Recursion Formulae for Hypergeometric Functions

By Jet Wimp

I. Notation. The series definition for the generalized hypergeometric function is

$$(1)\qquad {}_PF_Q\left(\begin{matrix} a_P \\ b_Q \end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(a_P)_k x^k}{(b_Q)_k k!}\,,$$

where

$$(2)\qquad (\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$$

is Pochhammer's symbol and the shorthand product notation above will be used throughout this paper. In general, where a parameter has a subscript which is a capital letter, the repeated product notation is understood:

$$(3)\qquad (a_P)_k = \prod_{j=1}^{P} (a_j)_k\,, \qquad (n + b_Q) = \prod_{j=1}^{Q} (n + b_j)\,, \qquad \text{etc.}\,,$$

and the * notation

$$(4)\qquad (1 + b_Q - b_h)^* = \prod_{j=1;\ j\neq h}^{Q} (1 + b_j - b_h)$$

indicates the term corresponding to $j = h$ is to be deleted.

If one of the $a_i = 0$ or a negative integer, then (1) always converges, since it terminates. Otherwise it converges for all finite x if $P \leqq Q$ and for $|x| < 1$ if $P = Q + 1$. In this case, however, the function can be analytically continued into the cut plane $|\arg (1 - x)| < \pi$, and we shall often denote by ${}_{Q+1}F_Q(x)$ not only the series (1), whenever it converges, but also the analytic continuation of the series. If $P > Q + 1$, the series does not converge (unless it terminates) and if one of the b_j is 0 or a negative integer, the series is not defined. If one of the a_i equals one of the b_j, ${}_PF_Q(x)$ reduces to ${}_{P-1}F_{Q-1}(x)$ and such a case is always excluded from consideration in this paper. We assume all ${}_PF_Q$'s are irreducible.

Equation (1) can be given an interpretation for $P > Q + 1$ by means of the G-function

$$(5)\qquad \frac{\Gamma(b_Q)}{\Gamma(a_P)}\, G^{1,P}_{P,Q+1}\left(-x \middle| \begin{matrix} 1 - a_P \\ 0, 1 - b_Q \end{matrix}\right)$$

and (5) is (1) (or its analytic continuation) if $P \leqq Q + 1$. The G-function can be defined by a Mellin-Barnes contour integral.

For a treatment of the generalized hypergeometric function and the G-function, see [1].

We also assume that (5), wherever it occurs, is irreducible, i.e., that no a_i equals any b_j, $i = 1, 2, \cdots, P, j = 1, 2, \cdots, Q$.

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II. Introduction. The subject of the recursion relations satisfied by hypergeometric functions occupies a prominent place in the literature of special functions. The functions of this type for which recursion formulae have been given are usually special cases of the functions

$$U_n(\lambda) = \frac{(a_P)_n\lambda^n}{(b_Q)_n(\gamma+n)_n}\, {}_{P+1}F_{Q+1}\left(\begin{matrix} n+a_{P+1} \\ n+b_Q,\, 2n+\gamma+1 \end{matrix}\,\middle|\,\lambda\right), \tag{6}$$

or of the polynomials

$$P_n(z) = \frac{1}{\Gamma(d_T)}\, {}_{R+2}F_T\left(\begin{matrix} -n,\, n+\gamma,\, c_R \\ d_T \end{matrix}\,\middle|\,z\right), \tag{7}$$

or

$$Q_n(z) = \frac{1}{\Gamma(d_T)}\, {}_{R+1}F_T\left(\begin{matrix} -n,\, c_R \\ d_T \end{matrix}\,\middle|\,z\right). \tag{8}$$

It can be shown that (6)–(8) obey linear recursion relationships of the form

$$\sum_{\nu=0}^{\rho} [k_\nu + xl_\nu]\Phi_{n+\nu} = 0\,, \tag{9}$$

where $x = 1/\lambda$ for (6), $x = z$ for (7) and (8), and $k_\nu = k_\nu(n)$, $l_\nu = l_\nu(n)$ depend on the particular function, but not on z or λ. Also, $k_0 = 1$, $l_0 = 0$, and ρ depends on the number of numerator and denominator parameters in the hypergeometric function: $\rho = \max[P+1, Q+2]$ for (6), $\rho = \max[T+1, R+2]$ for (7) and (8).

