Zeros of Sections of the Zeta Function. II

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1. Recapitulation. Paul Turán proved theorems connecting the locations of zeros of the Dirichlet polynomials

$$(1) \quad \zeta_N(s) = \sum_{n=1}^N n^{-s}$$

with the Riemann hypothesis. Let $s = \sigma + it$. One such theorem is that if all the zeros of every $\zeta_N(s)$ had real parts $\sigma \leq 1$, then the Riemann hypothesis would be true. Unfortunately, this very simple condition, which could perhaps have been worked with in an inductive fashion, was shown by Haselgrove [1] to fail infinitely often. In part I of this paper (Spira [2]), a description was given of a calculation of zeros of $\zeta_N(s)$ for various N up to 10^{10} . No zero with $\sigma > 1$ was found.

In this concluding part, we apply in Section 2 generalizations of basic theorems of Bohr to $\zeta_N(s)$, and find g.l.b. $|\zeta_N(s)|$ for $\sigma \geq 1$ and $N \leq 5$. In Section 3 we discuss a confirmation of Haselgrove's proof, and report on related calculations. In Section 4, we describe machine proofs that $\zeta_N(s)$ has no zeros with $\sigma \geq 1$ for $N \leq 9$, and proofs of the existence of such zeros for a variety of small N starting with $N = 19$. Finally, we discuss our attempts at finding such zeros.

2. Applications of Bohr's Theorems. Let p_j be the j th prime, so $p_1 = 2$. For $n > 1$, let $r_{n,j}$ be the highest power of p_j dividing n , and let q_n be the index of the largest prime dividing n . We then can write (1) as

$$(2) \quad \zeta_N(s) = \sum_{n=1}^N n^{-\sigma} \exp(-it(r_{n,1} \log 2 + r_{n,2} \log 3 + \cdots + r_{n,q_n} \log p_{q_n})),$$

where we interpret the sum in parentheses as 0 when $n = 1$. Introducing the new variables $x_j = t \log p_j$, we now define the *companion function* of $\zeta_N(s)$:

$$(3) \quad F_N = F_N(\sigma, x_1, x_2, \cdots) = \sum_{n=1}^N n^{-\sigma} \exp(-i(r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n})).$$

Since the $\log p_j$ are linearly independent over the rationals, and the $r_{n,j}$ are integers, we can apply generalizations (Spira [3]) of theorems of Bohr [4]. We can conclude first of all the set of values of $\zeta_N(s)$ for $\sigma \in (a, \infty)$ and $t \in (-\infty, \infty)$ is identical with the set of values of $F_N(\sigma, x_1, \cdots)$ where $\sigma \in (a, \infty)$ and each x_j runs independently over $[0, 2\pi)$. Thus, taking $a = 1$, $\zeta_N(s) = 0$ for some $\sigma > 1$ and some t if and only if $F_N(\sigma, x_1, \cdots) = 0$ for some (possibly different) $\sigma > 1$, and some values of the variables $x_1, x_2, \cdots$. We remark also that if $\zeta_N(s)$ has one zero with $\sigma > 1$, it has infinitely many (Spira [3]). Secondly, we can conclude that the values of

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$F_N(\sigma, x_1, \dots)$ where σ runs over a closed interval, and the x_j 's run independently over $[0, 2\pi)$, form a closed set. Thus, for the σ -interval $[1, 2]$, the distance of this set to the origin is a well-defined constant, d_N .

Now, from the last two lines of p. 542 of I, we have that for $\sigma > 1$,

$$(4) \quad |\zeta_N(s)| \geq 1 - 2^{-\sigma}(\sigma + 1)/(\sigma - 1)$$

and this last function increases with σ . At $\sigma = 2$, its value is $1/4$, so for $\sigma \geq 2$, $|\zeta_N(s)| \geq 1/4$. Hence, for $\sigma > 2$, $|F_N| \geq 1/4$, and if $d_N \leq 1/4$, d_N is the minimum distance of F_N to the origin for all $\sigma \geq 1$. For $5 \leq N \leq 50$ it turns out that $d_N \leq 1/4$, but it is also true that for $N \leq 4$, d_N is the minimum distance for all $\sigma \geq 1$.

