REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

With this issue, we introduce a decimal system of subject indexing for our reviews. We plan to use the following coarse subject listing in the January, April and July issues and to define and use a more detailed subject listing for the annual index to appear in October:

1. Biography and Bibliography (History)
2. Selected Topics in Numerical Analysis
   - 2.05 Approximation Theory
   - 2.10 Numerical Integration
   - 2.15 Numerical Differentiation
   - 2.20 Roots of Equations
   - 2.25 Evaluation of Series
   - 2.30 Continued Fractions
   - 2.35 Iteration Methods, Acceleration Techniques
   - 2.40 Differences, Divided Differences
   - 2.45 Algorithms, General Theory
   - 2.50 Inequalities
   - 2.55 Stability of Computation, Significance Arithmetic
3. Linear Algebra
4. Ordinary Differential Equations
5. Partial Differential Equations
6. Other Functional Equations
7. Special Functions
8. Probability and Statistics
9. Number Theory
10. Algebra and Combinatorial Theory
11. Geometry
12. Computers and Other Aids to Computation
13. Applications
    - 13.05 Physical and Chemical Sciences
    - 13.10 Astronomy, Astrophysics
    - 13.15 Engineering Sciences
    - 13.20 Earth Sciences, Atmospheric Sciences
    - 13.25 Biology and the Behavioral Sciences
    - 13.30 Economics and the Social Sciences
    - 13.35 Information Theory, Automata, Control Theory, Cybernetics
    - 13.40 Management Problems, Data Analysis and Processing
    - 13.45 Actuarial Science
    - 13.50 Humanities

1 [2.05].—R. S. Guter, L. D. Kudryavtsef & B. M. Levitan, *Elements of the Theory of Functions*, translated by H. F. Cleaves, edited by I. N. Sneddon, Pergamon Press, Oxford, 1966, xii + 219 pp., 23 cm. Price $8.50.

This book, a translation of the original Russian *Elementiy Teorii Funktsii* published in 1963 by Fizmatgiz in Moscow, should prove useful to people working in approximation theory.

The material comprises definitions, theorems and discussions. There are no proofs. The book has the strength and the weaknesses of the survey format.

Chapter I (pp. 1–85) was written by R. S. Guter and is entitled "Functions of a Real Variable." It is pretty much what you get in Natanson's book on that topic.

Chapter II (pp. 86–169), "Interpolation and Approximation," was written by L. D. Kudryavtsev and has the flavor of Natanson's book on Constructive Function Theory. However, mention is made of numerous recent results (particularly those concerned with the degree of approximation) by such authors as Favard, Stechkin, Akhiezer, Kreĭn, Timan, Dzyadyk, Nikolsky, and Kolmogorov. The theory of entropy is touched on, but nothing much is done with it.

Chapter III (pp. 170–205), written by B. M. Levitan, is on almost periodic functions.

The referencing is meager. One really wants a better job in a book of this kind. The translation is good. Names, of course, continue to give trouble. No great damage is done when H. Rademacher goes into two black translational boxes and comes back out as G. Radmacher, but what can the uninformed do when our own John Mairhuber goes in and comes back out as Mèrkh'yuber?

The numerical analysis or computational aspects of approximation theory are not treated.

PHILIP DAVIS

Computer Laboratory
Brown University
Providence, Rhode Island

2 [2.05, 2.35, 3, 4, 5, 6].—LOTHAR COLLATZ, *Functional Analysis and Numerical Mathematics*, translated by Hansjörg Oser, Academic Press, New York, 1966, xx + 473 pp., 24 cm. Price $18.50.

The purpose of this book is to indicate the role and utility of the concepts of functional analysis in numerical mathematics. The first major section (200 pages) is primarily devoted to developing the necessary functional analysis background. The treatment covers the usual elementary topics as well as partially ordered spaces, pseudo-metric spaces (the metric takes its values in a partially ordered linear space), and thirty pages on vector and matrix norms. A generous portion of examples, many from numerical analysis, are interwoven. The remainder of the book deals almost exclusively with the solution of linear and nonlinear operator equations with primary application to differential and integral equations (and associated eigenvalue problems), and secondary application to approximation problems. Topics given the most prominence are: constructive fixed-point theorems, the Gauss-Seidel and Jacobi iterations for linear and nonlinear systems of equations, Newton's method and the secant method for nonlinear operator equations and, especially, the setting of partially ordered spaces with discussion of monotone convergence of the Newton iterates, monotone operators and problems of monotone kind. Again, numerous general and concrete examples are interwoven into the development. The emphasis throughout is on obtaining error bounds.

Although the book is primarily concerned with the numerical analysis of nonlinear equations, it is not, and is not claimed to be, a definitive study of this topic. The development is based primarily on the contributions of the author, his students, and colleagues, and the relevant Russian and American work receives considerably less attention.

The translation reads well, and relatively few misprints were noted. There are 26 exercises.

J. M. O.

3 [2.05, 2.10, 2.20, 2.55, 3, 4].—Gerard P. Weeg & Georgia B. Reed, *Introduction to Numerical Analysis*, Blaisdell Publishing Co., Waltham, Mass., 1966, vii + 184 pp., 24 cm. Price $7.50.

This book is intended to serve as a text for a one-term introductory course in numerical analysis for sophomore and junior level students; the prerequisites are courses in calculus and introductory differential equations. The material is presented in eight chapters: (1) computational errors; (2) roots of algebraic and transcendental equations; (3) finite differences and polynomial approximation; (4) numerical integration; (5) numerical solution of ordinary differential equations; (6) linear algebraic equations; (7) least-squares approximation; (8) Gaussian quadrature. The level of sophistication is in accord with the stated prerequisites.

In recent years many good textbooks having essentially the same goals and prerequisites as this text, have appeared; among these are the books by W. Jennings and N. Macon. For this reason, a new textbook must justify itself either by presenting different or more recent material than is offered in other standard texts (à la Romberg integration) or by giving an outstandingly lucid and enlightening exposition. In the reviewer's opinion, this textbook does not completely justify itself in either respect. Although the material is basic and in accord with most standard texts, the book has some important defects in arrangement and emphasis. For example, Lagrange interpolation is introduced for the first time in Chapter 8, whereas Chapter 3 is devoted entirely to the Newton-Gregory form of the interpolation polynomial. The most serious drawback of this text lies in its manner of presentation. Many common methods and concepts, such as iteration, and approximation of functions, are hastily and inadequately developed. Similarly, one finds some basic theoretical results avoided, to the detriment of the student; thus from remarks on pp. 63 and 69, the reader would be led to think that only round-off errors might limit the use of an interpolation polynomial of high order in approximating a function on an interval; this is not true, and Runge's famous example should be mentioned.

In general, this book is not as readable and instructive as is required for an introductory text.

Alan Solomon

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

4 [2.05, 3, 5].—LOTHAR COLLATZ & WOLFGANG WETTERLING, *Optimierungsaufgaben*, Springer-Verlag, Berlin, 1966, ix + 181 pp., 21 cm. Price DM 10.80.

This book provides a clear and readable introduction into the fundamental principles of linear and convex programming, as well as the theory of matrix games. These principles provide a framework for a theory of Chebyshev approximations with applications to elliptic differential equations. This part of the book should be of particular interest. The difficult problems connected with the minimization of convex functions without constraints are not touched.

C. WITZGALL

Boeing Scientific Research Laboratories
Seattle, Washington 98124

5 [2.05].—MIECZYSLAW WARMUS, *Tables of Lagrange Coefficients for Quadratic Interpolations*, Polish Scientific Publishers, Warsaw, 1966, ix + 501 pp., 30 cm. Price Zl 180.

This volume, the second in a series of mathematical tables prepared at the Computing Centre of the Polish Academy of Sciences, gives values of the Lagrange interpolation coefficients $L_{-1}(t) = -t(1 - t)/2$, $L_1(t) = t(1 + t)/2$ to 11D; and $L_0(t) = 1 - t^2$ to 10D, all for $t = 0(0.00001)1$.

These tables are arranged in a condensed form, using the relations $L_{-1}(1 - t) = L_1(t)$, $L_0(1 - t) = L_0(t)$, and $L_1(1 - t) = L_{-1}(t)$.

Herein the argument-interval is one-tenth that of the previously largest similar table [1] and two more decimal places appear in each of the tabular entries.

The author points out in the preface that these tables provide an easy method of calculating the value of a function corresponding to an argument given to $k + 5$ decimal places from tabular values for arguments given to k decimal places, and he illustrates this with a single numerical example, which includes an estimate of the error arising from such interpolation.

The procedure followed in the calculation of these tables is not discussed, and no bibliography of earlier tables is given.

It seems appropriate to this reviewer to mention here the equally voluminous, unpublished 8D tables of Salzer & Richards [2] for quadratic and cubic interpolation by the Gregory-Newton and Everett formulas.

J. W. W.

1. NYMTP, *Tables of Lagrangian Interpolation Coefficients*, Columbia Univ., New York, 1944. (See *MTAC*, v. 1, 1943–1945, pp. 314–315, RMT **162**.)
2. HERBERT E. SALZER & CHARLES H. RICHARDS, *Tables for Non-linear Interpolation*, 1961. Copy deposited in UMT file. (See *Math. Comp.*, v. 16, 1962, p. 379, RMT **31**.)

6 [2.10, 3, 6, 7].—R. E. BELLMAN, R. E. KALABA & J. LOCKETT, *Numerical Inversion of the Laplace Transform*, American Elsevier Publishing Co., Inc., New York, 1966, viii + 249 pp., 24 cm. Price $9.50.

In numerous applied problems, characterized by ordinary differential equations, difference-differential equations, partial differential equations or other functional equations, the Laplace transform is often a powerful tool for obtaining a solution.

When the Laplace transform approach is applicable, getting the Laplace transform of the solution is relatively easy. The major problem is inverting the transform. It is often the case that closed-form representations in terms of tabulated functions for the inverse are not known, and so one must resort to numerical methods.

A comprehensive volume on the numerical inversion of Laplace transforms replete with examples would fill an important gap in the literature. Although the volume under review has much which is commendable, it is not comprehensive in its coverage, as the authors seem completely unaware of important segments of the literature. We return to this point later, but first we explore the contents of the volume and present a generalization of the basic tool used by the authors.