$U_n(\lambda)$ can be given an interpretation for $P > Q+1$ by means of the G-function

$$\begin{aligned} U_n(\lambda) &= \frac{\Gamma(b_Q)}{\Gamma(a_P)}\,(-)^n\tau_n G^{1,P+1}_{P+1,Q+2}\left(-\lambda\,\middle|\,\begin{matrix} 1-a_{P+1} \\ n,\, -n-\gamma,\, 1-b_Q \end{matrix}\right), \\ \tau_n &= (2n+\gamma)\Gamma(n+\gamma)/\Gamma(n+\beta+1)\,, \end{aligned} \tag{10}$$

provided a_i is not 0 or a negative integer, $i = 1, 2, \cdots, P+1$.

There exists a duality between the functions (7) and (10). For instance, we have, under a variety of conditions (see [2, Eq. (2.6)] and also related expansions in [3], [4]),

$$\begin{aligned} G^{P+R+1,1}_{Q+T+1,P+R+1}\left(-\frac{1}{\lambda z}\,\middle|\,\begin{matrix} 1,\, b_Q,\, d_T \\ c_R,\, a_{P+1} \end{matrix}\right) &= \frac{\Gamma(c_R)\Gamma(a_P)}{\Gamma(b_Q)} \sum_{n=0}^{\infty} (-)^n(n+1)_\beta \\ &\quad \times U_n(\lambda)P_n(z)\,, \end{aligned} \tag{11}$$

and if, in this multiplication formula, z is replaced by z/γ and $\gamma \to \infty$, a similar expansion in terms of $Q_n(z)$ results.

In fact, any function analytic at $z = 0$ can be expanded in a series of the polynomials P_n or Q_n, and Fields and Wimp studied such expansions from the standpoint of basic series in [6]. Linear combinations of P_n, Q_n also occur in classes of rational approximations to generalized hypergeometric functions, see [7] and the references given there.

For $R = 0$, $T = 1$, P_n is related to the Jacobi polynomial, as we have seen, and Q_n to the Laguerre polynomial. Here $\rho = 2$, and the recurrence formulae are classical. For $R = 0$, $T = 0$, P_n is the Bessel polynomial, whose recursion formula and other properties have recently been studied by a number of writers, see [8].

Recursion formulae for P_n for $R = 1, T = 2$ ($\rho = 3$) have been studied for various special values of the parameters, see [9]. For values of $\rho > 3$, i.e., larger values of R, T, no general results seem to exist in the literature, although general formulae for $\rho = 3$ have been derived but not published, [6].

When $P = 1$, $Q = 0$, then $\rho = 2$ and $U_n(\lambda)$ is related to the Jacobi function, $Q_n^{(\alpha,\beta)}$, whose recursion formula is given in [5]. No general formulae for larger values of P, Q seem to be known. However, for special values of γ and β, the recursion formula for $P = 2$, $Q = 0$ is given in [3], where it was also shown that $U_n(\lambda)$ could be computed by using (9) in the backward direction.

Since $U_n(\lambda)$ can often be computed by using (9) in the backward direction, and P_n and Q_n always by using (9) in the forward direction, it is quite desirable to have closed form expressions for l_ν, k_ν. It was previously doubted that such expressions existed, since the derivation of particular recursion formulae has hithertofore involved solving systems of algebraic equations whose complexity increases rapidly with P, Q, R and T.

In this paper, we determine closed form expressions for the coefficients in the recursion formula for $U_n(\lambda)$. These coefficients are terminating hypergeometric functions of unit argument. We show that $U_n(\lambda)$ satisfies one and only one recursion relation of type (9) of a certain order and none of a lower order. We next find a number of other solutions of (9), considered as a difference equation. It turns out that certain of these solutions are closely related to P_n, and by specialization of a certain parameter, we are able to determine the recursion formula for $P_n(z)$. Next, by taking a limit as $\gamma \to \infty$, we find the recursion formula for $Q_n(z)$.

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III. Results.

