$\sqrt{3}$, $(3/2)(-3 + \sqrt{5})$, $3\sqrt{2}/2$, $(3/190)(-15 + (35)^{1/2})$, $(15 - (15)^{1/2})/70$.

The over-all evidence suggests very strongly that in most practical situations method (A) is preferable to method (B).

TABLE

k	*Method* (A). $\lvert q\rvert$-*bound for convergence*	*Method* (B). q-*range for convergence*	*Method* (B). q-*range such that convergence factor* $\leq \cdot 1$
2	1.73	(−1.15 , 2.12)	(− .143 , .159)
3	1.43	(− .860 , 1.43)	(− .119 , .135)
4	1.33	(− .738 , 1.64)	(− .106 , .117)
5	1.21	(− .711 , 1.21)	(− .0994 , .102)
6	1.16	(− .687 , 1.50)	(− .0926 , .0866)
7	1.10	(− .576 , .813)	(− .0769 , .0686)
8	1.07	(− .493 , .475)	(− .0629 , .0517)

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3. A. C. R. NEWBERY, "Multistep integration formulas," *Math. Comp.*, v. 17, 1963, pp. 452–455. (See also corrigendum, *Math. Comp.*, v. 18, 1964, p. 536.) MR 27 #5362.

A Polynomial Approximation Converging in a Lens-Shaped Region[1]

By Jay A. Leavitt

The Taylor series expansion of $y = 1/(1 + x^2)$ about $x = 0$ has a radius of convergence $R = 1$, while the function itself is analytic for all real values of x. In order to represent $1/(1 + x^2)$ by a Taylor series for values of x outside the interval $(-1, 1)$, it is necessary to expand about a point of nonsymmetry.

In practice, given an analytic function $f(x)$, one uses only its truncated Taylor series $T_n(x)$. The expansion of such a truncated series of order n, i.e. $T_n(x)$, about the point b provides a polynomial, say $V_n(z)$ where $z = x - b$, which is of order n. But $V_n(z)$ converges to $f(x)$ only in the original circle of convergence of the $T_n(x)$. Nevertheless, this property is used to produce a sequence of even polynomials, $U_n(x)$, which have real coefficients and which converge to $y = 1/(1 + x^2)$ in a lens-shaped region that includes an extended interval of the real axis.

Let us expand $1/(x + i)$ about $x = (\lambda - 1)i$ and $1/(x - i)$ about $x = -(\lambda - 1)i$ and truncate; $\lambda \geq 1$ real.

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$$
\begin{aligned}
&\frac{1}{x+i} \simeq \frac{1}{\lambda i}[1 - (s/\lambda i) + (s/\lambda i)^2 - + \cdots + (-1)^n (s/\lambda i)^n] \equiv \frac{1}{\lambda i} P_n(s)\,, \\
&\frac{1}{x-i} \simeq \frac{-1}{\lambda i}[1 + (t/\lambda i) + (t/\lambda i)^2 + \cdots + (t/\lambda i)^n] \equiv \frac{1}{\lambda i} Q_n(t)\,,
\end{aligned} \tag{1}
$$

where $s = x - (\lambda - 1)i$ and $t = x + (\lambda - 1)i$.

$P_n(s)$ and $Q_n(t)$ approximate series that converge in the circles of radius $|\lambda|$ with centers $s = 0$, $t = 0$ respectively. The intersection of these circles is a lens lying between $-\sqrt{(2\lambda - 1)}$ and $+\sqrt{(2\lambda - 1)}$ on the real axis and between $\pm i$ on the imaginary axis.