Consider

$$h(p) = \int_0^\infty e^{-pt} f(t) dt \tag{1}$$

which we assume exists. Given $h(p)$ the Laplace transform of $f(t)$, the problem is to find $f(t)$. Elementary properties of $h(p)$ are treated in Chapter 1 and its numerical inversion is considered in Chapter 2. The volume is centered around a procedure which treats (1) as an integral equation. This is a useful approach since the techniques are applicable to the solution of linear integral equations of the form

$$u(x) = f(x) + \int_a^b k(x, y) u(y) dy . \tag{2}$$

The idea is to approximate the integral in (1) by a quadrature formula and then find approximate values of $f(t)$ by solving a system of linear equations. By appropriate choice of a quadrature formula, the solution can be expressed in a neat form without actually having to solve the system of linear equations in the usual sense. In (1), put

$$x = e^{-vt}, \qquad v > 0, \qquad vt = -\ln x , \tag{3}$$

so that

$$vh(p) = \int_0^1 x^{(p/v)-1} f(t) dx . \tag{4}$$

From the theory of orthogonal functions [1], we have

$$h(p) = \sum_{j=1}^{n} w_j x_j^{(p/v)-1} z(x_j) + F_n , \tag{5}$$

where

$$\begin{gathered} w_j = \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{n!\Gamma(n + \lambda) x_j^{\beta+1} (1 - x_j)^{\alpha+1} v [R_n^{(\alpha,\beta)\prime}(x_j)]^2}, \qquad \lambda = \alpha + \beta + 1 , \\ R_n^{(\alpha,\beta)}(x_j) = 0 , \qquad x_j = e^{-vt_j} , \qquad z(x_j) = f(t_j) , \end{gathered} \tag{6}$$

where F_n is a remainder term and $R_n^{(\alpha,\beta)}(x)$ is the shifted Jacobi polynomial, i.e., $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x - 1)$. Let p take on n distinct values and put $F_n = 0$. Then (4) gives rise to n equations in n unknowns. By a judicious choice of the p values, the solution of the linear equation system with $F_n = 0$ is easily constructed. With

(7)
$$p/v = k + 1\,, \qquad k = 0, 1, \cdots, n-1\,,$$
$$a_k = h(v[k+1])\,, \qquad w_j z(x_j) = y_j$$

the linear equation system to solve is

(8)
$$\sum_{j=1}^{n} x_j^{\,k} y_j = a_k\,.$$

It can be shown that

(9)
$$w_j z(x_j) = \sum_{k=0}^{n-1} a_k q_{k,j}\,,$$
$$\sum_{k=0}^{n-1} q_{k,j} x^k = \frac{R_n^{(\alpha,\beta)}(x)}{(x - x_j) R_n^{(\alpha,\beta)\prime}(x_j)}\,.$$

The authors treat the case $\alpha = \beta = 0$ only, in which event $P_n^{(0,0)}(x)$ is the Legendre polynomial. To facilitate use of the formulas, tables of x_j, $R_n^{(0,0)\prime}(x_j)$ and $q_{k,j}$ are provided for $n = 3(1)15$ to 17S, of which the first 15 figures are believed to be correct.

Observe that in the above general development, the parameters v, α and β are free, and ideally this should be exploited to smooth out irregularities in the behavior of the functions involved and so improve the efficiency of the inversion process. This points up a shortcoming in the analysis, since for each choice of α and β tables of x_j, $R_n^{(\alpha,\beta)\prime}(x_j)$ and $q_{k,j}$ must be prepared. Use of the Chebyshev polynomials $T_n^*(x)$ and $U_n^*(x)$ (except for normalization constants, these are the cases $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, respectively) would shorten some of the computational effort, since the zeros x_j and weights w_j are easily expressed. Of course, other orthogonal polynomials with weights chosen to reflect singularities could also be used.

Several examples are treated in Chapters 3 and 4 to show that the procedure for $\alpha = \beta = 0$ can lead to satisfactory results. A number of examples are also developed to show how the ideas may be extended to solve other functional equations, both linear and nonlinear. It is a virtue of the volume that it warns the reader that there is no panacea and that pitfalls abound. The point is this. If (8) is expressed in matrix form as $Ay = b$, A^{-1}, of course, is known from (9). However, there may be serious difficulties as the matrix A is ill-conditioned. Thus $A^{-1}b$ may be meaningless unless b is known to high accuracy. Chapter 5 studies applications of dynamic programming techniques for the solution of ill-conditioned systems. It would be interesting to know if the detrimental effects of ill-conditioning can be removed or mitigated by use of other choices of α and β.

An appendix lists some FORTRAN IV programs for "The Heat Equations," "Routing Problem" and "Adaptive Computation." These pages are a total loss, as I do not find any information as to which specific problems the programs apply.

I now return to the statement made earlier concerning material which deserves a place in a comprehensive treatment on the inversion of transforms. We divide our discussion, which is necessarily brief and by no means complete, into three parts.

First, there are two important papers by A. Erdélyi [2], [3]. There it is shown that if $f(t)$ in (4) is expanded in a series of the Jacobi polynomials $R_n^{(\alpha,\beta)}(t)$, then the coefficients in this expansion, call them a_n, can be expressed as a finite sum of

$n + 1$ terms, each of which depends on a different value of $h(p)$. Notice that this procedure yields a continuous-type approximation as opposed to the discrete-type approximation described by [2]–[9]. Attention should also be called to papers by Tricomi [4], who got a continuous-type approximation based on Laguerre polynomials.

In the above approaches, the problem is viewed as that of solving an integral equation. As the inverse Laplace transform has an integral representation, it is natural to seek the inverse transform by a direct quadrature. Examples of this approach are given in three papers by Salzer [5], [6], [7].

Finally, we note a valuable technique which is slightly touched upon by the authors. However, no references to the literature are given. The idea is to approximate $h(p)$ by the ratio of two polynomials and then invert this approximation in the usual fashion. Only a few examples of this approach are known; see the papers by Luke [8]–[10] and a paper by Fair [11]. In each instance the accuracy of the results is quite remarkable. Furthermore, the approximation for $f(t)$ is a sum of exponentials. This is especially valuable in numerous problems where integrals and other expressions involving $f(t)$ are required.

Y. L. L.

1. A. Erdélyi, W. Magnus, F. Oberhettinger & F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York, 1953.
2. A. Erdélyi, "Inversion formulae for the Laplace transformation," *Philos. Mag.*, (7), v. 34, 1943, pp. 533–536.
3. A. Erdélyi, "Note on an inversion formula for the Laplace transformation," *J. London Math. Soc.*, v. 18, 1943, pp. 72–77.
4. F. G. Tricomi, "Transforzione di Laplace e polinomi di Laguerre," *Rend. Accad. Naz. dei XL*, (6), v. 13, pp. 232–239, 420–426.
5. H. E. Salzer, "Equally-weighted quadrature formulas for inversion integrals," *MTAC*, v. 11, 1957, pp. 197–200.
6. H. E. Salzer, "Tables for the numerical calculation of inverse Laplace transforms," *J. Math. and Phys.*, v. 37, 1958, pp. 89–109.
7. H. E. Salzer, "Additional formulas and tables for orthogonal polynomials originating from inversion integrals," *J. Math. and Phys.*, v. 40, 1961, pp. 72–86.
8. Y. L. Luke, "Rational approximations to the exponential function," *J. Assoc. Comput. Mach.*, v. 4, 1957, pp. 24–29.
9. Y. L. Luke, "On the approximate inversion of some Laplace transforms," *Fourth U. S. Congr. Appl. Mech.*, 1962, Amer. Soc. Mech. Engrs., New York, pp. 269–276.
10. Y. L. Luke, "Approximate inversion of a class of Laplace transforms applicable to supersonic flow problems," *Quart. J. Mech. Appl. Math.*, v. 17, 1964, pp. 91–103.
11. W. Fair, "Padé approximation to the solution of the Riccati equation," *Math. Comp.*, v. 18, 1964, pp. 627–634.

7 [2.35, 4, 5, 6, 13.15].—Yu. V. Vorobyev, *Method of Moments in Applied Mathematics*, translated from Russian by B. Seckler, Gordon and Breach Science Publishers, New York, 1965, x + 165 pp., 23 cm. Price $12.50.

This monograph presents a study with applications of the method of moments for the approximate solution of functional equations in Hilbert spaces involving (mostly completely continuous and self-adjoint bounded) linear operators. The method is based on a variational principle and is closely related to the Chebyshev-Markov classical problem of moments. The representation of the approximate operators constructed in the method of moments shows that the author's method falls within the general framework of the projection or the abstract Ritz-Galerkin method. It differs merely in the choice of the projections, that is, the method of moments gives a specific and very often a useful way of determining the coordinate elements used in the approximations which are closely connected with the problem being studied and which can also be used in accelerating the convergence of iterative methods of Picard-Neumann-Banach type.

The book consists of seven chapters in which the theory and the application of the method of moments is investigated. In Chapter I, "Approximation of bounded linear operators," the author introduces the concepts of an abstract Hilbert space, bounded linear operators, discusses without proofs their properties, formulates the method of moments in a Hilbert space and shows its relation to the projection method or the abstract Ritz-Galerkin method. In Chapter II, "Equations with completely continuous operators," the method of moments is first formulated for completely continuous operators and then it is applied to the solution of nonhomogeneous equations with completely continuous linear operators and to the determination of their eigenvalues. It is also shown how the method of moments can be used in the acceleration of convergence of iterative methods of Picard-Neumann-Banach type. In Chapter III, "The method of moments for self-adjoint operators," the problem of moments is first formulated for equations involving self-adjoint operators and then the method is applied to the determination of the spectrum of a self-adjoint operator and to the solution of nonhomogeneous linear equations involving bounded self-adjoint operators. In Chapter IV, "Speeding up the convergence of linear iterative processes," the author first discusses the linear iterative processes, $x_{n+1} = Ax_n + f$, and various methods for their acceleration in the solution of the equation $x = Ax + f$ and then shows how the method of moments may be used to speed up the convergence of the above linear iterative processes. He applies this technique to the solution of the finite-difference equations arising in the numerical solution of the first boundary-value problem for an elliptic operator with constant coefficients. In Chapter V, "Solution of time-dependent problems by the method of moments," the author applies the method of moments to the solution of various classes of nonstationary linear problems (e.g., oscillatory systems with a finite number of degrees of freedom, heat conduction in an inhomogeneous rod, the transient in an automatic control system, etc.). In Chapter VI, "Generalization of the method of moments," it is shown that the author's method is applicable to certain classes of linear equations involving unbounded operators. In Chapter VII, "Solution of integral and differential equations," the method of moments is applied to the solution of certain classes of linear integral equations in L_2 spaces and boundary-value problems for ordinary and partial differential equations. The numerical results obtained by the method of moments are illustrated by actually solving approximately the problems associated with bending of a beam of variable cross-section and with the field of an electrostatic electron lens.

Finally, it should be noted that the monograph is clearly written and well motivated. Its English translation is quite satisfactory.

W. V. PETRYSHYN

University of Chicago
Chicago, Illinois 60637

8 [2.55].—RAMON E. MOORE, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1966, xi + 145 pp. Price $9.00.

In the days when numerical computations were made by hand on desk computers, it was generally customary to monitor the degree of precision of all intermediate and final results by a simple set of rough rules; the precision of the result of each step was estimated in terms of the precision of the operands, and the last significant digit was indicated by an underline when this result was written down. Proceeding in this way, one obtained an estimate of the significance of the final results. The procedure failed to take into account the accumulation of rounding errors, but it provided a check on possible loss of significance by subtraction, which is an ever-present danger in a computation of any size, and it certainly was enormously better than the complete absence of any significance-monitoring procedure, which is almost universal practice with today's multi-million-dollar automatic digital computers.