THEOREM 1. *Let P, Q, n be integers $\geqq 0$. Let β, γ, a_i, b_j, $i = 1, 2, \cdots, P$, $j = 1, 2, \cdots, Q$ be complex constants such that none of the quantities $\beta + 1$, a_i, b_j, γ are negative integers or zero. Let λ be a complex variable, finite and $\neq 0$, and let $a_i = \beta + 1$ for $i = P + 1$. Then the following statements are true*:

(1) *the functions $U_n(\lambda)$ as given by* (10) *satisfy the difference equation*

$$\sum_{\nu=0}^{\sigma} \left[A_\nu + \frac{B_\nu}{\lambda} \right] \Phi_{n+\nu}(\lambda) = 0\,, \qquad \sigma = \max\,[P + 1, Q + 2]\,, \tag{12}$$

where

$$\begin{aligned} A_\nu = {} & \frac{(-)^\nu (2n+\gamma)_\nu}{\nu!(n+\gamma)_\nu}(n+\beta+1) \\ & \times {}_{\nu P+3}F_{P+2}\left(\begin{matrix} -\nu, 2n+\gamma+\nu, n+a_{P+1}+1 \\ 2n+\gamma+\sigma+1, n+a_{P+1} \end{matrix} \middle| 1 \right), \end{aligned} \tag{13}$$

$$\begin{aligned} B_\nu = {} & \frac{(-)^\nu (2n+\gamma)_{\nu+1}(n+\beta+1)_\nu (n+b_Q)}{\Gamma(\nu)(n+\gamma)_\nu (n+a_{P+1})} \\ & \times {}_{Q+2}F_{Q+1}\left(\begin{matrix} 1-\nu, 2n+\gamma+\nu+1, n+b_Q+1 \\ 2n+\gamma+\sigma+1, n+b_Q \end{matrix} \middle| 1 \right) \end{aligned} \tag{14}$$

$(A_0 = 1, B_0 = B_\sigma = 0)$;

(2) *other solutions of* (12) *are*

Case A. $\sigma = Q + 2; p < Q + 1$ *or* $P = Q + 1$, $|\arg(1 - \lambda)| < \pi$; (*here* $U_n(\lambda)$ *is given by* (6));

$$\psi_n(\lambda) = \frac{(-)^{n(P+1)}\tau_n\lambda^{-n}}{\Gamma(b_Q - n - \gamma)\Gamma(n + \gamma + 1 - a_{P+1})\Gamma(1 - \gamma - 2n)} \times {}_{P+1}F_{Q+1}\left(\begin{matrix} a_{P+1} - n - \gamma \\ b_Q - n - \gamma, 1 - \gamma - 2n \end{matrix} \middle| \lambda\right), \tag{15}$$

$$\phi_n^{[h]}(\lambda) = \frac{\tau_n}{\Gamma(2 - b_h - n)\Gamma(n + \gamma + 2 - b_h)\Gamma(1 + b_Q - b_h)} \times {}_{P+1}F_{Q+1}\left(\begin{matrix} 1 + a_{P+1} - b_h \\ (1 + b_Q - b_h)^*, 2 - b_h - n, n + \gamma + 2 - b_h \end{matrix} \middle| \lambda\right), \tag{16}$$

$h = 1, 2, \cdots, Q$;

Case B. $\sigma = P + 1; P > Q + 1$ *or* $P = Q + 1$, $|\arg(1 - 1/\lambda)| < \pi$;

$$\theta_n^{[h]}(\lambda) = \frac{\tau_n(a_h)_n(-)^n}{\Gamma(n + \gamma + 1 - a_h)\Gamma(1 + a_h - a_{P+1})} \times {}_{Q+2}F_P\left(\begin{matrix} n + a_h, -n - \gamma + a_h, 1 - b_Q + a_h \\ (1 + a_h - a_{P+1})^* \end{matrix} \middle| \frac{(-)^{Q+P+1}}{\lambda}\right), \tag{17}$$

$h = 1, 2, \cdots, P + 1$;

(3) *none of the functions above satisfy any other difference equation of type* (12), *with* $A_0 = 1, B_0 = B_\sigma = 0$, *of order* $\leqq \sigma$.

