On First Appearance of Prime Differences

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There has long been interest in strings of consecutive composite numbers appearing among the natural numbers. Most elementary texts on number theory include a discussion of how arbitrarily large gaps between consecutive primes can be constructed, for example [1]. Such constructive techniques lead to rather large numbers, however, and lower occurrences have been studied [2], [3] to gain insight into the subject.

In 1961, Gruenberger and Armerding examined the first six million primes (up to $P = 104{,}395{,}289$) [4] on a computer and produced certain statistics covering these primes [5]. They tabulated the primes forming the lower boundary for the first appearance of prime differences of prescribed lengths, where all intervening numbers are composite, up to the limit of the primes list. The largest difference found between two consecutive primes was 220, and the smallest difference whose first appearance was not found was 186.

An algorithm for direct search for prime-differences (usable on a computer of limited storage capacity) proceeds as follows:

(a) Start at a known prime, say P_a, below which all differences of interest are known.

(b) Form $P_a + D$, where D is the smallest difference whose first appearance is unknown.

(c) From the point $P_a + D$, test the successively smaller numbers for primality by trial division or other technique until a prime P_b is found.

(d) If $P_b > P_a$, replace P_a by P_b and repeat the algorithm.

(e) If $P_b = P_a$, start testing at $P_a + D$, and proceed to successively larger numbers until a prime P_c is reached. $P_c - P_a$ is then a difference $\geq D$ between successive primes, and is recorded, unless such a difference has already occurred.

(f) Update D, if necessary, to the next larger difference whose first appearance is unknown; replace P_a by P_c, and repeat the algorithm.

A computer program for the CDC 3200 was written to implement this algorithm, and Table I through the range $0 < P < 1.46 \times 10^9$ represents the data obtained from this program.

The algorithm itself guarantees that no difference of interest (i.e., $\geq$ smallest difference whose first appearance is unknown) will escape notice, while a separate check was run on the data in Table I. This check took the form of another computer program which read the Table I data as input, established the primality of P_a *and* P_b by testing for divisibility by primes up to the square root of P_a or P_b, and explicitly exhibited all the prime factors of each odd number between the two primes. Thus, the differences listed are verified to be exactly as long as stated. Since all previous results in [5] were exactly duplicated (items of Table I for $D \leq 184$ and $D = 196{,}198{,}210{,}220$), the data may be regarded as accurate.

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The primality testing process was designed to operate without using an extensive table of primes, while, at the same time, being made as rapid as practicable. First, the numbers to be tested were required to be prime to 210. Since a sequence of consecutive numbers was being tested, a single division by 210 followed by a table lookup in a table of 210 positions sufficed to exclude all numbers not prime to 210. Each entry of the table actually pointed to the next eligible number to be tested. Secondly, division by a few small primes was used. Since the range of interest quickly exceeded the single-precision word length of the computer (24 bits), the 48-bit hardware arithmetic of the machine was used. However, in order to avoid double-precision division as long as possible, the numbers being tested were reduced to single precision by subtraction of self-adjusting multiples of groups of small primes prior to division by those primes. The final step was division by all odd numbers prime to 6 (by alternately adding 2, then 4, to an appropriate starting prime) and less than the square root of the number tested, in order to verify the primality of the end points for the algorithm, and to determine that intervening numbers were composite. Of course, as soon as any eligible number being tested was found to be composite, it was rejected, and the next eligible number was selected for testing. Since there was no room in the program to store a table of pseudoprimes to the base 2, experiments with the converse of Fermat's Theorem to detect composite numbers were dropped when it was noted that the program spent the majority of its running time verifying the primality of the end points, rather than eliminating composite numbers between the end points.

With the availability of a larger computer memory in which to store a table of primes and their starting points with respect to a fixed field of bits, it becomes feasible to use a sieve technique for extending this search. However, with a very limited computer memory, the algorithm given above has the advantage of requiring only a table of previously found differences, and a starting point for each run, and thus could be used as a small background problem.

