A New Approximation Related to the Error Function

By W. R. Schucany and H. L. Gray

In a recent note R. G. Hart [1] gives the following expression:

$$P(x) = \frac{\exp(-x^2/2)}{x} \tag{1}$$
$$\times \left[1 - \frac{(1+bx^2)^{1/2}/(1+ax^2)}{P_0 x + [P_0^2 x^2 + \exp(-x^2/2)(1+bx^2)^{1/2}/(1+ax^2)]^{1/2}}\right],$$

where

$$P_0 = (\pi/2)^{1/2}$$
$$a = \frac{1 + (1 + 6\pi - 2\pi^2)^{1/2}}{2\pi}$$
$$b = 2\pi a^2,$$

which closely approximates

$$F(x) = \int_x^\infty \exp(-t^2/2)dt. \tag{2}$$

A new and more simple expression has been developed and may be employed for $x > 2$. We shall define

$$E(x) = \frac{x \exp(-x^2/2)}{x^2 + 2}\left[\frac{x^6 + 6x^4 - 14x^2 - 28}{x^6 + 5x^4 - 20x^2 - 4}\right], \tag{3}$$

and the motivation for such a definition may be found in [2] and [3].

It is readily apparent that $E(x)$ and $F(x)$ have the following properties in common:

1. For $x > 2$, $E(x)$ is real, positive and finite.
2. For $x > 2$, dE/dx is real, negative and finite.
3. As $x \to \infty$, $E(x) \to 0$.
4. As $x \to \infty$, $x \exp(x^2/2)\, E(x) \to 1$.
5. As $x \to \infty$, $dE/dx \to 0$.

The following table compares $E(x)$ to $P(x)$. $P(x)$ is a better fit than Hasting's approximation [4] for $x > 2$ and $E(x)$ appears to be superior to $P(x)$. The relative error, $\epsilon_\phi(x)$ is the quantity selected as a basis for comparison and

$$\epsilon_\phi(x) = \left|\frac{F(x) - \phi(x)}{F(x)}\right|. \tag{4}$$

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The numbers in parenthesis are powers of ten, e.g., $(-2).33837 = .33837 \times 10^{-2}$.

x	$F(x)$	$\epsilon_P(x)$	$\epsilon_E(x)$
2	(−1) .57026	(−3).49	(−2).20
3	(−2) .33837	(−3).20	(−5).99
4	(−4) .79388	(−4).48	(−4).10
5	(−6) .71853	(−5).43	(−5).39
6	(−8) .24730	(−5).56	(−5).13
8	(−14).15594	(−5).54	(−6).21
10	(−22).19100	(−5).32	(−7).42

For the range of values of x given in the table the expression for P is obviously more complicated than E and, in fact, on a digital computer requires about thirty percent more computation time. Unfortunately, however, the worth of $E(x)$ is questionable for $x < 2$, which is not the case for $P(x)$.

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Improvement in Recurrence Techniques for the Computation of Bessel Functions of Integral Order

By Fr. Mechel

The Bessel functions satisfy recurrence relations which are very convenient for the generation of these functions, especially when a great number of functions with varying index is needed, [1], [2], [3].

To start with the spherical Bessel functions $j_n(x)$, $y_n(x)$, the recurrence relation

$$f_{n-1}(x) + f_{n+1}(x) = ((2n + 1)/x)f_n(x) \tag{1}$$

is valid, which can be used either in the upward direction of the index n or in the downward direction. For the generation of the spherical Neumann functions $y_n(x)$ it must be used in the upward direction with the starting functions

$$y_0(x) = -\cos x/x \quad \text{and} \quad y_1(x) = -\sin x/x - \cos x/x^2 . \tag{2}$$

The computation of the spherical Bessel function $j_n(x)$ with Eq. (1) is more difficult, since now the recurrence relation must be used in the downward direction.

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