Note. We assume U_n is not reducible for all n, i.e., no b_i equals any a_j or $\beta + 1$. However, for particular values of n, U_n may be reducible. Such will be the case if any $a_j = r + \gamma + 1, j = 1, 2, \cdots, P + 1$, r an integer $\geqq 0$.

Proof. First we note that

$$ {}_{M+2}F_{M+1}\left(\begin{matrix} -\nu, \nu + \mu, 1 + a_M \\ \mu + r, a_M \end{matrix} \middle| 1\right) = 0, \qquad \nu, r = 0, 1, 2, \cdots, \tag{18}$$

for $M < r \leqq \nu$, as can be seen by writing out the νth difference with respect to x of $\prod_{t=1}^{\nu-r}(x + r + \mu - 1 + t)\prod_{j=1}^{M}(x + a_j)$ at $x = 0$. This shows that, if (13) and (14) are true, then $A_\nu = 0, \nu > \sigma$ and $B_\nu = 0, \nu \geqq \sigma$, in particular, that $B_\sigma = 0$, as stated.

Next, we remark that if $P < Q + 1$, or $P = Q + 1$ and $|\arg(1 - \lambda)| < \pi$, then $U_n(\lambda)$ is precisely (6). If $P > Q + 1$ or $P = Q + 1$ and $|\arg(1 - 1/\lambda)| < \pi$, then $U_n(\lambda)$ is a sum of the functions $\theta_n^{[h]}(\lambda)$, $h = 1, 2, \cdots, P + 1$. See [10].

Let $P < Q + 1$ or $P = Q + 1$ and $|\lambda| < 1$. By substituting $U_n(\lambda)$ into the difference equation and equating to zero the coefficient of λ^{n+k}, we find that the theorem demands that

$$S_1(k) + S_2(k) \equiv 0, \tag{19}$$

where

$$S_1(k) = (n + b_Q + k)\sum_{\nu=0}^{\sigma} \frac{\tau_{n+\nu}A_\nu}{\Gamma(k - \nu + 1)\Gamma(2n + \nu + k + \gamma + 1)}, \tag{20}$$

$$S_2(k) = (n + a_{P+1} + k)\sum_{\nu=1}^{\sigma-1} \frac{\tau_{n+\nu}B_\nu}{\Gamma(k - \nu + 2)\Gamma(2n + \nu + k + \gamma + 2)}. \tag{21}$$

Now substitute the functions $\phi_n{}^{[h]}$ into (12) and equate to zero the coefficient of λ^k. The result is

$$(22)\quad S_1(k+1-n-b_h) + S_2(k+1-n-b_h) \equiv 0\,, \qquad h = 1, 2, \cdots, Q\,,$$

with the same value of σ as above.

Substituting $\psi_n(\lambda)$ into (12) and equating to zero the coefficient of λ^{-n+k}, we see we must have

$$(23)\qquad S_1(k-2n-\gamma) + S_2(k-2n-\gamma) \equiv 0\,.$$

Finally, let $P > Q+1$ or $P = Q+1$ and $|\lambda| > 1$ and consider the functions $\theta_n{}^{[h]}(\lambda)$. Proceeding as above, we see that we must have

$$(24)\quad S_1(-k-a_h-n) + S_2(-k-a_h-n) \equiv 0\,, \qquad h = 1, 2, \cdots, P+1\,.$$

If (19) is multiplied by $\Gamma(k+1)\Gamma(2n+\sigma+k+\gamma+1)$ which is defined for all k in some right half-plane, then (19) becomes a polynomial in k, and we see that a necessary and sufficient condition for (19) to hold is that

$$(25)\qquad (n+b_Q+k)f_1(k) + (n+a_{P+1}+k)f_2(k) \equiv 0\,,$$

$$(26)\qquad f_1(k) = \sum_{\nu=0}^{\sigma} (-)^{\nu}(-k)_{\nu}(2n+k+\nu+\gamma+1)_{\sigma-\nu}\overline{A}_{\nu}\,,$$

$$(27)\qquad f_2(k) = \sum_{\nu=1}^{\sigma-1} (-)^{\nu-1}(-k)_{\nu-1}(2n+k+\nu+\gamma+2)_{\sigma-\nu-1}\,\overline{B}_{\nu}\,,$$

where k is a generally complex-valued variable, and

$$(28)\qquad \overline{A}_{\nu} = \tau_{n+\nu}A_{\nu}\,, \qquad \overline{B}_{\nu} = \tau_{n+\nu}B_{\nu}\,.$$