To discuss these $N \leq 4$ cases, and for the sequel, we define:

$$(5) \quad \pi_N = \text{the set of primes } p \text{ satisfying } N/2 < p \leq N,$$

$$(6) \quad \zeta_N^*(s) = \sum_{n=1; n \notin \pi_N}^N n^{-s}, \quad \pi_N(s) = \sum_{n \in \pi_N} n^{-s},$$

$$(7) \quad F_N^* = F_N^*(\sigma, x_1, \dots) = \text{the companion function of } \zeta_N^*(s),$$

$$(8) \quad P_N = P_N(\sigma_1, x_1, \dots) = \text{the companion function of } \pi_N(s).$$

We have $\zeta_N(s) = \zeta_N^*(s) + \pi_N(s)$, $F_N = F_N^* + P_N$. Note that σ is the only variable F_N^* and P_N have in common. The first four $\zeta_N^*(s)$ are $1, 1, 1, 1 + 2^{-s} + 4^{-s}$. In general, we have

$$(9) \quad \begin{aligned} \zeta_N^*(s) &= \zeta_{N-1}^*(s) \text{ if } N \text{ is a prime,} \\ \zeta_N^*(s) &= \zeta_{N-1}^*(s) + (N/2)^{-s} + N^{-s} \text{ if } N \text{ is twice a prime,} \\ \zeta_N^*(s) &= \zeta_{N-1}^*(s) + N^{-s} \text{ otherwise,} \end{aligned}$$

and there are similar equations for F_N^* . It is clear that the variables in P_N can be chosen so that P_N is a vector in any assigned direction of length $\sum_{p \in \pi_N} p^{-\sigma}$. We avoid the general question of whether values of $\sigma, x_1, \dots$ which minimize $|F_N^*|$ also minimize $|F_N|$ after suitable selection of the variables in P_N . For $N \leq 5$, this turns out to be true, since the minima for $\sigma \geq 1$ of $|F_N^*|$ lie at the extreme $\sigma = 1$. The remarks above on the relations of the sets of values of $\zeta_N(s)$ and F_N also carry over to the functions $\zeta_N^*(s)$ and F_N^* .

For $N = 1$, it is trivial that $d_N = 1$. A short calculation shows that for $N = 2$, we obtain a minimum $d_2 = 1/2$ at $\sigma = 1, x_1 = \pi$, and for $N = 3$, $d_3 = 1/6$ and is attained at $\sigma = 1, x_1 = x_2 = \pi$. It is also easy to see that these three minima hold for $\sigma \geq 1$. We sketch the calculation of d_4 .

We consider first F_4^* . We have $|F_4^*| \geq 1 - 2^{-\sigma} - 4^{-\sigma}$, which is > 0 for $\sigma \geq 1$. Thus, for some σ in $[1, 3]$ and for some x_1 in $[0, 2\pi)$, $|F_4^*|$ takes on a positive minimum. In finding such σ and x_1 , we can study $g(\sigma, x_1) = |F_4^*|^2$ instead of $|F_4^*|$. At a minimum, we must have $\partial g / \partial x_1 = 0$, (since g is periodic of period 2π in x_1 , we do not have to consider extreme values in that variable). A short calculation gives

$$(10) \quad g(\sigma, x_1) = 1 + 2^{-2\sigma} + 4^{-2\sigma} + 2^{1-\sigma}[1 + 4^{-\sigma}] \cos x_1 + 2 \cdot 4^{-\sigma} \cos 2x_1,$$

$$(11) \quad \partial g / \partial x_1 = -2^{1-\sigma}(\sin x_1)[1 + 4^{-\sigma} + 2^{2-\sigma} \cos x_1].$$

Thus, $\partial g / \partial x_1 = 0$ if and only if $x_1 = 0$ or π , or $\cos x_1 = -(2^\sigma + 2^{-\sigma})/4$. For $x_1 = \pi$,

setting $x = 2^{-\sigma}$, we obtain $g(\sigma, \pi) = (x^2 - x + 1)^2$. It is easily seen that $x^2 - x + 1 > 0$, and has its minimum at $x = 1/2$ or $\sigma = 1$, where $|F_4^*| = 3/4$.