For the enormous numerical calculations that are commonly attempted today, quite sophisticated significance-monitoring procedures will clearly be needed before one can take such calculations seriously. Before one can develop generally satisfactory procedures of this kind, an extensive study of the overall problem of significance-monitoring in large calculations is necessary. Professor Moore's development of methods using interval-arithmetic over the past seven to eight years represents substantial progress in this area. The book under review presents a detailed theory and analysis of these methods and is undoubtedly the most important single work to appear in the field.

As Moore says, practically all ways of expressing the accuracy of a quantity involve an interval $[a, b]$ (e.g., $[p - e, p + e]$ where p equals expected value, e equals probable error, or the like) together with the statement that the true value lies in that interval or lies in it with a stated probability. Roughly speaking, there are three classes of methods for assigning an interval to the result of an arithmetic operation, when the intervals assigned to the operands are given. At one extreme, there is "significance arithmetic" (see Ashenhurst and Metropolis, "Unnormalized floating-point arithmetic," *J. Assoc. Comput. Mach.*, v. 6, 1959, p. 415), in which the accumulation of errors is ignored, and it is simply assumed that the error of a result is influenced by one operand, but not by both. (For example, in multiplication, the number of significant digits of the product is taken as the number of significant digits of the less accurate factor.) Intermediate is the procedure, attributed to L. H. Thomas, in which a probable error is carried along with each quantity, and when two quantities are combined the error of the result is calculated on the assumption that the errors of the operands are normally distributed and uncorrelated. (Here, if n nearly identical quantities are added, the estimated error of the result increases as $\sqrt{n}$.) At the second extreme is Moore's "interval arithmetic," in which the interval assigned to the result is *certain* to contain the true value of the result, if the same was true of the operands. For example, if two intervals $[a, b]$ and $[c, d]$ are added, the resulting interval is $[a + c, b + d]$. When the calculation is done on a finite computer, the upper limit of the interval is then rounded up (when necessary) and the lower limit is rounded down (when necessary). In this way, guaranteed error bounds for the result of a computation are obtained, if guaranteed error bounds were available for the input data. The procedure is fully automatic when suitably programmed, two numbers a and b being stored for each quantity, corresponding to the interval $[a, b]$ assigned to the quantity. (Here, if n nearly identical quantities are added, the estimated error of the result increases as n.) In any of these systems, the errors of finite-difference methods or other approximate numerical methods must be analyzed independently.

The chief difficulties that have arisen in connection with the first two classes of methods mentioned above appear to arise from the fact that even if one stores in the computer, in some fashion, an estimate of the probable error of each quantity, this gives no information about the correlations of the errors of pairs of quantities; these correlation effects can produce cumulative results in some computational algorithms. For this reason, the study of interval analysis (which in effect allows for the worst possible degree of correlation at each step) seems especially important.

One of the main problems attacked is this: For the calculation of a quantity or set of quantities (inverse of a matrix, value of an algebraic function, root of an algebraic equation, solution of a differential equation, coefficients of a power series, or the like), there are generally many algorithms that are equally satisfactory from the point of view of the required amount of calculation and the accuracy that would be attained if the input data were infinitely precise and all arithmetic steps were performed with infinite accuracy (infinitely many binary or decimal places); the problem is to find, among these, the algorithms that give the narrowest intervals for the final results in a calculation by interval arithmetic. [This is just a precisely defined form of the central problem in any theory of analysis of numerical errors.] Often the optimal algorithm is not a previously known one, and considerable ingenuity is required to construct such algorithms. This activity, of course, overlaps to some extent similar activity on the part of many numerical analysts (in the old-fashioned sense) over the years, but it seems to the reviewer that Moore's approach has several valuable new ingredients: (1) its complete rigor, (2) the power of various theoretical concepts and tools introduced by Moore (e.g., a metric topology for intervals, and interval integrals), and (3) the ability to make precise evaluation and comparison of complicated algorithms by computations on computers programmed for interval arithmetic. Furthermore, errors introduced by mathematical approximation can often be expressed in terms of intervals and thereby incorporated directly and automatically in a calculation; for example, the remainder after n terms of a Taylor's series involves the $(n + 1)$th derivative at an argument x^* for which a precise interval $[a, x]$ is known.

Although interval analysis is in a sense just a new language for inequalities, it is a very powerful language and is one that has direct applicability to the important problem of significance in large computations.

The range of subjects treated is indicated by the chapter headings, which are: Introduction, Interval numbers, Interval arithmetic, A metric topology for intervals, Matrix computation with intervals, Values and ranges of values of real functions, Interval contractions and root finding, Interval integrals, Integral equations, The initial-value problem in ordinary differential equations, The machine generation of Taylor coefficients, Numerical results with the Kth order methods, Coordinate transformations for the initial-value problem. There are two appendices.

The reviewer looked in vain for a comparison of interval arithmetic with other modes of significance arithmetic. It is difficult to resist the conjecture that in some calculations, where errors are mostly uncorrelated, interval arithmetic must give overly pessimistic estimates of the final errors; it would be of value to know whether this can often be the case (it may be quite rare), and if so whether anything can be done about it (for example, possibly a temporary switch to one of the other modes of significance arithmetic).

R. D. RICHTMYER

Department of Mathematics
University of Colorado
Boulder, Colorado

9 [2.55, 4, 5].—I. BABUSKA, M. PRAGER, I. VITASEK, *Numerical Processes in Differential Equations*, John Wiley & Sons, Inc., New York, 1966, x + 351 pp., 24 cm. Price $9.50.

This translation from the 1964 Czech edition reads quite well and has the following chapters:

1. Introduction, 4 pages
2. Stability of Numerical Processes and Some Processes of Optimization of Computations, 44 pages
3. Initial-Value Problems for Ordinary Differential Equations, 56 pages
4. Boundary-Value Problems for Ordinary Differential Equations, 150 pages
5. Boundary-Value Problems for Partial Differential Equations of the Elliptic Type, 50 pages
6. Partial Differential Equations of the Parabolic Type, 35 pages.

The stability chapter contains some nice examples of the loss in accuracy due to finite word length. Definitions of stability are given and applied. They boil down to continuous dependence on the inhomogeneous data or specified (polynomial) growth of errors.

Much of the standard convergence and stability (in the sense of Dahlquist) theory for linear multistep and one-step methods is presented in a neat form. Unfortunately, predictor-corrector methods are never mentioned.

Only linear two-point boundary-value problems are considered for second- and fourth-order equations. "Factorization" methods in which the boundary-value problem is replaced by several initial-value problems for first-order equations (in both directions) are studied in some detail. As is later shown, these methods are suggested by the factorization of the tri- and qui-diagonal matrices so familiar in difference methods for such problems. The stability of the initial-value problems is shown under appropriate conditions. The "shooting" method is but briefly mentioned and the usual warning of possible instability is based on an example. A detailed treatment of finite-difference methods, including higher-order accurate schemes, is given. Finally, the Ritz method for self-adjoint positive-definite problems is considered in some generality.

The material on elliptic problems is devoted mainly to setting up difference equations for second-order self-adjoint problems with no mixed derivatives and to solving the difference equations by some of the standard iterative methods (not including alternating directions).

Linear parabolic problems in one- and two-space dimensions are treated briefly using maximum estimates for implicit and explicit schemes and Lee's energy estimates for the Crank-Nicolson scheme in one dimension. Alternating directions are described.

This book presents a good introduction to the several topics which it treats. However, the level of presentation is rather mixed, since about one third of the material assumes a knowledge of functional analysis.

H. B. Keller

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

10 [4, 6].—Ivar Stakgold, *Boundary Value Problems of Mathematical Physics, Vol.* I, The Macmillan Company, New York, 1967, viii + 340 pp., 24 cm. Price $12.95.

This is the first of two volumes intended for a graduate course in mathematical physics. Although the topics discussed are mathematical in nature, it is written in a clear and pleasant style by a man who knows how to talk to physicists and engineers and who enjoys doing so.

While the easier results are proved, more difficult theorems or those requiring lengthy proof are motivated heuristically, in such a way that the reader at least gets the feeling of how the proof goes. In such cases it is clearly stated that a proof is needed, and whether the proof is difficult or easy.

Chapter 1 deals with ordinary differential equations. In addition to the standard material on this subject, there is a beautiful discussion of one-dimensional distribution theory. Its purpose is to provide a firm foundation for the Dirac delta function, which is then used to define fundamental solutions and Green's functions.

Chapter 2 is an introduction to linear spaces, with particular emphasis on linear transformations in a Hilbert space.

These concepts are applied in Chapter 3 to the study of linear integral equations with symmetric kernels. This chapter includes some discussion of variational methods both for nonhomogeneous problems and for eigenvalue problems. In particular, eigenfunction expansions are discussed, and the Rayleigh-Ritz method and the eigenvalue inclusion theorem are presented.

Chapter 4 deals with singular self-adjoint boundary value problems for second-order ordinary differential operators. It includes a proof of Weyl's limit point-limit circle theorem, and a discussion of the general spectral representation.

The author has been able to concoct a large set of exercises which are nontrivial and educational, but still not too difficult for students taking the course for which the book is designed.

Hans F. Weinberger

University of Minnesota
Minneapolis, Minnesota

11 [4, 5, 7].—A. W. Babister, *Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations*, The Macmillan Company, New York, 1967, xi + 414 pp., 24 cm. Price $14.95.

Consider

$$(1) \qquad L(D)y(x) = f(x)\,, \qquad L(D) = \sum_{k=0}^{n} p_k(x)D^{n-k}\,, \qquad D = d/dx\,.$$

The general theory for obtaining particular solutions of (1) is treated in a number of texts and references. In the case where detailed properties of $y(x)$ are desired for specific operators $L(D)$ and functions $f(x)$, very little data is to be found in the literature. This is particularly the situation where

$$(2) \qquad M(\delta)y(x) = xf(x)\,,$$

$$(3) \qquad \begin{aligned} M(\delta) &= \delta(\delta + \rho_1 - 1)(\delta + \rho_2 - 1)\cdots(\delta + \rho_q - 1) \\ &\quad -x(\delta + \alpha_1)(\delta + \alpha_2)\cdots(\delta + \alpha_q)\,, \qquad \delta = xD\,, \end{aligned}$$

$$(4) \qquad M(\delta)\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| x\right) = 0\,, \qquad {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| x\right) = {}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_p\\ \rho_1, \rho_2, \cdots, \rho_q\end{matrix}\middle| x\right),$$

and $f(x)$ has the form

$$(5) \qquad f(x) = tx^{\gamma}(1 - ax)^{\epsilon}e^{\lambda x}$$

for special values of a, γ, ϵ and λ. Here t is free of x, and the ${}_pF_q$ is the notation for the generalized hypergeometric function. Throughout the discussion we follow the notation of [1].

Many of the special functions of mathematical physics are characterized by the ${}_pF_q$ symbol. The history of such functions and of (4) is long and varied. This includes not only theoretical developments stemming from the pure side of mathematics, but also the analysis of diverse physical problems where (4) appears. Physically, when the order of $M(\delta)$ is 2, $f(x)$ can be interpreted as a forcing function, and characterization of the ensuing response due to the presence of $f(x)$ is very important. Thus, for this and numerous other problems, the volume under review should provide a useful compendium of results.