Thus, if $\overline{A}_{\nu}$, $\overline{B}_{\nu}$ can be chosen so that (25) holds, the functions U_n, ψ_n, $\phi_n{}^{[h]}$, $\theta_n{}^{[h]}$ will satisfy the difference equation whenever the series defining them converge, since (19)–(24) are all equivalent to (25)–(27).

We now discuss the quantity σ, which up till now has been unspecified.

Note that $f_1(k)$ is a polynomial in k of degree σ at most and, since no b_i equals any a_i or $\beta+1$, has zeros at $k = -n-a_i$, $i = 1, 2, \cdots, P+1$.

Or

$$(29)\qquad f_1(k) \equiv (n+a_{P+1}+k)M_r(k)\,,$$

where $M_r(k)$ is a polynomial of degree r in k. Neither f_1 nor M_r can be identically zero, since

$$(30)\qquad f_1(0) = (2n+\gamma+1)_{\sigma}\overline{A}_0\,.$$

Equation (29) shows that, for some integer m_1, $m_1 \geqq 0$, $\sigma - m_1 = P+r+1$ or $\sigma \geqq P+1$.

Likewise, f_2 is a polynomial of degree $\sigma - 2$ at most and

$$(31)\qquad f_2(k) = (n+b_Q+k)N_s(k)\,,$$

where N_s is a polynomial of degree s in k. Setting $k = 0$ in (25) gives

$$(32)\qquad \overline{B}_1 = -(n+b_Q)(2n+\gamma+1)_2\,\overline{A}_0/(n+a_{P+1})$$

and clearly this is the only possible value of $\overline{B}_1$.

Furthermore,

$$f_2(0) = -(n + b_Q)(2n + \gamma + 1)_\sigma \overline{A}_0/(n + a_{P+1}) \tag{33}$$

so $N_s(k) \not\equiv 0, f_2(k) \not\equiv 0$; (31) shows that, for some integer $m_2 \geqq 0$, $\sigma - m_2 - 2 = Q + s$ or $\sigma \geqq Q + 2$.

Thus, the smallest possible value of σ is

$$\sigma = \max [P + 1, Q + 2] . \tag{34}$$

Assume σ has this value. We will show that $\overline{A}_\nu$, $\overline{B}_\nu$ (hence, A_ν, B_ν) are then uniquely determined by (25) and that $A_\sigma \not\equiv 0$, which means that no other recursion relationship of order $\leqq \sigma$ exists for any of the given functions, i.e., statement (3) of the theorem. (It is clear, however, that larger values of σ are possible, e.g., add to (12) the recursion relationship obtained by replacing n by $n + 1$ and the result is a recursion formula of order $\sigma + 1$.)

LEMMA 1. *Let the conditions of the theorem hold. Then* (25) *is true if and only if* $\overline{A}_\nu$, $\overline{B}_\nu$ *are such that*

$$f_1(k) \equiv (2n + \gamma + 1)_\sigma (n + a_{P+1} + k)\overline{A}_0/(n + a_{P+1}) , \tag{35}$$

$$f_2(k) \equiv -(2n + \gamma + 1)_\sigma (n + b_Q + k)\overline{A}_0/(n + a_{P+1}) . \tag{36}$$

If k is assigned σ distinct values in (35) and $\sigma - 2$ distinct values in (36), then $\overline{A}_\nu$, $\nu = 1, 2, \cdots, \sigma$ and $\overline{B}_\nu$, $\nu = 2, 3, \cdots, \sigma - 1$ are uniquely determined and so, by (28), are A_ν, B_ν. Also, $A_\sigma \not\equiv 0$.