For $x_1 = 0$, we obtain $g(\sigma, 0) = (x^2 + x + 1)^2$, which is greater than $(x^2 - x + 1)^2 (=g(\sigma, \pi))$ if $x > 0$, which we can assume as $x = 2^{-\sigma}$. Thus, the minimum cannot be attained for $x_1 = 0$.

In the final case, using $\cos 2x_1 = 2 \cos^2 x_1 - 1$, if $\cos x_1 = -(2^\sigma + 2^{-\sigma})/4$, we obtain $\cos 2x_1 = (4^\sigma - 6 + 4^{-\sigma})/8$ and $g(\sigma, x_1) = \frac{3}{4}(1 - 4^{-\sigma})^2$, which is least at $\sigma = 1$, and indeed gives the least possible $|F_4^*|$ of $3\sqrt{3}/8$, at $\cos x_1 = -5/8$. This gives $d_4 = (9\sqrt{3} - 8)/24$ and $d_5 = (45\sqrt{3} - 64)/120$, where we choose x_2 and x_3 so that the vectors $\exp(-ix_2)$ and $\exp(-ix_3)$ point opposite to $F_4^*(1, \arccos(-5/8))$.

If $F_N(\sigma, x_1, \dots) = 0$, and if the appropriate Jacobian does not vanish, we can solve the equation for σ , and thus obtain a σ interval in which F_N vanishes. The zeros of $\zeta_N(s)$ will have real parts dense in this interval. The empirical results suggest the conjecture that to each $\zeta_N(s)$ there is a single such interval, though it could possibly arise from overlapping F_N σ -intervals.

If we take F as the companion function of a general Dirichlet series, it is not clear what we should do about such solvability conditions, since F will have infinitely many variables.

3. Calculations Related to Haselgrove's. The Dirichlet polynomial

$$(12) \quad L_N(s) = \sum_{n=1}^N \frac{\lambda(n)}{n^s}$$

where $\lambda(n)$ is Liouville's function, is equivalent in the sense of Bohr [4] to $\zeta_N(s)$, and hence assumes the same set of values in any half plane $\sigma > \sigma_0$. If s is real and large, then $L_N(s)$ is near 1. Thus, if $L_N(1) < 0$, then there would be a real root of $L_N(s)$ larger than 1, and hence also a root of $\zeta_N(s)$ with $\sigma > 1$.

The author found that $L_N(1) > 0$ for $N \leq 824,000$, and found $L_{293}(1) = .0051122775$, $L_{1000}(1) = .0289948068$, values slightly different from those appearing in Turán [6]. The lowest value obtained was $L_{96862} = .00011996$. R. Sherman Lehman's [5] values of $L_N(0)$ for $N = 200,000(200,000)800,000$ were verified.

To study $L_N(1)$ further, one may use analytic expressions, derived by the calculus of residues (Haselgrove [1], Lehman [5]). Indeed, setting

$$(13) \quad L_1(x) = \sum_{n \leq x} \frac{\lambda(n)}{n}$$

the expression

$$(14) \quad B_T(u) = \frac{-1}{\zeta(\frac{1}{2})} + \sum_{|\gamma_n| < T} \frac{\zeta(2\rho_n)}{(\rho_n - 1)\zeta'(\rho_n)} \exp(i\gamma_n u),$$

where $\rho_n = \frac{1}{2} + i\gamma_n$ are roots of $\zeta(s)$, under various unproved hypotheses, can be shown to represent $e^{u/2} L_1(e^u)$, with some blurring. The focusing improves as T increases. For example, for $T = 200$ and $u \leq 2$, one can readily see rather sharp changes (without a Gibbs effect) as $L_1(x)$ makes a step. Lehman [5] used a function similar to (14) to successfully guess where $L(x) = \sum_{n \leq x} \lambda(n)$ changed sign.