The bulk of the present volume is concerned with solutions of (2) for the cases $p = 0,\ q - 1,\ p = q = 1$ and $p = 2,\ q = 1$. For each of these situations the order of $M(\delta)$ is 2. The volume contains other material on which we shall comment in due course.

There are 11 chapters. Each concludes with a set of problems, and where appropriate, problems are taken from physical applications.

Chapter 1 takes up general methods for the solution of (1), including variation of parameters, Cauchy's method, solution in terms of integrals, series expansions, and Green's functions. When the coefficients $p_k(x)$ in (1) are constant, then a solution is often facilitated by use of the Laplace transform. This is the subject of Chapter 2.

Perhaps the best known and oldest example of (2)–(5) is the case of Lommel functions. Thus with $p = 0$, $q = 1$, $\rho_1 = \nu + 1$, and appropriate transformations, we have

$$(6) \qquad (z^2D^2 + zD + z^2 - \nu^2)h(z) = z^{\mu+1}$$

which is satisfied by $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$, see [1, Vol. 2, Chapter 6] or [2]. Both of the above functions satisfy the same difference and difference-differential formulas. Many other properties of these functions are known, e.g., integral representations, integrals involving them, and asymptotic expansions. The delineation of such properties is given in Chapter 3. As the homogeneous solutions of (6) are the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, some properties of these functions needed in the analysis are also developed. Actually, this chapter begins with the analysis of Struve's function, which is the special case $\mu = \nu$, since $s_{\nu,\nu}(z) = 2^{\nu-1}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} + \nu)H_\nu(z)$. The chapter then concludes with an analysis of the Lommel functions. Thus there is some repetition. Now $s_{\mu,\nu}(z)$ is essentially a ${}_1F_2$. Hence many of its properties are a consequence of the general theory for the ${}_pF_q$. However, for the most part this general approach is not followed. In this connection, if in (5) $f(x) = \mu(\mu + \rho_1 - 1) \times (\mu + \rho_2 - 1) \cdots (\mu + \rho_q - 1)x^{\mu-1}$, it can be readily shown that

$$(7) \qquad y(x) = x^{\mu}{}_{p+1}F_{q+1}\left(\begin{matrix}1, \mu + \alpha_p \\ \mu + 1, \mu + \rho_q\end{matrix}\middle| x\right)$$

so that numerous properties of $y(x)$ follow on appeal to the general theory for the ${}_pF_q$. The relation (7) is given in Chapter 8.

The kinds of properties known for Lommel functions serve as models for properties sought for solutions of other nonhomogeneous differential equations studied in the text. Further, as is to be expected, properties of $y(x)$ as defined by (2) are akin to those for the homogeneous solutions for this equation.

Solutions of (2) with $p = q = 1$, $\alpha_1 = a$, $x = z$, $t^{-1}f(z) = e^{z/2}$, $e^{z/2}z^{1-c}$, $z^{\sigma-1}$ and $e^{\rho z}z^{\sigma-1}$ for appropriate t, that is

$$(8) \qquad (zD^2 + (c - z)D - a)y(z) = f(z) ,$$

are taken up in Chapter 4. Here the homogeneous solutions are the confluent hypergeometric functions ${}_1F_1(a; c; z)$ and $\psi(a; c; z)$, and properties of these functions are also given. The author seems unaware that the special case $t^{-1}F(z) = z^{\sigma-1}$, $c = 2a$ has been previously studied, see [3].

In (8), put $y(z) = z^{-c/2}e^{z/2}g(z)$, $c = 2u + 1$, $a = \frac{1}{2} - k + u$, then

$$(9) \qquad [D^2 + (-\tfrac{1}{4} + k/z + (\tfrac{1}{4} - u^2)z^2)]\, g(z) = e^{-z/2}z^{u-1/2}f(z) = r(z) .$$

The homogeneous solutions of the latter are also confluent hypergeometric functions. They are known as Whittaker functions. The cases $r(z) = z^{u-1/2}$ and $z^{\nu-1/2}$ are treated in Chapter 5. This chapter also contains material on the particular integrals of the nonhomogeneous equation for the functions of the paraboloid of revolution and for the parabolic cylinder functions.

Chapters 6 and 7 take up (5) with $p = 2$, $q = 1$. Thus the homogeneous solutions are Gaussian hypergeometric functions which include Legendre functions. Here several forms of $f(x)$, as given by (5) with $\lambda = 0$, are considered.

Generalizations of some of the material in the previous chapters is given in Chapter 8 by putting $f(x) = t_{p-1}F_{q-1}(\alpha_1 + 1, \cdots, \alpha_{p-1} + 1; \rho_1, \cdots, \rho_{q-1}; \frac{1}{2}x)$, t a constant. Thus if $p = q = 1$, $f(x) = te^{x/2}$, see Chapter 4; and if $p = 2$ and $q = 1$, $f(t) = t(1 - x/2)^{-\alpha_1-1}$, see Chapter 6.

The homogeneous differential equations encountered in Chapters 3–8 have at most three regular singularities. Now second-order differential equations with four regular singularities at $z = 0, 1, a$ and ∞ can be reduced to a canonical form which goes by the name of Heun. This includes the differential equation for Lamé and Mathieu functions. Chapter 9 discusses Heun's equation, call it $H(D)y = 0$, and particular integrals of $H(D)y = [(z - 1)(z - a)]^{-1}z^{\sigma-2}$.

Nonhomogeneous linear partial differential equations is the subject of Chapter 10. Chapter 11 is devoted to nonhomogeneous partial differential equations of mathematical physics. In particular, the methods of Chapter 10 are used to solve the equations:

(10) $\Delta V = f(x, y, z)$,

(11) $\Delta V - c^{-2}\, \partial^2 V/\partial t^2 = f(x, y, z, t)$,

(12) $\Delta V - k^{-1}\, \partial V/\partial t = f(x, y, z, t)$,

where Δ is the Laplacian, that is

(13) $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

The Laplacian is expressed in terms of some commonly used systems of orthogonal curvilinear coordinates, and if V is separable, then for each coordinate system the three resulting differential equations are presented. If $f(x, y, z)$ is separable and has a certain form in a given coordinate system, then a solution of (10) is also separable and the three resulting ordinary nonhomogeneous differential equations are set forth. Special cases relating this material to that of the earlier chapters are noted. Similar material for (11), (12) is also developed.

Y. L. L.

1. A. Erdélyi, W. Magnus, F. Oberhettinger & F. G. Tricomi, *Higher Transcendental Functions*, Vols. 1, 2, McGraw-Hill, New York, 1953.
2. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 1945.
3. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962.

12 [7, 9].—M. Lal, *First* 39000 *Decimal Digits of* $\sqrt{2}$, deposited in UMT file.

Previous results on the decimal expansion of $\sqrt{2}$ to 19600D [1] are here extended to 39000D.

In the present investigation, the Newton-Raphson method was used to find an improved approximation to $\sqrt{2}$. Let x_1 be an approximate value of $\sqrt{2}$, then a value of x_2, accurate to twice the number of digits as x_1, is given by

$$x_2 = \tfrac{1}{2}\, x_1 + x_1^{-1} .$$

Here x_1 was known to 19600D, and in order to double the accuracy the reciprocal of x_1 must be carried to 2×19600 digits. The process of dividing 1 by x_1 was carried out on the 1620, Model II, 60K at Queen's University in two parts, and 39075D of x_2 were obtained. The accuracy of the value of x_2 was checked by squaring the output of 39076 digits with the decimal point disregarded. This multiplication, which was carried out in five sections of approximately 8K digits each, showed a 1 followed by an unbroken string of 39074 nines. This test establishes that this value of $\sqrt{2}$ is accurate to 39074D. The first 39000D are recorded here.

In order to examine the internal randomness of digits in an unsophisticated way, the frequency distribution of digits in 39 blocks of 1000 digits, and also in the total 39000 digits, was computed. The chi-square test for the goodness of fit reveals no abnormal behavior in the distribution of digits in this sample. These data are also recorded here.

Author's Summary

1. M. Lal, *Expansion of* $\sqrt{2}$ *to* 19600 *Decimals*, reviewed in *Math. Comp.*, v. 21, 1967, pp. 258–259, RMT **17**.

13 [7, 13.15].—HERBERT SHIPLEY, *Standard Tables for Circular Curves*, Edwards Brothers, Inc., Ann Arbor, Mich., 1963 (second printing, November 1965), 736 pp., 24 cm. Price $24.00. (Obtainable from Curve Data, P.O. Box 542, Kingman, Arizona 86401.)

These voluminous tables, prepared specifically for the use of civil engineers, provide 8D approximations to five lengths (tangent, exsecant, arc, segment height, and chord) associated with the central angle in a circle of unit radius. In trigonometric notation the tabulated quantities are, respectively, $\tan(\Delta/2)$, $\sec(\Delta/2) - 1$, Δ in radians, $1 - \cos(\Delta/2)$, and $2 \sin(\Delta/2)$. The argument Δ assumes the values $0°(10'')120°$, which range, according to the author, suffices for all practical applications in civil engineering. For each tabulated quantity, average first differences are provided at intervals of $1'$ in the argument. There is appended a 10-page conversion table which gives 8D equivalents in degrees of angles to 1° expressed in minutes and seconds.

The underlying calculations were performed to 9D on a UNIVAC system, and the results were then rounded to 8D for printing. The retention of only a single guard figure has naturally led to a relatively large number of rounding errors; none, however, is as large as two final units, so far as this reviewer could ascertain from a comparison of several hundred entries with corresponding data derived from Peters' definitive 8D tables [1].

Photo-offset printing of these tables from edited computer output has resulted in nonuniform typographic quality; nevertheless, all the tabular entries seem to be legible.

Examination of the tabular literature [2] reveals that the tables under review exceed all others of their kind with respect to range, precision, and fineness of argument. They should be of significant value to practicing civil engineers and others requiring these specific data in a convenient compilation.

J. W. W.

1. J. PETERS, *Eight-Place Tables of Trigonometric Functions for Every Second of Arc*, Chelsea, New York, 1963. (See *Math. Comp.*, v. 18, 1964, p. 509, RMT 65.)
2. A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD & L. J. COMRIE, *An Index of Mathematical Tables*, second edition, Addison-Wesley Publishing Co., Reading, Mass., 1962, v. 1, pp. 189–191.

14 [7, 13.15].—HERBERT SHIPLEY, *Areas of Curve Elements*, Edwards Brothers, Inc., Ann Arbor, Mich., 1962 (second printing, November 1965), 131 pp., 24 cm. Price $12.00. (Obtainable from Curve Data, P.O. Box 542, Kingman, Arizona 86401.)

This is a companion to the author's *Standard Tables for Circular Curves*, described in the preceding review. It gives 8D values (without differences) of the areas of six configurations determined by various combinations of radii, tangents, chords, and arcs associated with central angles in a circle of unit radius. As explicitly stated in the table headings, multiplication of the tabular entries by a proportionality factor R^2 yields the corresponding areas for a circle of radius R.

