Proof that Every Integer $\leqq$ 452,479,659 is a Sum of Five Numbers of the Form $Q_x \equiv (x^3 + 5x)/6$, $x \geqq 0$

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Watson [1] proved that every positive integer is a sum of eight tetrahedral numbers $T_x \equiv (x^3 - x)/6, x \geqq 1$, as well as of eight numbers $Q_x \equiv T_x + x = (x^3 + 5x)/6$, $x \geqq 0$, and states that "a similar result holds" for $R_x \equiv T_x - x = (x^3 - 7x)/6$, $x = 0$ or $x \geqq 3$. He also points out that T_x, Q_x and R_x are the only expressions of the form $T_x + Dx$, D integral, which can take the value 1 and permit a universal result for summands $\geqq 0$. In view of the results obtained by the authors in [2], which gave overwhelming evidence that every integer required only five values of T_x, it is interesting to see whether a similar conjecture is justified for Q_x and R_x. There is an immediate lack of comparative interest in R_x whose nonnegative values are 0, 1, 6, 15, 29, 49, 76, 111, . . . because six such addends are needed for the following values of $n \leqq 100$: 11, 26, 40, 54, 69. The remaining form of possible interest, namely Q_x, whose values run 0, 1, 3, 7, 14, 25, 41, 63, 92, 129, 175, . . . does not appear offhand as promising or "nice looking" as T_x to allow every integer to be a sum of five, even though Watson [1] verified that for $n \leqq 210$. However, it was quite a surprise to find that, defining an "exceptional number" as a number requiring more than four summands, when the test was made up to 1,000,000, for Q_x there were vastly fewer exceptional numbers than for T_x. Thus, whereas in [1] the authors found as many as 241 exceptional numbers for T_x, the largest being as high as 343,867, in the present investigation only 21 exceptional numbers were found for Q_x, the largest being only 28415.

Following are the only numbers $\leqq$ 1,000,000 that are not the sum of four numbers Q_x:

TABLE I

Exceptional numbers $\leqq$ 1,000,000

37	372	2861	5898	28415
115	541	3340	6522	
122	1805	4148	6529	
166	2532	4980	7557	
334	2773	5157	10915	

From Table I it is immediately apparent that every integer $\leqq$ 1,000,000 is a sum of five numbers Q_x. The size of the gap between 28415 and 1,000,000 enables us to find a number N much larger than 1,000,000 for which every $n \leqq N$ is a $\sum_5$, or sum of five numbers Q_x. The basic principle in finding such an N is not new, having been employed by both Watson [1] and the authors [2] in a sort of loose manner. Apparently the sharpest form of that principle is formulated in the lemma below, which is also applicable to T_x and a wide class of similar functions.

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Lemma. *Let E be the largest exceptional number found in a test extending through $L > E$. Let x be the largest x for which $\Delta Q_x \equiv Q_{x+1} - Q_x < I = L - E$. Suppose that from the tabulation of exceptional numbers it is apparent that every $n \leqq E$ is a $\sum_5$. Then any $n \leqq N \equiv Q_{x+1} + L$ is a $\sum_5$.*

Proof. For $n \leqq L$, the result is in the hypothesis. If $L < n < Q_{x+1}$,* n − some Q_i, $i \leqq x - 1$, will come closest above L, so that $n - Q_{i+1} \leqq L$. Since $Q_{i+1} - Q_i \leqq Q_x - Q_{x-1} < Q_{x+1} - Q_x < I$, $n - Q_{i+1}$ falls within the interval (E, L), so that n is a $\sum_5$. For $n = Q_{x+1}$, or $n = N \equiv Q_{x+1} + L$, the result is immediate, since L is the largest tested $\sum_4$. For $Q_{x+1} < n < N \equiv Q_{x+1} + L$, since $n - Q_{x+1} < L$, if $n > L$, n − some Q_i, $i \leqq x$, comes closest above L, so that $n - Q_{i+1} \leqq L$, and from $Q_{i+1} - Q_i \leqq Q_{x+1} - Q_x < I$, $n - Q_{i+1}$ falls within the interval (E, L), so that n is a $\sum_5$. Q.E.D.

If we try to push the lemma to apply beyond $N \equiv Q_{x+1} + L$, say up to $Q_{x+1} + L + e$, it fails because for some n beyond $Q_{x+1} + L$ the i making $n - Q_i$ come closest above L must be $\geqq x + 1$, and we have no assurance that $n - Q_{i+1}$ falls within the interval (E, L). The reason is that $Q_{i+1} - Q_i \geqq Q_{x+2} - Q_{x+1} \geqq I$, and if the number by which $Q_{x+2} - Q_{x+1}$ exceeds I is greater than the number by which $n - Q_i$ exceeds L, then $n - Q_{i+1} < L - I = E$.

Applying this lemma to Q_x, where the condition $\Delta Q_x < I$ is equivalent to $x^2 + x + 2 < 2I$, from Table I, $E = 28415$, $L = 1{,}000{,}000$, $2I = 2(L - E) = 1{,}943{,}170$, and $x = 1393$ is the largest x for which $x^2 + x + 2 = 1{,}941{,}844 < 2I$. Thus, every $n \leqq N = Q_{1394} + L = 451{,}479{,}659 + 1{,}000{,}000 = 452{,}479{,}659$ is a $\sum_5$.

We may apply this lemma also to T_x for which it was found in [1] that $E = 343{,}867$ when the test for exceptional numbers extended as far as $L = 1{,}043{,}999$. From the tabulation of exceptional numbers in [1] it was apparent that every $n \leqq E$ is a $\sum_5$ for T_x. The condition $\Delta T_x < I$ is equivalent to $x^2 + x < 2I$. The largest x satisfying $x^2 + x < 2I = 2(L - E) = 1{,}400{,}264$ is $x = 1182$ ($x = 1183$ for which $x^2 + x = 1{,}400{,}672$ is just slightly too big). Thus, every $n \leqq T_{1183} + L = 275{,}932{,}384 + 1{,}043{,}999 = 276{,}976{,}383$ is a sum of five tetrahedral numbers. This is a substantial improvement over the 250,000,000 obtained previously in [1] from a looser use of the main idea in the above lemma instead of its optimally sharpened formulation given above.

Table I was calculated with a program similar to that employed in [1] to find exceptional numbers with respect to T_x. The first run, using 1,000,000 words of memory was done on an IBM 360-75. The print-out was checked by using a different machine, an IBM 360-65, and by varying the code to perform in five groups of 200000 words of memory.

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1. G. L. Watson, "Sums of eight values of a cubic polynomial," *J. London Math. Soc.*, v. 27, 1952, pp. 217–224. MR **14**, 250.

2. H. E. Salzer & N. Levine, "Table of integers not exceeding 10 00000 that are not expressible as the sum of four tetrahedral numbers," *MTAC*, v. 12, 1958, pp. 141–144. MR **20** #6194.

* Q_{x+1} may be less than L when I is small. But the result for the case $Q_{x+1} < n < L$ is contained in the hypothesis.