An expression the same as (14) but with the factor $(1 - \gamma_n/T)$ inside the sum was used by Haselgrove to show that $L_1(x)$ is negative infinitely often. We designate

this sum by $B_T^*(u)$. For corresponding sums for $L(x)$ we use the notation $A_T(u)$ and $A_T^*(u)$, as used by Lehman [5]. Finally, by $C_T(u)$ and $C_T^*(u)$ we mean the corresponding sums for the function $M(x) = \sum_{n \leq x} \mu(n)$, where $\mu(n)$ is the Möbius function. We have

$$(15) \quad C_T(u) = \sum_{|\gamma_n| < T} \frac{\exp(i\gamma_n u)}{\rho_n \zeta'(\rho_n)},$$

which represents, hopefully, $e^{-u/2} M(e^u)$. Formulas for $A_T(u)$ and $A_T^*(u)$ can be found in Lehman [5], and for $B_T^*(u)$ and $C_T^*(u)$ in Haselgrove [1].

All six of these functions were calculated in double precision for $T = 100$, $u = 0(.01)500$, and for $T = 200, 500$, and 1000 in selected ranges. The coefficients for these functions were calculated in double precision, calculating first improved γ_n from the 6D values in Haselgrove and Miller [7]. We first discuss the tables in Haselgrove [1]. Write $\alpha_n = \zeta(2\rho_n)/\rho_n \zeta'(\rho_n)$. In Table I, the γ_n are correct to within 3 units in the 10th significant figure. For the first six $|\alpha_n|$, terminal digits 19,8,5,3,2,993, were obtained rather than 23,6,8,5,4,878. For $n = 9, 15, 18, 34, 48$ the terminal digits 4,4,6,6,3 were obtained rather than 5,3,5,7,4. The quantity $(ph \alpha_n)/\pi$ was not checked. The values of $A_{1000}^*(u)$ in Table II were confirmed within 1 unit except for the five values starting with 831.837, which are two units low. Also, the value at 831.857 was found to be $-.06320$. In Lehman's [5] paper the value $A_{1000}(814.492)$ was found to have terminal digit 0, and $A_{1000}(831.847)$ was found to be .0049448.

Two new places were found where $A_T^*(u) > 0$: $A_{1000}^*(310.8276) = .0109$, $A_{1000}^*(384.690) = .0316$. High maxima also occur at $u = 33.495, 44.591$ and 214.404 .

For $L_1(x)$, the author found that $B_{1000}^*(853.853) = -.0321$ and $B_{1000}^*(996.980) = -.0450$ and $B_{1000}^*(996.981) = -.0457$, confirming Haselgrove's proof. It was also found that $B_{1000}^*(171.4938) = -.0009$ and $B_{1000}^*(331.9602) = -.0170$, giving two new places where this function is negative. Low minima occur at $u = 43.897, 54.624, 124.843, 188.830$, and 437.758 .

To disprove Mertens hypothesis it would be sufficient to find u and T such that $|C_T^*(u)| > 1$. No such values were found. Table I gives places where $|C_{1000}^*(u)|$ rises above .5.

TABLE I. *Approximate Values for $e^{-u/2} M(e^u)$*

u	$C_{1000}^*(u)$	u	$C_{1000}^*(u)$
22.7730	+ .5003	441.5100	+ .5145
43.8965	- .5199	480.6430	+ .5069
97.5260	+ .5355	814.4910	+ .5061
310.8258	+ .5301	853.852	- .6027