A program for the CDC 6600 was written to implement a sieve technique for generating and examining gaps in primes. This program occupied considerably more memory but ran significantly faster (partially due to an increase in computer speed) than the program described above. The sieve program allocated a block of computer memory in which consecutive bits represented the successive odd integers. A table of the first ten thousand primes was generated and stored by the program during initialization. Another table of starting points (i.e., index of the first bit in the field corresponding to a multiple of each prime in the stored table) for marking by each prime in the sieve field was also generated and saved. The program then cycled through successive bit fields marking bits corresponding to the odd composite numbers, then searched the field for gaps of interest. End effects at the boundaries of the sieve fields were noted so that gaps of interest would not be missed. Table I for $1.46 \times 10^9 < P < 1.096 \times 10^{10}$ presents the results obtained from this program.

In private correspondence Daniel Shanks suggested the possibility of extending Table I in [2] over the new differences found. Accordingly, Table I shows $\log P_b/(D - 1)^{1/2}$, with each maximal gap D marked with an asterisk. Maximal gaps, according to Shanks, are those larger than any preceding gap in the sequence of

TABLE I

D	P_a	P_b	$\log P_b/(D-1)^{1/2}$
2*	3	5	1.609
4*	7	11	1.384
6*	23	29	1.506
8*	89	97	1.729
10	139	149	1.668
12	199	211	1.614
14*	113	127	1.344
16	1831	1847	1.942
18*	523	541	1.526
20*	887	907	1.562
22*	1129	1151	1.538
24	1669	1693	1.550
26	2477	2503	1.565
28	2971	2999	1.541
30	4297	4327	1.555
32	5591	5623	1.551
34*	1327	1361	1.256
36*	9551	9587	1.550
38	30593	30631	1.698
40	19333	19373	1.581
42	16141	16183	1.514
44*	15683	15727	1.474
46	81463	81509	1.686
48	28229	28277	1.495
50	31907	31957	1.482
52*	19609	19661	1.384
54	35617	35671	1.440
56	82073	82129	1.526
58	44293	44351	1.417
60	43331	43391	1.390
62	34061	34123	1.336
64	89689	89753	1.437
66	162143	162209	1.488
68	134513	134581	1.443
70	173359	173429	1.452
72*	31397	31469	1.229
74	404597	404671	1.511
76	212701	212777	1.417
78	188029	188107	1.384
80	542603	542683	1.486
82	265621	265703	1.388
84	461717	461801	1.432
86*	155921	156007	1.297
88	544279	544367	1.416
90	404851	404941	1.369
92	927869	927961	1.440
94	1100977	1101071	1.443
96*	360653	360749	1.313
98	604073	604171	1.352
100	396733	396833	1.296

TABLE I—*Continued*

D	P_a	P_b	$\log P_b/(D-1)^{1/2}$
102	1444309	1444411	1.411
104	1388483	1388587	1.394
106	1098847	1098953	1.357
108	2238823	2238931	1.414
110	1468277	1468387	1.360
112*	370261	370373	1.217
114*	492113	492227	1.233
116	5845193	5845309	1.453
118*	1349533	1349651	1.305
120	1895359	1895479	1.325
122	3117299	3117421	1.359
124	6752623	6752747	1.418
126	1671781	1671907	1.282
128	3851459	3851587	1.346
130	5518687	5518817	1.367
132*	1357201	1357333	1.234
134	6958667	6958801	1.366
136	6371401	6371537	1.348
138	3826019	3826157	1.295
140	7621259	7621399	1.344
142	10343761	10343903	1.360
144	11981443	11981587	1.363
146	6034247	6034393	1.297
148*	2010733	2010881	1.197
150	13626257	13626407	1.346
152	8421251	8421403	1.298
154*	4652353	4652507	1.241
156	17983717	17983873	1.342
158	49269581	49269739	1.414
160	33803689	33803849	1.375
162	39175217	39175379	1.378
164	20285099	20285263	1.318
166	83751121	83751287	1.420
168	37305713	37305881	1.349
170	27915737	27915907	1.319
172	38394127	38394299	1.335
174	52721113	52721287	1.352
176	38089277	38089453	1.320
178	39389989	39390167	1.315
180*	17051707	17051887	1.245
182	36271601	36271783	1.294
184	79167733	79167917	1.344
186	147684137	147684323	1.383
188	134065829	134066017	1.368
190	142414669	142414859	1.366
192	123454691	123454883	1.348
194	166726367	166726561	1.363
196	70396393	70396589	1.294
198	46006769	46006967	1.257
200	378043979	378044179	1.400

