on this order and here we

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