4. Machine proofs. We first describe the proofs that $\zeta_N(s) \neq 0$ for $\sigma \geq 1$, $N \leq 9$. The idea of such a proof is, for one variable, based on the formulas

$$(16) \quad |f(x_0 + h)| \geq |f(x_0)| - \max |h| \cdot \max |f'(\xi)|,$$

$$(17) \quad |f(x_0 + h)| \geq |f(x_0)| - \max |h| |f'(x_0)| - (\max |h|^2/2!) \max |f''(\xi)|,$$

which are easily derivable from the Taylor's expansion under suitable restrictions on f . Thus, from formula (16), if $|f'(\xi)|$ is suitably bounded, and $f(x_0) \neq 0$, we can conclude that $f(x) \neq 0$ for a small interval about x_0 . Formula (17) is useful when $|f(x)|$ has a small minimum, as in the cases we consider. We then get some help from $|f'(x_0)|$ being small near the minimum, and from the $|h|^2$ in the next term. These formulas easily generalize to the case of f being a real or complex function of several real variables. Roundoff also must be taken into account.

In our particular case, we can take advantage of the special nature of our functions F_N , and consider only the variables appearing in F_N^* . If we take the variables $\sigma, x_1, \dots$, to lie in a cube C , we have at any point in C ,

$$(18) \quad |F_N| \geq |F_N^*| - |P_N| \geq \text{g.l.b.}_C |F_N^*| - \text{l.u.b.}_C |P_N|.$$

Thus, $|F_N| > 0$ in C provided $\text{g.l.b.}_C |F_N^*| > \text{l.u.b.}_C |P_N|$ ($= \sum_{p \in \pi_N} p^{-\min \sigma}$). Now we can apply formulas of the type (16) or (17). Write $F_N^* = u + iv$. A formula corresponding to (16) is

$$(19) \quad \begin{aligned} & |F_N^*(\sigma + h, x_1 + h_1, \dots)| \\ & \geq |F_N^*(\sigma, x_1, \dots)| - \max |h| \left[\max \left| \frac{\partial u}{\partial \sigma} \right| + \max \left| \frac{\partial v}{\partial \sigma} \right| \right] \\ & \quad - \max |h_j| \left[\sum_i \left(\max \left| \frac{\partial u}{\partial x_i} \right| + \max \left| \frac{\partial v}{\partial x_i} \right| \right) \right], \end{aligned}$$

and one corresponding to (17) is

$$(20) \quad \begin{aligned} & |F_N^*(\sigma + h, x_1 + h_1, \dots)| \geq |F_N^*(\sigma, x_1, \dots)| \\ & \quad - \max |h| \left[\left| \frac{\partial u}{\partial \sigma}(\sigma, x_1, \dots) + i \frac{\partial v}{\partial \sigma}(\sigma, x_1, \dots) \right| \right] \\ & \quad - \max |h_j| \left[\sum_i \left| \frac{\partial u}{\partial x_i}(\sigma, x_1, \dots) + i \frac{\partial v}{\partial x_i}(\sigma, x_1, \dots) \right| \right] \\ & \quad - \frac{\max |h|^2}{2!} \left[\max \left| \frac{\partial^2 u}{\partial \sigma^2} \right| + \max \left| \frac{\partial^2 v}{\partial \sigma^2} \right| \right] \\ & \quad - \frac{\max |h_j|^2}{2!} \left[\sum_{i,j} \left(\max \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| + \max \left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right| \right) \right] \\ & \quad - \max |h h_j| \left[\sum_i \left(\max \left| \frac{\partial^2 u}{\partial \sigma \partial x_i} \right| + \max \left| \frac{\partial^2 v}{\partial \sigma \partial x_i} \right| \right) \right]. \end{aligned}$$

It was not possible to avoid using (20). The expressions in (20) can be simplified. We have, writing π_N^* as $[1, N] - \pi_N$,

$$(21) \quad \max \left| \frac{\partial u}{\partial \sigma} \right| + \max \left| \frac{\partial v}{\partial \sigma} \right| \leq 2 \sum_{n \in \pi_N^*} (\log n) n^{-\min \sigma}$$

and using the notation of (2),

$$(22) \sum_i \left(\max \left| \frac{\partial u}{\partial x_i} \right| + \max \left| \frac{\partial v}{\partial x_i} \right| \right) \leq 2 \sum_{n \in \pi_N^*} (r_{n,1} + r_{n,2} + \cdots + r_{n,q_n}) n^{-\min \sigma}.$$