TABLE I—*Continued*

D	P_a	P_b	$\log P_b/(D-1)^{1/2}$
202	107534587	107534789	1.304
204	112098817	112099021	1.301
206	232423823	232424029	1.345
208	192983851	192984059	1.326
210*	20831323	20831533	1.166
212	215949407	215949619	1.321
214	253878403	253878617	1.326
216	202551667	202551883	1.304
218	327966101	327966319	1.331
220*	47326693	47326913	1.194
222*	122164747	122164969	1.253
224	409866323	409866547	1.328
226	519653371	519653597	1.338
228	895858039	895858267	1.368
230	607010093	607010323	1.336
232	525436489	525436721	1.321
234*	189695659	189695893	1.249
236	216668603	216668839	1.252
238	673919143	673919381	1.320
240	391995431	391995671	1.280
242	367876529	367876771	1.270
244	693103639	693103883	1.306
246	555142061	555142307	1.286
248*	191912783	191913031	1.214
250*	387096133	387096383	1.253
252	630045137	630045389	1.279
254	1202442089	1202442343	1.314
256	1872851947	1872852203	1.337
258	1316355323	1316355581	1.310
260	944192807	944193067	1.284
262	1649328997	1649329259	1.314
264	2357881993	2357882257	1.331
266	1438779821	1438780087	1.295
268	1579306789	1579307057	1.296
270	1391048047	1391048317	1.284
272	1851255191	1851255463	1.296
274	1282463269	1282463543	1.269
276	649580171	649580447	1.224
278	4260928601	4260928879	1.332
280	1855047163	1855047443	1.278
282*	436273009	436273291	1.187
284	1667186459	1667186743	1.262
286	2842739311	2842739597	1.289
288*	1294268491	1294268779	1.238
290	1948819133	1948819423	1.258
292*	1453168141	1453168433	1.237
294	5692630189	5692630483	1.312
296	5260030511	5260030807	1.303
298	8650524583	8650524881	1.328
300	4758958741	4758959041	1.289

TABLE I—*Continued*

D	P_a	P_b	$\log P_b/(D-1)^{1/2}$
302	6675573497	6675573799	1.304
304	2433630109	2433630413	1.242
306	3917587237	3917587543	1.265
308	5490459101	5490459409	1.280
310	4024713661	4024713971	1.258
312	6570018347	6570018659	1.282
314	8948418749	8948419063	1.295
318	4372999721	4373000039	1.247
320*	2300942549	2300942869	1.207
322	7961074441	7961074763	1.272
324	10958687879	10958688203	1.286
326	5837935373	5837935699	1.247
330	6291356009	6291356339	1.244
332	5893180121	5893180453	1.237
336*	3842610773	3842611109	1.206
340	8605261447	8605261787	1.242
354*	4302407359	4302407713	1.181
382*	10726904659	10726905041	1.183

primes. These data tend to support the conjectured relation in [2], namely that $\log P_b \sim (D - 1)^{1/2}$ for maximal gaps, and, also, possibly, for all gaps at the point of their first appearance. For example, $D = 316$ is the first difference that does not appear in our table, but since $\log P_b/(D - 1)^{1/2}$ is consistently $< 4/3$ for $D > 256$, it is not unreasonable to guess that $D = 316$ will appear before $\exp(4(\sqrt{315})/3) = 10^{10 \cdot 277}$.

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