The tabular argument is the central angle, Δ, which assumes the values $0°(1')120°$. In trigonometrical notation the tabulated quantities are, respectively, $\tan(\Delta/2)$, $\frac{1}{2}\sin\Delta$, $\Delta/2$ in radians, $\tan(\Delta/2) - \Delta/2$, $\tan(\Delta/2) - \frac{1}{2}\sin\Delta$, and

$(\Delta - \sin \Delta)/2$. The main table is supplemented by an 8D conversion table of angles in minutes and seconds to degrees.

The same criticisms apply to this set of tables as apply to the other set by this author; namely, numerous rounding errors and partially indistinct figures, though still legible.

Despite these flaws, this compilation should prove especially useful to civil engineers (for whom it is mainly intended) because it is the most extensive of its kind.

J. W. W.

15 [7].—WILHELM MAGNUS, FRITZ OBERHETTINGER & RAJ PAL SONI, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966, viii + 508 pp., 24 cm. Price $16.50.

This is a new and enlarged English edition of a previous work by the first two authors which appeared under the title *Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik*; see *MTAC*, v. 3, 1948, pp. 103–105, 368–369, 522–523. A great deal of the present edition did not appear in the earlier editions. As in the previous editions, there are no proofs. The style of references has been changed. These are restricted to books and monographs and are placed at the end of each pertinent chapter. On occasion, references to papers are given in the text following the associated results. The authors justify this change in that, 20 years ago, much of the material was scattered over numerous single contributions, while in recent times, much of the material has been included in books with quite extensive bibliographies.

The volume covers a vast amount of ground as evidenced by the description of its contents which follows. Chapter I is devoted to the gamma function and related functions. The hypergeometric function is the subject of Chapter II. Nearly all the results are for the ${}_2F_1$—the Gaussian hypergeometric function. Generalized hypergeometric series are touched upon in two pages. There are no results on Meijer's G-function and other generalizations of the ${}_2F_1$. Bessel functions and Legendre functions are detailed in Chapters III and IV respectively. Chapter V takes up orthogonal polynomials. Chapter VI presents the confluent hypergeometric function—the ${}_1F_1$, and Chapter VII deals with Whittaker functions which are also confluent hypergeometric functions. The next two chapters deal with special cases of confluent hypergeometric functions, namely, parabolic cylinder functions (Chapter VIII) and the incomplete gamma functions and related functions (Chapter IX). Chapter X presents elliptic integrals, theta functions and elliptic functions. Integral transforms is the subject of Chapter XI. Here examples are given for Fourier cosine, sine and exponential transforms, and the transforms associated with the names of Laplace, Mellin, Hankel, Lebedev, Mehler and Gauss. This chapter contains a section giving closed-form solutions for integral equations of the form $f(s) = \int_a^b K(s, t)y(t)dt$, where a and b are finite, and $K(s, t)$ has an integrable singularity in the range of integration. An appendix to the chapter gives representations of some elementary functions in the form of Fourier series, partial fractions and infinite products. Chapter XII deals with transformations of systems of coordinates and their application to numerous partial differential equations of mathematical physics. A list of special symbols is provided. There is also a list of special functions which gives the location of their definition in the text.

Since the first two authors were part of the team of A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi which produced *Higher Transcendental Functions*, Vols. 1, 2, 3, 1953–1955, see *MTAC*, v. 10, 1956, pp. 252–254, and *Tables of Integral Transforms*, Vols. 1, 2, 1954, see *MTAC*, v. 11, 1957, pp. 114–116, it is natural to make some comparisons with the volume under review (call it MOS for short) and the five volumes noted above (call it EMOT for short). Virtually all of the material in MOS will be found in EMOT. The amount of material in MOS which is not in EMOT arises from results which appeared in the literature after 1953. This is a very small portion of the total work. EMOT in addition to being a compendium, sketches proofs of many important results. It also gives a more detailed set of references. This is important to locate related material and to check results for typographical errors and the like. EMOT includes a number of topics relating to the special functions of mathematical physics not found in MOS; for instance, detailed treatment of Lamé functions, Mathieu functions, spheroidal wave functions, and tables of integral transforms. As noted, MOS essentially does not give results on generalizations of the hypergeometric functions, while EMOT does. In the latter, the confluent hypergeometric function is presented in a single chapter. In MOS, two chapters are devoted to the topic. Of course, given results for the Whittaker functions and the formulae which connect them to the ${}_1F_1$, results for the latter are easily obtained and vice-versa. As both notations appear rather widely, the reader will appreciate the dual presentation. Each key equation in EMOT is given a number. This practice is not followed in MOS and consequently reference to specific equations is awkward.

Past experience indicates that in spite of numerous precautions to avoid errors in mathematical text, the avoidance of all is virtually impossible. It seems one can never proofread enough, and the reader should always impose some check on a formula before using it. We have examined a rather sizeable portion of MOS and in view of the vast amount of material covered, the number of essential errata seems rather small.*

Applied workers will find this volume very useful, but I would advise using it as a compendium about the kind of results which are available rather than as a collection of guaranteed data.

Y. L. L.

---

* In particular, on pp. 1–3, 13–16, 25–28, 283–286 we found 2, 0, 1, 8 errors out of 32, 19, 28 and 41 entries respectively.

16 [7, 8].—S. H. Khamis, *Tables of the Incomplete Gamma Function Ratio*, Justus von Liebig Verlag, Darmstadt, Germany, 1965, il + 412 pp., 20 cm. Price DM 42.00.

These fundamental tables consist of 10D values (without differences) of the incomplete gamma function ratio or the gamma cumulative distribution function, represented by the integral

$$P(n, x) = \frac{1}{2^n\Gamma(n)} \int_0^x t^{n-1} e^{-t/2} dt \,, \qquad n > 0 \,, \qquad x \geqq 0 \,.$$

The range covered is $n = 0.05(0.05)10(0.1)20(0.25)70$, $x = 0.0001(0.0001)0.001(0.001)0.01(0.01)1(0.05)6(0.1)16(0.5)66(1)166(2)250$. Occasionally, tabular values are listed for a few additional values of x. On the other hand, tabular values that round to 0 or 1 to 10D have been omitted. The underlying calculations were performed on an IBM 7090 system under the direction of Wilhelm Rudert at the Technische Hochschule Institut für Praktische Mathematik, at Darmstadt.

As the author points out in the introduction to these tables, the tabulated function $P(n, x)$ coincides with the well-known χ^2 cumulative distribution function for $2n$ degrees of freedom whenever $2n$ is a positive integer, and it is related to the Poisson cumulative distribution when n is a positive integer.

Indeed, the calculation of these impressive tables was motivated by a desire to provide more adequate tables for these two statistical distributions. Thus, earlier tables such as those of K. Pearson [1], Hartley & E. S. Pearson [2], Molina [3], and Kitigawa [4] are now superseded by this more extensive tabulation.

The author includes all these tables in his list of 13 references.

The introduction is written in English as well as in German. This section of the book includes a description of the tables; a brief description of the procedure followed in their calculation; a discussion of some of their general uses; a discussion, with illustrative examples, of interpolation (direct and inverse) and extrapolation; and an auxiliary 10D table of $\Gamma(n)$ for $n = 1(0.025)2$.

Attractively bound and printed, these tables constitute a significant contribution to the mathematical and statistical tabular literature.

J. W. W.

1. K. Pearson, *Tables of the Incomplete Γ-Function*, H. M. Stationery Office, London, 1922; reissued by *Biometrika* Office, University College, London, 1934.
2. H. O. Hartley & E. S. Pearson, "Tables of the χ^2-integral and of the cumulative Poisson distribution," *Biometrika*, v. 37, 1950, pp. 313–325.
3. E. C. Molina, *Tables of Poisson's Exponential Limit*, Van Nostrand, New York, 1942.
4. T. Kitigawa, *Tables of Poisson Distributions*, Baifukan, Tokyo, 1951.

17 [7].—Joyce Weil, Tadepalli S. Murty & Desiraju B. Rao, *Zeros of* $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda)$ *and* $J_n'(\lambda)Y_n'(\eta\lambda) - J_n'(\eta\lambda)Y_n'(\lambda)$, ms. of 20 computer sheets deposited in UMT file, also in microfiche section of this issue.

The first ten positive zeros of the two functions specified in the title are tabulated to 5D for $n = 0(1)10$ and $\eta = 0(0.05)0.95$. Details of the underlying computational procedure have been published [1] by the authors. The zeros of the second function were found in the same manner as those of the first, after use was made of the relation $Z_n'(x) = nZ_n(x)/x - Z_{n+1}(x)$, where Z_n represents either J_n or Y_n.

J. W. W.

1. Joyce Weil, Tadepalli S. Murty & Desiraju B. Rao, "Zeros of $J_n(\lambda)Y_n(\eta\lambda) - J_n(\eta\lambda)Y_n(\lambda)$," *Math. Comp.*, v. 21, 1967, pp. 722–727.

18 [7].—Henry E. Fettis & James C. Caslin, *Elliptic Functions for Complex Arguments*, Report ARL 67-0001, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, January 1967, iv + 404 pp., 28 cm. Copies obtainable from the Defense Documentation Center, Cameron Station, Alexandria, Va. 22314.

These unique tables consist of 5D values of the Jacobian elliptic functions $\mathrm{sn}(w, k)$, $\mathrm{cn}(w, k)$, and $\mathrm{dn}(w, k)$, where $w = u + iv$, as functions of Jacobi's nome q, which equals $\exp(-\pi K'/K)$, where K and K' are the quarter-periods (the complete elliptic integrals of the first kind of modulus k and of complementary modulus k', respectively).

The range of parameters in the table is: $q = 0.005(0.005)0.480$, $u/K = 0(0.1)1$, and $v/K' = 0(0.1)1$. For larger values of q the authors give in the introduction approximations to the elliptic functions by circular and hyperbolic functions.

These tables were computed on an IBM 7094 system by the method of modulus reduction based on Gauss's transformation. Essentially the same subroutine was used here as in the calculation of an earlier table [1] of Jacobian elliptic functions by the same authors.

Reference should also be made to a manuscript table [2] of elliptic functions for complex arguments by these authors, which, however, has $\sin^{-1} k$ for an argument in place of q.

J. W. W.

1. Henry E. Fettis & James C. Caslin, *Ten Place Tables of the Jacobian Elliptic Functions*, Report ARL 65–180, Part 1, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, September 1965. (See *Math. Comp.*, v. 21, 1967, pp. 264–265, RMT **25**.)
2. Henry E. Fettis & James C. Caslin, *Jacobian Elliptic Functions for Complex Arguments*, ms. deposited in UMT file. (See *Math. Comp.*, v. 21, 1967, p. 508, RMT **65**.)

19 [7].—V. M. Beliakov, R. I. Kravtsova & M. G. Rappaport, *Tablitsy ellipticheskikh integralov*, Tom II (*Tables of Elliptic Integrals*, Vol. II), Izdatel'stvo Akademii Nauk SSSR, Moscow, 1963, xii + 783 pp., 27 cm. Price 6 rubles, 18 kopecks.