For formula (20) we can use

$$(23) \max \left| \frac{\partial^2 u}{\partial \sigma^2} \right| + \max \left| \frac{\partial^2 v}{\partial \sigma^2} \right| \leq 2 \sum_{n \in \pi_N^*} (\log n)^2 n^{-\min \sigma},$$

$$(24) \sum_{i,j} \left(\max \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| + \max \left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right| \right) \leq 2 \sum_{n \in \pi_N^*} (r_{n,1} + \cdots + r_{n,q_n})^2 n^{-\min \sigma},$$

$$(25) \sum_i \left(\max \left| \frac{\partial^2 u}{\partial \sigma \partial x_i} \right| + \max \left| \frac{\partial^2 v}{\partial \sigma \partial x_i} \right| \right) \leq 2 \sum_{n \in \pi_N^*} (\log n) (r_{n,1} + \cdots + r_{n,q_n}) n^{-\min \sigma}.$$

To save computation, the machine proof was attempted simultaneously for those N 's for which the F_N^* have the same number of variables x_j . The process of proof starts with the cube $\sigma: [1, 2]$, $x_j: [0, 2\pi]$, $j = 1, \cdots$. One could limit one of the x_j 's to $[0, \pi]$, since $|F_N(\sigma, x_1, x_2, \cdots)| = |F_N(\sigma, 2\pi - x_1, 2\pi - x_2, \cdots)|$. One breaks the cube into smaller cubes, and checks by (19) and then (20) to see if $|F_N| > 0$ throughout each smaller cube. The smaller cubes not satisfying this are further refined. The final program used integer pair coordinates for the x_j 's, $(m, n) = 2m\pi/n$, where n was chosen a power of 2. All the cubes of a size were examined together, so that the sines and cosines could be computed just once for a given set of cubes. Also, the coordinates of $\min |F_N|$ were saved. If one attempts to use condition (19) alone, the number of cubes rises to an impractical level.

Table II gives results of the proofs for $N = 6$ to 9 at several stages. The roundoff leeway was taken as 10^{-5} . Column 1 contains the σ -width of the cubes. Column 2 has the number of division of 2π for the x_j edge length. Column 3 gives the σ -coordinate of the center of the cube with minimum $|F_N|$, which turned out to be the same for $N = 6$ to 9. The next three columns give $\min |F_6|$ and the integer first coordinates of x_1 and x_2 where this minimum was attained. The second coordinate is twice the value in column 3, (since we are calculating at the *centers* of cubes, the program needs a mesh half the edge). For example, in the first row, we are considering cubes of σ -width .25, and x_j width $2\pi/16$. The $\min |F_6|$ is .28384 and is attained at $x_1 = 11 \cdot (2\pi/32)$, $x_2 = 15 \cdot (2\pi/32)$. The next nine columns give corresponding data for $N = 7, 8, 9$. The last two columns give the letters Y and N according to whether a proof was obtained or not. The letters are in order corresponding to $N = 6$ to 9. The first of these columns gives the proof results obtained using (19) above, and the second, the results obtained using (20) also. The results in the $N = 8$ columns indicate that the proof first sought out a secondary minimum, which was later calculated, and then finally found a cube where there was a lower minimum as the mesh refined. Each set of cubes was processed completely to find $\min |F_N|$, rejecting when possible, using the current $\min |F_N|$, and then a second pass made using the final values of $\min |F_N|$ to reject further cubes. The program also saved cubes where there was a possibility of lower $\min |F_N|$ within the cube. The total run time was less than 20 minutes.