If we translate $P_n(s)$ and $Q_n(t)$ to the origin, the expansion

$$\frac{1}{2\lambda}[P_n(s) - Q_n(t)] = \frac{1}{2\lambda}[P_n(x - (\lambda - 1)i) - Q_n(x + (\lambda - 1)i)] \equiv U_n(x)$$

is a polynomial approximation for $1/(1 + x^2)$ in this lens. Furthermore, this polynomial is real and symmetric in x because the coefficients of x^k vanish for k odd, and are real for k even,

$$
\begin{aligned}
U_n(x) &= \frac{1}{2\lambda}\sum_{j=0}^{n}\left[\left(\frac{x + (\lambda - 1)i}{\lambda i}\right)^j + (-1)^j\left(\frac{x - (\lambda - 1)i}{\lambda i}\right)^j\right] \\
&= \frac{1}{2\lambda}\sum_{j=0}^{n}\frac{1}{\lambda^j}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{x}{i}\right)^k(\lambda - 1)^{j-k}[1 + (-1)^k]\,.
\end{aligned}
$$

This approximation can also be obtained by using a theorem by Appell.[2]

By summing the geometric series (1), we find that the error, R_{n+1}, is given by:

$$
\begin{aligned}
R_{n+1} &\equiv \frac{1}{1+x^2} - \frac{1}{2\lambda}[P_n(s) - Q_n(t)] \\
&= \frac{i}{2}\left[\frac{(t/\lambda i)^{n+1}}{\lambda i - t} + (-1)^{n+1}\frac{(s/\lambda i)^{n+1}}{\lambda i + s}\right].
\end{aligned}
$$

This can be re-expressed as

$$R_{n+1} = \frac{i}{2}\left[\frac{\left(\dfrac{x + (\lambda - 1)i}{\lambda i}\right)^{n+1}}{i - x} + (-1)^{n+1}\frac{\left(\dfrac{x - (\lambda - 1)i}{\lambda i}\right)^{n+1}}{i + x}\right]$$

which reduces to

$$R_{n+1} = \left[\left(\frac{x}{\lambda}\right)^2 + \left(\frac{\lambda - 1}{\lambda}\right)^2\right]^{(n+1)/2}\left[\frac{\cos[(n+1)\theta] - x\sin[(n+1)\theta]}{x^2 + 1}\right]$$

where $\theta = \arg((\lambda - 1)/\lambda + xi/\lambda)$.

Below is a comparison between the standard Taylor approximation and the method of this paper. The degree is 27 and $\lambda = 2$. The odd coefficients are zero and the even are given by:

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[2] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math. Soc. Colloq. Publ., vol. 20, Amer. Math. Soc., Providence, R. I., 1965, p. 19.

$.9999999963$
$-.9999984838$
$.9999100044$
$-.9981404170$
$.9821509309$
$-.9075333290$
$.7142059058$
$-.4252770096$
$.1724642254$
$-.4357927665 \times 10^{-1}$
$.6270475686 \times 10^{-2}$
$-.4561170936 \times 10^{-3}$
$.1372024417 \times 10^{-4}$
$-.1080334187 \times 10^{-6}$

x	$\frac{1}{1+x^2}$	$\frac{1}{1+x^2} - T_{27}(x)$	R_{28}
0.0	1.0000000000	0.0	$.37 \times 10^{-8}$
.1	.9900990099	$.99 \times 10^{-28}$	$-.41 \times 10^{-8}$
.2	.9615384615	$.16 \times 10^{-18}$	$.53 \times 10^{-8}$
.3	.9174311927	$.21 \times 10^{-14}$	$-.67 \times 10^{-8}$
.4	.8620689655	$.62 \times 10^{-11}$	$.11 \times 10^{-8}$
.5	.8000000000	$.30 \times 10^{-8}$	$.48 \times 10^{-7}$
.6	.7352941176	$.45 \times 10^{-6}$	$-.24 \times 10^{-6}$
.7	.6711409369	$.31 \times 10^{-4}$	$.34 \times 10^{-6}$
.8	.6097560975	$.12 \times 10^{-2}$	$.22 \times 10^{-5}$
.9	.5524861878	$.29 \times 10^{-1}$	$-.83 \times 10^{-5}$
1.0	.5000000000	.5	$-.31 \times 10^{-4}$
1.1	.4524886878	—	$.93 \times 10^{-4}$
1.2	.4098360656	—	$.61 \times 10^{-3}$
1.3	.3717472119	—	$.39 \times 10^{-3}$
1.4	.3378378378	—	$-.65 \times 10^{-2}$
1.5	.3076923077	—	$-.30 \times 10^{-1}$

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