This set of tables is a continuation of the systematic tabulation (without differences) of the elliptic integral of the third kind $\prod (n, k^2, \phi)$ to 7S which was carried out for negative values of the parameter n in the first volume [1]. In this second volume, n assumes nonnegative values; specifically, $n = 0(0.1)1$, 1.2, 1.5(0.5) 5(1)10, 12, 15(5)40(10)100, while the ranges of k and ϕ are the same as in the first volume; namely, $k^2 = 0(0.01)1$ and $\phi = 0°(1°)90°$.

To this main table of 728 pages there are appended six supplementary tables. The first four of these are of $T_n^{\epsilon}(k^2, \phi) = \int_\epsilon^\phi \sin^{-2n} \alpha(1 - k^2 \sin^2 \alpha)^{-1/2} d\alpha$ for $k^2 = 0(0.01)1$, $35° \leqq \phi \leqq 90°$, $\epsilon = 35°$, and $n = 1, 2, 3, 4$, respectively. The precision of these four tables is 7S, 5S, 3S and 2S, respectively. They are provided to facilitate the evaluation of

$$\prod\nolimits_\epsilon (n, k^2, \phi) = \int_\epsilon^\phi (1 + n \sin^2 \alpha)^{-1}(1 - k^2 \sin^2 \alpha)^{-1/2}$$

$$= \sum_{m=1}^{\infty} (-1)^{m+1} T_m^{\epsilon}(k^2, \phi) n^{-m},$$

where $\epsilon = 35°$ and $n > 100$.

The calculation of $\prod (n, k^2, \phi)$ when $0° < \phi \leqq 35°$ and $n > 100$ can be effected by two series (according as $k^2 \leqq 0.7$ or $k^2 \geqq 0.7$) that involve, respectively, $A_m(\phi) = \int_0^\infty \sin^{2m} \alpha d\alpha$ and $R_m(\phi) = \int_0^\phi \tan^{2m} \alpha \sec \alpha d\alpha$. These functions are tabulated to 8D or 8S in the first volume for $m = 1(1)10$ and $m = 0(1)8$, respectively.

A detailed discussion of these series appears in the Introduction to the present tables (pp. v–vi).

The remaining two auxiliary tables consist, respectively, of sin ϕ, cos ϕ, $\phi = 0°(1°)90°$, 10D; tan ϕ, cot ϕ, $\phi = 0°(1°)90°$, 11S and 12S; and of $R_0 = \ln \tan (\phi/2 + \pi/4)$, $\phi = 0°(1°)89°$, 9D.

Appended to the Introduction is a bibliography of 11 items, which, however, does not include a reference to the tables of Paxton & Rollin [2]. Moreover, since the publication of these Russian tables, Fettis & Caslin have calculated 10D tables of elliptic integrals of all three kinds [3]; however, therein the tabulation of the integral of the third kind is over a broader mesh in k and ϕ and is restricted to $-1 \leqq n \leqq 1$, in contrast to the tables under review, which are by far the most elaborate of their kind calculated to date.

J. W. W.

1. V. M. Belîakov, R. I. Kravt͡sova & M. G. Rappaport, *Tablit͡sy elliptiches͡kikh integralov*, Tom I, Izdatel'stvo Akademii Nauk SSSR, Moscow, 1962. (See *Math. Comp.*, v. 18, 1964, pp. 676–677, RMT **93**; v. 19, 1965, p. 694, RMT **127**.)
2. F. A. Paxton & J. E. Rollin, *Tables of the Incomplete Elliptic Integrals of the First and Third Kind*, Curtiss-Wright Corporation, Research Division, Quehanna, Pa., June 1959. (See *Math. Comp.*, v. 14, 1960, pp. 209–210, RMT **33**.)
3. Henry E. Fettis & James C. Caslin, *Tables of Elliptic Integrals of the First, Second, and Third Kind*, Applied Mathematics Research Laboratory Report ARL 64–232, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, December 1964. (See *Math. Comp.*, v. 19, 1965, p. 509, RMT **81**. For errors, see *Math. Comp.*, v. 20, 1966, pp. 639–640, MTE **398**.)

20 [9].—M. Lal, *Decimal Expansion of Mersenne Primes*, ms. of 19 pp., dated June 20, 1967, deposited in the UMT file.

This manuscript contains the exact values in the decimal system of those thirteen known Mersenne primes M_p for which $p > 100$, calculated on an IBM 1620 at Dalhousie University.

A table is included which gives for each of these primes the frequency distribution of the digits, with the corresponding χ^2 value, and the total number of digits. No significant departure from a random distribution of the digits can be inferred from this statistical analysis.

The author notes in his introductory remarks that Sierpiński [1] gives the number of digits in M_{1279} and M_{11213} incorrectly as 376 and 3381, respectively, instead of 386 and 3376. In addition, the present author has observed that Hardy & Wright [2] give the latter number incorrectly as 3375.

For supplementary information the author refers the reader to papers by Gillies [3] and Uhler [4], [5].

J. W. W.

1. W. Sierpiński, *Elementary Theory of Numbers*, Państwowe Wydawnictwo Naukowe (Polish Scientific Publishers), Warsaw, and Hafner Publishing Co., New York, 1964, p. 341.
2. G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., 1960, reprinted 1965, p. 16.
3. D. B. Gillies, "Three new Mersenne primes and a statistical theory," *Math. Comp.*, v. 18, 1964, pp. 93–95.
4. H. S. Uhler, "A brief history of the investigation on Mersenne numbers and the latest immense primes," *Scripta Math.*, v. 18, 1952, pp. 122–131.
5. H. S. Uhler, "On the 16th and 17th perfect numbers," *Scripta Math.*, v. 19, 1953, pp. 128–131.

21 [9, 11].—M. F. Jones, *Isoperimetric Right-Triangles*, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, April 1967. Computer output deposited in the UMT file.

Let $\pi_n(P)$ be the number of integers $p \leqq P$ such that exactly n different primitive Pythagorean triangles exist having the same perimeter p. Let $T(P)$ be the total number of primitive triangles with perimeter $\leqq P$. Clearly,

$$T(P) = \sum_{n=1}^{\infty} n\pi_n(P) .$$

For example,

$$\pi_2(1715) = 0 , \qquad \pi_2(1716) = 1 ,$$

since

$$(748, 195, 773) \quad \text{and} \quad (364, 627, 725)$$

constitutes the pair of isoperimetric primitive triangles with the smallest perimeter.

In [1], we find $T(P)$ for $P = 10^3(10^3)120 \cdot 10^3$. Later, [2], the value $T(120{,}000) = 8430$ was corrected to 8432, and all 8432 triangles were listed. In [3], there are listed $175 \leqq \pi_3(10^6)$ triples and $7 = \pi_4(10^6)$ quadruples of isoperimetric triangles of perimeter $\leqq 10^6$. Subsequently, [4], in connection with the massive complete listing of $T(500{,}000) - T(120{,}000) = 26{,}683$ triangles, ten other triples with $p < 10^6$ were found, and added to the UMT **107** of [3].

In Table Errata E-419 of this issue, still six more such triples are listed. It is asserted that this is now complete, so that we have $\pi_3(10^6) = 191$ exactly.

In the present table we find, first,

$$\pi_n(P) \quad \text{for} \quad n = 2(1)5, \qquad P = 5 \cdot 10^4(5 \cdot 10^4)25 \cdot 10^5 .$$

From this table we excerpt the following:

	n			
$P \cdot 10^{-5}$	2	3	4	5
5	1751	65	1	–
10	3819	191	7	–
15	6021	311	13	–
20	8323	433	27	4
25	10690	549	47	5

All triples, quadruples, and quintuples with $p \leqq 2{,}533{,}500$ are listed. Listings of such multiplets are continued to $p \leqq 5{,}060{,}250$, but are *not complete* here because of the computation method. Similarly, the $\pi_n(P)$ listed for $P > 25 \cdot 10^5$ are merely lower bounds. The criterion for listing multiplets is that at least one triangle of the multiplet, with sides $a^2 \pm b^2$ and $2ab$, has both generators a and b less than 1126.

The functions $T(P)$ and $\pi_1(P)$ are not discussed, and $\pi_n(P) = 0$ over this range for $n > 5$.

D. S.

1. A. S. Anema, UMT **106**, *MTAC*, v. 4, 1950, p. 224.
2. A. S. Anema, UMT **111**, *MTAC*, v. 5, 1951, p. 28.
3. A. S. Anema & F. L. Miksa, UMT **107**, *MTAC*, v. 4, 1950, p. 224.
4. F. L. Miksa, UMT **133**, *MTAC*, v. 5, 1951, p. 232.

22 [9].—M. F. Jones, 22900D *Approximation to the Square Roots of the Primes less than 100*, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, June 1967. Copy of computer printout deposited in the UMT file.

There are given here values of $\sqrt{p}$ accurate to 22900D for each prime p less than 100. These values were computed on an IBM 1620 by "twelve stages" of Newton's method starting with 25D approximations. (Since ten iterations should suffice, one presumes that the first stage here is merely the initial approximation, and that the twelfth was performed to check the eleventh.) A few of the high-accuracy digits in $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ here were compared with those of other recent calculations [1], [2], [3], [4] and no discrepancy was found.

The decimal-digit distribution over the *entire* range of 22900D is also given, together with corresponding values of χ^2. No counts are given for smaller blocks. For $p = 17, 19, 67, 37$ one finds

$$\chi^2 = 17.74, 17.32, 16.43, \text{ and } 15.41 ,$$

respectively, and the author concludes: "On the basis of this test, it can be said with a 95% confidence that the tested digits of $\sqrt{17}$ and $\sqrt{19}$ are not random and further that $\sqrt{67}$ and $\sqrt{37}$ come very close to the rejection region."

Nonstatisticians often find the χ^2 statistic as elusive as nonphysicists find entropy; dubious conclusions similar to the foregoing are even found in published papers; and while the reviewer is himself a nonstatistician, he feels called upon to comment. The χ^2-statistic for a random sequence, according to the theory, should be *distributed* around a mean nearly equal to the number of degrees of freedom, here equal to 9, according to a prescribed distribution if a sufficient number of samples of χ^2 are computed. Now, 95% of such values (and this is the figure that the author alludes to) should have $\chi^2 < 16.9$. But that is merely another way of saying that *one time out of twenty* the χ^2 will be larger. If, with *one trial only*, one obtains a χ^2 somewhat greater than 16.9, this is hardly something to be alarmed at, since nothing is shown to indicate that this trial was not that "one time." That the author is being unduly concerned about the *large* χ^2 found for $\sqrt{(17)}$ is also shown by his lack of concern for certain small values. Thus, 5% of the time (only) the χ^2 should be <3.3, but the χ^2 for the $\sqrt{5}$ here is 3.05, and therefore the $\sqrt{5}$ is equally "nonrandom"—that is, not at all—as the $\sqrt{(17)}$ is. Actually, all experience has shown that similar conclusions as those here, for example, von Neumann's concern about the low χ^2 for the first 2000 digits of e, are generally rectified when a larger sample of χ^2 values is computed.

D. S.

1. M. Lal, *Expansion of* $\sqrt{2}$ *to* 19600 *Decimals*, reviewed in *Math. Comp.*, v. 21, 1967, pp. 258–259, UMT **17**.