The $\min |F_N|$ in the table were recalculated in double precision, and a separate confirmatory calculation was performed along $\sigma = 1$ which found x_1 and x_2 which minimized $|F_N(1, x_1, x_2)|$ for a mesh of $2\pi/650$. Values of x_1, x_2 which gave $|F_N(1, x_1, x_2)|$ slightly greater than the minimum were also saved, and studied, and

TABLE II
Proof Data, N = 6, 7, 8, 9

σ mesh	x mesh	σ min	$N = 6$		$N = 7$		$N = 8$		$N = 9$		Pr. 1	Pr. 2
.250	16	1.125	.28384	11 15	.17182	11 15	.23154	15 15	.25615	15 11	NNNN	NNNN
.125	32	1.0625	.23256	21 31	.10607	21 31	.17370	31 31	.20056	31 21	NNNN	YNY Y
.0625	64	1.03125	.20289	43 61	.06846	43 61	.14455	63 63	.16898	61 43	YNNN	YYYY
.03125	128	1.015625	.18778	85 121	.04920	85 121	.12941	71 123	.15287	121 85	YNY Y	YYYY

TABLE III
Local Minima of $|F_N^*|$ on $\sigma = 1$ and Angles where Attained

N	local mini- mum	$\sum_{p \in \pi_N p^{-1}}$ less	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1	1	+ 1									
2	1	+ 1/2									
3	1	+ 1/6									
4	$3\sqrt{3/8}$	+ .316186*	2.245928†								
5		+ .116186*									
6	.372759	+ .172759	2.091870	2.953272							
7		+ .029902									
8	.452582	+ .109724	1.721394	3.044474							
9	.479230	+ .136372	2.986152	2.062967							
10	.286430	+ .143573	1.739896	2.496772	3.168891						
11		+ .052664									
12	.330873	+ .097107	1.633311	2.376010	3.218477						
13		+ .020184									
14	.173630	+ .005798	1.586010	2.404622	3.250068	3.250068					
15	.183411	+ .015578	1.626395	2.278784	2.974448	3.283475					
16	.232361	+ .064529	1.471382	2.331302	3.030376	3.321327					

17		+	.005705																	
18	.264527	+	.037871	1.462765	2.159240	3.059133	3.346347													
19		-	.014760																	
20	.287264	+	.007976	1.397744	2.199895	2.891836	3.378705													
21	.287901	+	.008614	1.429136	2.110608	2.917566	3.114078													
22	.180110	-	.008267	1.399971	2.127013	2.932031	3.131736	3.411964												
23		-	.051746																	
24	.210415	-	.021442	1.341302	2.094990	2.956244	3.156690	3.430748												
25	.216213	-	.015643	1.369312	2.126559	2.614426	3.174392	3.452011												
26	.1232	-	.0317	1.3466	2.1355	2.6293	3.1880	3.4641	3.4641											
27	.1472	-	.0078	1.3704	1.9584	2.6683	3.2073	3.4750	3.4750											
28	.1595	+	.0046	1.3294	1.9880	2.6900	3.0328	3.4947	3.4947											
29		-	.0299																	
30	.1741	-	.0153	1.3277	1.9442	2.5888	3.0524	3.5086	3.5086											
31		-	.0476																	
32	.2003	-	.0214	1.2591	1.9723	2.6129	3.0663	3.5224	3.5224											
33	.1957	-	.0260	1.2812	1.9135	2.6242	3.0796	3.2710	3.5297											
34	.1225	-	.0403	1.2609	1.9239	2.6351	3.0879	3.2797	3.5382	3.5382										
35	.1249	-	.0379	1.2818	1.9417	2.5294	2.9352	3.2905	3.5516	3.5515										

* $3\sqrt{3}/8$ less $1/3$ and then less $1/5$.† are $\cos(-5/8)$.

TABLE III (Continued)