2. Koki Takahashi & Masaaki Sibuya, *The Decimal and Octal Digits of* $\sqrt{n}$, reviewed in *Math. Comp.*, v. 21, 1967, pp. 259–260, UMT **18.**
3. M. Lal, *Expansion of* $\sqrt{3}$ *to* 19600 *Decimals*, reviewed in *Math. Comp.*, v. 21, 1967, p. 731, UMT **84.**
4. M. Lal, *First* 39000 *Decimal Digits of* $\sqrt{2}$, reviewed in *Math. Comp.*, v. 22, 1968, p. 226, UMT **12.**

23 [9].—Mohan Lal & James Dawe, *Tables of Solutions of the Diophantine Equation* $x^2 + y^2 + z^2 = k^2$, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, February 1967, xiii + 60 pp., 28 cm. Price $7.50.

Table 3 of this attractively printed and bound volume lists all integral solutions of

$$x^2 + y^2 + z^2 = k^2 \qquad (0 < x \leqq y \leqq z) \tag{1}$$

for $k = 3(2)381$. The brief introduction points out that F. L. Miksa published such a table to $k = 207$, but neglects to mention that he also extended this himself to $k = 325$ [1]. The present extension does not, therefore, constitute a large increase in the upper limit for k, but since the number of solutions of (1) is roughly proportional to k, the number of listed solutions is increased over [1] by a somewhat larger factor.

The imprimitive solutions—those where x, y, z, and k all have a common divisor >1—are marked with an asterisk. (In [1] this was done only for $k > 207$.)

Table 1 lists the number of solutions for each k, and Table 2 lists the number of primitive solutions. (In [1], this data was not given.) The introduction makes no reference to theoretical treatments of the numbers in Tables 1, 2, cf. [2], nor are any empirical observations made concerning these numbers. It is quite convincing, however, from a brief examination of these results, and without reference to the theory, that if k is a prime of the form $8n \pm 1$ or $8n \pm 5$, then there are exactly n solutions, all of which, of course, are primitive.

The introduction points out that none of the listed solutions of (1) are of the form

$$x^4 + y^4 + z^4 = k^4\,, \tag{2}$$

and this proves that (2) has no solutions for $k < 20$. But M. Ward had already proved that result for $k \leqq 10^4$, and recently [3] this was extended to $k \leqq 22 \cdot 10^4$.

D. S.

1. Francis L. Miksa, "A table of integral solutions of $A^2 + B^2 + C^2 = R^2$, etc.," UMT **82**, *MTAC*, v. 9, 1955, p. 197.
2. Leonard Eugene Dickson, *History of the Theory of Numbers*, Volume II, Chapter VII, Stechert, New York, 1934.
3. L. J. Lander, T. R. Parkin & J. L. Selfridge, "A survey of equal sums of like powers," *Math. Comp.*, v. 21, 1967, p. 446.

24 [12, 13.35].—J. Hartmanis & R. E. Stearns, *Algebraic Structure Theory of Sequential Machines*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966, viii + 211 pp., 23 cm. Price $9.00.

This excellent little book brings together in one place most known results on the algebraic structure theory of sequential machines. By a structure theory for sequential machines the authors mean "an organized body of techniques and results which deal with the problems of how sequential machines can be realized from sets of smaller component machines, how these component machines have to be interconnected, and how 'information' flows in and between these machines when they operate."

The book is written so as to be understandable by people of either an engineering or mathematical background, and it is essentially self-contained. The necessary mathematical concepts are introduced in Chapter 0; the fundamental concepts of sequential machines are given in Chapter 1. Chapter 2 introduces the substitution property, as applied to partitions of the state set, and shows how it relates to two-machine serial and parallel decompositions and to state reduction. Partition pairs and pair algebra are presented in Chapter 3 and are applied to the state assignment problem. A general concept of a network of machines is also given in this chapter. Chapter 4 is concerned with the relationship between the lattice of the partitions with the substitution property and "loop-free" decompositions of a machine (i.e., with decompositions in which the component machines are not connected in a circle). This reviewer found these results to be a particularly fine example of the structural approach. Multiple coding of states, or "state splitting" is investigated in Chapter 5. The notion of set systems (or "overlapping partitions") is introduced in this section and serves as a primary tool for the analysis in this and subsequent chapters. It seems to this reviewer that by introducing the concept of a set system earlier in the development, the authors could have made their treatment somewhat more concise and elegant. Chapter 6 is concerned with the mathematical treatment of the intuitive notion of feedback and its role in machine decomposition. A study is also made of the propagation of state-transition and input errors. Chapter 7 consists of a partial treatment of the results of Krohn and Rhodes [1] on the relationship between the semigroup and the decompositions of a sequential machine. The treatment follows that of H. Zeiger [2], but the more "involved and uninformative" proofs are omitted.

As a rule, the book is well written and is well supplied with motivation and examples. The book contains a fair number of typographic errors, but no particularly serious ones were encountered. There is, however, an error in the statement (and thus the proof) of Lemma 7.7, which was brought to my attention by Ann Penton. In brief, the function λ^*, defined at the bottom of p. 203, has a range which properly contains, rather than is, the set ψ. Penton [3] suggests that this difficulty can be overcome by introducing the notion of a *cover* of the set S of states, that is a collection of subsets of S whose union is S. Extending the concepts of machine, the substitution property, and the Max operation, to covers, we may replace 7.7 with the following statement:

Given a machine $M = (S, I, \delta)$, a set system $\phi > 0$ with substitution property on S and a cover ϕ' for S such that Max $\phi' = \phi$ and a machine $M_{\phi'} = (S', I, \delta')$ for ϕ' such that under a mapping $\lambda' : S \to \phi'$ we have

$$\lambda'(\delta(s', a)) = \delta(\lambda'(s'), a)$$

for all $s' \in S'$ and $a \in I$, then

1. there exists a set system $\psi < \phi$ with substitution property and a cover ψ' for S such that Max $\psi' = \psi$, and

2. there exists a *P-R* machine $M_{\phi/\psi}$ such that the group part of the semigroup of $M_{\phi/\psi}$ is a factor group of a subgroup of M, and

3. the serial connection of $M_{\phi'}$ and $M_{\phi/\psi}$ is a machine $M_{\psi'}$ for cover ψ' with function λ^* mapping the states of $M_{\psi'}$ *onto* the set ψ'.

(Thus $M_{\psi'}$ replaces M_ψ and, ψ' replaces ψ in the definition of λ^*.)

Eric G. Wagner

IBM Thomas J. Watson Research Center
Yorktown Heights, New York 10598

1. K. B. Krohn & J. L. Rhodes, *Algebraic Theory of Machines*, Proc. Sympos. Math. Theor. Automata, New York, April 25–26, 1962, *Microwave Research Institute Symposium Series*, Vol. XII, Polytechnic Press, Brooklyn, New York, 1963.

2. H. P. Zeiger, *Loop-free Synthesis of Finite State Machines*, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, 1964.

3. Ann Penton, *Algebraic Study of Sequential Machine Decomposition*, Master's Thesis, Wesleyan University, Middletown, Conn., 1967.

25 [12].—Frank Bates & Mary L. Douglas, *Programming Language/One*, Prentice-Hall, Inc., Englewood Cliffs., N. J., 1967, viii + 375 pp., 25 cm. Price $5.95.

This book is a welcome addition to the literature of PL/I. It is written in a clear and concise style covering a wide field. In spite of the uncertainties about how the PL/I language will finally be implemented (witness the frequently changing specifications used by the manufacturer), this book manages to convey an idea of the power of the PL/I language and to develop in the reader a facility for writing clear, efficient PL/I code.

The examples are easily understood and to the point. Several of the problems provide good practice in the fine art of debugging. The answers seem to be correct and complete.

A very fortunate feature of this book is that technical points (e.g., the inaccuracy caused by representing a decimal fraction in a base other than ten) are reserved until the end of the appropriate chapter, where they appear in sets of notes. This is commendable, since it provides the reader with useful technical information without disturbing the flow of the more basic material.

The book contains several useful appendices, including PL/I character sets, keywords and abbreviations, built-in functions, conditions and format specifications. These tables, combined with a thorough index, make this book valuable as a reference work for the experienced programmer as well as useful as an introductory text to the subject.

Tom Keenan
Henry Mullish

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

26 [12].—Charles R. Bauer, Anthony P. Peluso & William S. Worley, Jr., *IItran*/360: *Self-Instructional Manual and Text*, Addison-Wesley Publishing Co., Reading, Mass., 1967, xi + 212 pp., 28 cm. Price $4.95.

This book is an introduction to a new breed of computer languages called IITRAN, an acronym for Illinois Institute of Technology Translator, a language greatly resembling PL/I and one which processed all of the worked examples on an IBM 360 Model 40.

Unlike so many other books in the field, this text does not assume a high level of familiarity with computer languages, and the exercises seem to be well geared to introduce the novice to the basic notions without his having to seek the assistance of outside help. The ideas are presented clearly and a variety of techniques is employed, thus making the book both informative and pleasant to read—a most unusual combination.

The contents of the book could be thoroughly digested in a period of only a few hours, and the reader should be able to write a computer program after a short while. But, as the authors state in the preface, a mastery of the material covered will not transform one into a "programmer." Considerable practice and experience will, indeed, be necessary before proficiency can be attained.

HENRY MULLISH

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

27 [12].—CHARLES PHILIP LECHT, *The Programmers ALGOL: A Complete Reference*, McGraw-Hill Book Co., New York, 1967, xxiii + 251 pp., 28 cm. Price $8.95.

This book might be more appropriately called an elementary primer. Like a primer, it has large type, excessive white space, "controlled" introduction of material and incessant repetition. Unlike a "complete reference" it lacks an index and a reasonable glossary (it has an unreasonable glossary of 19 entries).

According to the preface, the book was derived not from the official report of the ALGOL authors, but from the manual for the GE 625/635 ALGOL compilers. ALGOL coding is printed in a hardware representation rather than in the reference language. About 30 percent of the book is devoted to one manufacturer's approach to input/output. Of the seven primitive I/O procedures recommended by IFIP/WG 2.1, only three are mentioned.

Although the author claims the advantages of basing his work on an actual compiler, he frequently fails to clarify what position "his" compiler takes on well-known ambiguities in the ALGOL report. He attempts to make ALGOL easier to swallow—cutting it up into bite-size pieces by unwinding some of the recursion in the definitions—a format which requires much repetition. As a result, some rules have been stated in a short form which is harder to take than the original official formulation. For example, it takes some ten pages to get to the description of statements of the form: *if* b_1 *then* s_1 *else if* b_2 *then* s_2 *else* . . . *if* b_{n-1} *then* s_{n-1} *else* s_n.

For the most part, it is not easy to find in this book any discussions of the subtler or more uncertain points of ALGOL such as those raised by Knuth and Merner in "ALGOL 60 Confidential" or those explicitly left unresolved in the revised ALGOL report (which of course had to be resolved somehow in the compiler on which the book is based).

Some important omissions are the following:

(a) No mention is made of the initial values of *own* variables.

(b) The description of *own* variables do not distinguish whether the static or dynamic interpretation is intended.

(c) The effect of a "go to" into a conditional statement is not spelled out, although an example (p. 75) shows a simple case.

(d) It is not stated whether all the primaries of a simple Boolean are evaluated every time (for consistency in the operation of side effects).

(e) Simple arithmetic expressions are not defined well enough to forbid Bottenbruch's counterexamples: $2a + b$, $a \uparrow -2$, $2(x + y)$.

Following are some misprints and confusing words that were noted:

(a) pp. 34, 76, 251; the characters (, +, 0, . , 5, and) following ENTIER should be taken from the hardware representation character set.

(b) p. 38, rules 3, 4: two characters are chopped off at the end of each of these lines.

(c) p. 84, rule 6: "is by their appearance" means "is by the order of their appearance."

(d) p. 86, ex. 4: for "(A, I, N)" read "$(A, 1, N)$".

(e) p. 100, rule 4: since call by name is being described, delete "or assigned the value of."

(f) p. 111, rule 8: replace second sentence with: "If there is, rules 4 and 8 on pages 106 and 107 are applicable."

M. GOLDSTEIN

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

28 [12].—R. E. SMITH, *The Bases of FORTRAN*, Control Data Institute, Minneapolis, Minn., 1967, vii + 253 pp., 26 cm. Price $3.50.

A cursory glance through the contents of this book on Fortran would lead one to believe that it is a professional's manual for baseball players. Upon closer inspection, however, one finds that it is an attempt to present the basic concepts of Fortran programming in an appealing, informal way, without causing the reader to become overwhelmed by the subject matter as in the rigid system followed by most books on the subject.

Indeed, the author's approach to programming is quite unusual. The book is studded with most interesting anecdotes, challenging problems and thought-provoking questions seemingly unrelated to programming. Actually they are very much to the point. The ideas are cogent, and the diligent reader might well succeed in his efforts. It is questionable, however, considering the limited number of programs and techniques discussed in detail, that the average reader will be able to cope with the various problems presented in the text. Nevertheless, the approach is commendable. But after having mastered this book, the interested reader would be advised to follow up on this Fortran I presentation.

HENRY MULLISH

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

29 [12].—Charles M. Thatcher & Anthony J. Caputo, *Digital Computer Programming: Logic and Language*, Addison-Wesley Publishing Co., Reading, Mass., 1967, xi + 159 pp., 28 cm. Price $3.95 paperbound.

This soft-cover book is one of the many to appear on the computer programming scene in recent months. It is significantly different from several of its competitors for various reasons.

In the first place, much time is devoted to a discussion of a hypothetical computer (a good idea in itself) called HICOP, an acronym for a Highly Imaginary Computer for Orientation Purposes. Unfortunately, the beginner is just as ignorant of the need for a HICOP as he is about the workings of any nonhypothetical computer. His task is not made easier by being thrown directly into a host of definitions and a technical discussion of flow-charts followed by an explanation of iterative looping.

Furthermore, the manner in which the material is presented is couched in the jargon of the sophisticate—hardly the language of the average person to whom the authors rightfully address themselves in the preface. Taking, almost at random, a paragraph out of the otherwise excellent preface, will serve to illustrate the reviewer's objections since this kind of technical mumbo-jumbo can only confuse and confound a beginner.

"Significantly, the hypothetical computer accepts the problem-oriented language directly as its own machine language; but subsequent modifications to the computer's capability make it more realistic in this respect. These modifications ultimately lead to a typical machine language, in terms of which the internal operation of real computers is summarized." The object of such a book as this surely should be the enlightenment of the reader.

In the introduction to indexing, the unwary reader is presented with the following gem:

"Any repetitive operation lends itself to iterative looping. It might therefore be anticipated that the iterative technique can be used to good advantage when many separate items of data are to be subjected to the same manipulation, even though no recurrence formula can be written, since the items are distinctly separate."

I know a good number of experienced programmers who would CALL EXIT at this kind of description.

Henry Mullish

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

30 [12].—Gerald M. Weinberg, *PL/I Programming Primer*, McGraw-Hill, Inc., New York, 1966, ix + 278 pp., 24 cm. Price $5.95.

As stated in the Preface, this book is intended to be "an introduction to PL/I, to fill promptly the need for an introductory text on a new language. . . ." As such, it provides a fairly complete and readable introduction to the features of the language that it covers, but its value, even to the beginner, is severely diminished by omissions. Since this is one of the first books on the subject, and was written at a time (1966) when the PL/I language specifications were hazy and in a state of flux, many of these omissions are understandable. However, the complete absence of at least two important PL/I statements (LOCATE and REVERT) and of numerous minor but important techniques, (e.g., indexing a structure BY NAME), prevent the reader from fully appreciating the power of the PL/I language.

The organization of the material, based upon the elaboration of a few basic programs to incorporate more advanced features, is highly commendable. It is far superior to the obviously contrived and trivial examples often used in textbooks to illustrate programming techniques. The exercises are adequate to illustrate and reinforce the major points of the book, and the answers provided seem to be correct and complete.

There is a conspicuous lack of the usual technical appendices (the only one provided is the PL/I character set). This is unfortunate since one would appreciate at least a list of legal statements and their formats. This is especially vital since several statements are never mentioned, or, if they are, they are not included in the index. Presumably, the reader is at the mercy of the IBM-provided language specification or Mr. Weinberg's promised "second volume" which "is contemplated for the time when the most advanced features of the language are more firmly specified." Perhaps this volume will continue in the excellent style of the first, and will provide a reasonably complete introduction to PL/I. In the meantime, there are several other texts on the market, notably Frank Bates and Mary Douglas, *Programming Language/One*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1967, which seems better fare for both the novice and the experienced programmer.

HENRY MULLISH

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

31 [12].—GUENTHER HINTZE, *Fundamentals of Digital Machine Computing*, Springer-Verlag, New York, 1966, ix + 225 pp., 24 cm. Price $6.40.

As pointed out in the preface, this book is "the outgrowth of class notes developed over several years for a basic course on digital computers. . . ". The approach is nearer that of the computer designer than that of the programmer, starting with the representation of numbers, and progressing through logical design, instruction codes, and assembly language to an introduction to automatic programming. The book is somewhat dated, showing the computer family tree ending at the IBM 7094 and the CDC 3600, and the most recent reference being the revised Algol report of January 1963. It contains a number of generalizations which may have been reasonably accurate several years ago, but which appear very curious in the light of recent developments. For example, the sentence "Typical timing figures for presently used circuits, coordinating pulses and voltage levels, are pulse width .25 microseconds, response time for flip-flop approximately 1.5 microseconds, and pulse intervals 2 microseconds" is in no way consistent with commercially available packages which are more than 10 times as fast, and machines like the CDC 6600 using flip-flops which switch in 20 nanoseconds.

The personal preference of this reviewer is for course material which contains factual information about machines or programs to which the student has access, supplemented by a more general text if one is available. This book is in the category of general texts and is quite readable, but the material will need careful updating by the instructor.

MALCOLM C. HARRISON

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

32 [12, 13.05].—ALFRED M. BORK, *Fortran for Physics*, Addison-Wesley Publishing Co., Reading, Mass., 1967, viii + 85 pp. 23 cm. Price $1.95 paperbound.

This rather thin (85 pages) booklet is devoted to the physicist who has had no exposure to the mysteries of computer programming. In particular, the author addresses himself to classical mechanics. Indeed, the first three chapters are devoted to the subject of classical mechanics to the complete exclusion of Fortran. By this time the reader is left wondering whether Fortran *is* Physics. However, in Chapter 4 the reader is presented with an IBM 1620 Fortran II program which, regrettably, requires a substantial textbook on Fortran II to understand it.

HENRY MULLISH

New York University
Courant Institute of Mathematical Sciences
New York, New York 10012

33 [13.05].—JOHANN JAKOB BURCKHARDT, *Die Bewegungsgruppen der Kristallographie*, Birkhäuser Verlag, Basel, Switzerland, 1966, 209 pp., 25 cm. Price F 37.50.

There are very few readable, carefully developed derivations of crystallographic space groups available. This book is one of them. From the few known derivations of space groups, the author selects one which he has helped to develop. This derivation relies heavily on the concept of an arithmetic crystal class (which is to be distinguished from the more common geometric crystal class, or point group), and on the Frobenius congruences. The development is such as to require a minimum of mathematical background (even the required linear algebra is developed in the text). The theoretical development is accompanied by many examples, pictures, and tables.

This second edition does not differ markedly from the first, although several sections have been rewritten.

The author defines a point lattice L to be a subset of a Euclidean space R^ν which spans R^ν, which is closed under subtraction, and which has the property that there exists a positive, real number ϵ such that $x, y \in L \Rightarrow ||x - y|| > \epsilon$. A symmetry of a point lattice $L \subset R^\nu$ is then a function $f: R^\nu \rightarrow R^\nu$ such that $f(L) \subset L$, and $f(x) = Ax + a$, where A is a real $\nu \times \nu$ orthogonal matrix, and a is a real $\nu \times 1$ column matrix. The author develops some properties of lattices, and proceeds to define and develop properties of crystal classes, geometric and arithmetic crystal classes, and space groups. A few of the results obtained can be summarized in the following table.

dimension	*number of geometric crystal classes*	*number of arithmetic crystal classes*	*number of space groups*	*number of symmorphic space groups*
2	10	13	17	13
3	32	73	230	73

There are also brief discussions of space groups in higher dimensions, and of color groups.

It seems to this reviewer that a careful discussion of Bravais classes of lattices would have been desirable. Also, some discussion of isomorphism of crystallographic groups might have been worthwhile (the idea does, more or less, arise on pp. 128 and 179).

Some related but omitted topics are: (1) double groups, (2) representations of crystallographic groups, (3) experimental crystallography, (4) tensors in crystals, (5) lattice dynamics, (6) wave functions in crystals.

John Lomont

The University of Arizona
Tucson, Arizona 85721

34 [13.15, 13.35, 13.40].—S. H. Hollingdale, Editor, *Digital Simulation in Operational Research*, American Elsevier Publishing Co., New York, 1967, xv + 392 pp., 23 cm. Price $14.50.

The scope and contents of this volume are described by the editor in his foreword, from which we quote the following:

"This volume records the Proceedings of a Conference held in Unilever-Haus, Hamburg, from 6th to 10th September, 1965. It was sponsored by the N.A.T.O. Advisory Panel on Operational Research under the aegis of the Scientific Affairs Division of N.A.T.O. About 180 people, drawn from 13 countries, participated in the Conference; 41 papers (one opening address, three survey lectures and 37 short presentations) were presented in 14 sessions.

"The purpose of the conference was two-fold; to provide an opportunity for discussion and exchange of information between practitioners of the art of digital simulation, and to inform and stimulate those who have not yet made use of the technique. With computers now widely available, the possibility of using simulation methods has come within the reach of most operational research organisations.

"It is because of the dual objectives of the Conference that the papers in this volume cover so broad a spectrum—from descriptive accounts of specific applications to specialist expositions of methodological topics—and deal with so wide a range of industrial, commercial and military applications. The contributors themselves are drawn from N.A.T.O. and Governmental organisations, industry, commerce, universities and research institutes."

E. I.