36	.1457	— .0171	1.2611	1.8813	2.5547	2.9571	3.3053	3.5597	3.5592		
37		— .0441									
38	.0798	— .0573	1.2463	1.8848	2.5610	2.9645	3.3136	3.5698	3.5716	3.5702	
39	.0743	— .0629	1.2608	1.8496	2.5738	2.9724	3.3166	3.3138	3.5631	3.5669	
40	.0910	— .0462	1.2219	1.8684	2.5191	2.9883	3.3300	3.3288	3.5819	3.5822	
41		— .0706									
42	.1018	— .0599	1.2195	1.8416	2.5328	2.8826	3.3419	3.3426	3.5926	3.5940	
43		— .0831									
44	.1044	— .0805	1.1909	1.8551	2.5440	2.8990	3.1698	3.3567	3.6064	3.6061	
45	.1173	— .0675	1.2064	1.8015	2.4921	2.9096	3.1784	3.3622	3.6074	3.6076	
46	.0622	— .0792	1.2010	1.8018	2.4845	2.9079	3.1822	3.3727	3.6229	3.6238	3.6259
47		— .0805									
48	.0793	— .0834	1.1624	1.7952	2.5120	2.9342	3.1987	3.3798	3.6215	3.6205	3.6191
49	.0808	— .0819	1.1836	1.8184	2.5186	2.7110	3.1941	3.3758	3.6193	3.6229	3.6235
50	.0906	— .0721	1.1840	1.8268	2.3905	2.7145	3.2001	3.3875	3.6373	3.6392	3.6405

other checking computations were performed.

It would not be difficult now to repeat the computational proof.

Minima of $|F_{10}^*|$ and $|F_{12}^*|$ were also sought with a mesh of $2\pi/150$, and of $|F_{14}^*|$ through $|F_{21}^*|$ with a mesh of $2\pi/50$. Local minima were then sought by a minimum search program, using starting values obtained from these initial searches. The searching program simply stepped each variable by a quantity h , halving h , for a number of times, as the minimum was found with mesh h . For $N > 21$, further minima were also sought, using as starting values the quantities x_j at which $|F_{N-1}^*|$ was a minimum, taking π for the initial value of any new variable.

Table III gives the results of these computations. The x_j are given in radians. The quantities for $N \leq 35$ should be accurate to one unit in the last place, as they were computed in double precision. For $35 < N \leq 50$, the quantities should be correct to within 3 units in the last place. It is not claimed that the local minima are the true absolute minima of $|F_N^*|$.

Since $|F_N^*| \rightarrow 1$ and $\sum_{p \in \pi_N} p^{-\sigma} \rightarrow 0$ as $\sigma \rightarrow \infty$, if we find values of the variables $x_1, \dots$, so that $|F_N^*(1, x_1, \dots)| - \sum_{p \in \pi_N} 1/p < 0$, then $F_N(\sigma, x_1, \dots)$ has a root with $\sigma > 1$. Thus, from Table III, for $N = 19, 23$ to 27 and 29 to 50 , F_N has roots with $\sigma > 1$.

It is possible that such searches for minima could be speeded by using a gradient method. If this were so, one could write a general program for computing successive minima of the $|F_N^*|$ and study whether this situation of zeros for $\zeta_N(s)$ for $N \geq 29$ continues to hold. We remark that it follows from Rosser and Schoenfeld's [8] inequalities for $\sum_{p \leq x} 1/p$ that $\sum_{p \in \pi_N} 1/p$ is approximately $\log 2/\log N$ and tends to zero as $N \rightarrow \infty$.

To find a zero of $\zeta_{19}(s)$ with $\sigma > 1$, one can seek t for which $t \log p_j \equiv x_j + \epsilon_j \pmod{2\pi}$, $j = 1, \dots, 4$, where the x_j have the values of Table III for $N = 19$ and ϵ_j is small. For $j = 1$, we can make $\epsilon_1 = 0$ by choosing $t = (x_1 + 2k\pi)/\log 2$, $k = 0, \pm 1, \pm 2, \dots$. Then for each k we can check if $t \log p_j \equiv x_j \pmod{2\pi}$ within ϵ_j , for $j = 2, 3, 4$, where we preassign the ϵ_j 's. If one accumulated a sufficiently large number of such cases, and if 11^{-s} , 13^{-s} , 17^{-s} and 19^{-s} are randomly pointed, one could hope to find a case of a zero of $\zeta_{19}(s)$ beyond $\sigma = 1$. Efforts along these lines produced the zero of $\zeta_{23}(s)$ $.9705 + i 10716133.0062$, which has real part somewhat beyond that of the lowest zero $.9325 + i 1.6975$.

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