Proof. First assume $P > Q + 1$, $\sigma = P + 1$. Then $f_1(k)$ is a polynomial of degree $P + 1$ at most. But since $f_1(k) \not\equiv 0$, (29) shows it must be exactly of degree $P + 1$, and

$$f_1(k) = K(n + a_{P+1} + k) . \tag{37}$$

Letting $k = 0$ and using (30) determines K, and when (35) is substituted into (25), (36) follows.

Let $P \leqq Q + 1$, $\sigma = Q + 2$; $f_2(k)$ is a polynomial in k of degree Q at most. As before, $f_2(k) \not\equiv 0$ and so

$$f_2(k) = K'(n + b_Q + k) . \tag{38}$$

Letting $k = 0$ and using (33) we find K' whence (36) follows. When (36) is substituted into (25), (35) results.

Now let σ distinct values k_i, $i = 1, 2, \cdots, \sigma$ be assigned to k in (35). The result is σ nonhomogeneous equations in the σ unknowns $\overline{A}_\nu$, $\nu = 1, 2, \cdots, \sigma$. Now this system has a unique solution which is independent of the values of k assigned.

Let V_R denote the alternate determinant

$$V_R(x_R) = |x_i^{\,j-1}|_{i,j=1,2,\cdots,R} = \prod_{m=2}^{R} \prod_{l=1}^{m-1} (x_m - x_{m-l}) . \tag{39}$$

Here and in what follows, τ_{ij} is the element in the ith row and jth column of the determinant $|\tau_{ij}|_{i,j=1,2,\cdots,R}$. The determinant of the system formed from (35) is

$$D = |(-)^{j-1}(1 - k_i)_{j-1}(2n + k_i + j + \gamma + 1)_{\sigma-j}|_{i,j=1,2,\cdots,\sigma} \tag{40}$$

which, by [11], is

$$D = KV_\sigma(k_\sigma) \tag{41}$$

and K is independent of the k_i's. To determine K, let $k_i = i$. The resulting determinant is triangular, and we find

$$D = V_\sigma(k_\sigma) \prod_{i=1}^{\sigma} (2n + 2i + \gamma + 1)_{\sigma-i} \tag{42}$$

so, under our hypotheses, $D \neq 0$. If the system is solved by Cramer's rule, it can be verified that $V_\sigma(k_\sigma)$ also factors out of each numerator determinant, leaving a quantity independent of the k_i's. Thus, $\bar{A}_\nu$ is uniquely determined by (35), and similarly one can show that $\bar{B}_\nu$ is uniquely determined by (36), with $\bar{B}_1$ given by (32). $\bar{A}_\sigma$, hence A_σ, can be found by putting $k = -\sigma - \gamma - 2n$ in (35), and the result is displayed in Theorem 2, Eq. (52). Under our hypothesis, $A_\sigma \neq 0$.

It remains to prove that A_ν, B_ν are indeed given by (13) and (14). For this, we require two more lemmas.

LEMMA 2. *Let k, b and z be complex quantities, $b + k + 1 \neq 0, -1, -2, \cdots$, and s an integer $\geqq 0$. Then*

$$\sum_{\nu=0}^{s} \frac{(b + 2\nu)(-k)_\nu(b + z)_\nu}{(1 - z)_\nu(b + k + 1)_\nu} = \frac{z(k + b) + \dfrac{(-k)_{s+1}(b + z)_{s+1}}{(b + k + 1)_s(1 - z)_s}}{(z - k)}. \tag{43}$$

Remark. Since the left-hand side and the right-hand side of (43) are the same meromorphic function of z, they have the same residues at the simple poles $z = 1, 2, \cdots, s$ and possess the same limit as $z \to k$.

Proof. By induction on s.

LEMMA 3. *If*

$$f_k = \sum_{\nu=0}^{k} \frac{(-k)_\nu g_\nu}{(a + k)_\nu}, \qquad k = 0, 1, 2, \cdots, M \geqq 0, \tag{44}$$

then

$$g_\nu = \frac{(a + 2\nu - 1)}{\nu!} \sum_{s=0}^{\nu} \frac{(-\nu)_s(a + s)_\nu f_s}{s!(a + s + \nu - 1)} \tag{45}$$

provided $a \neq 0, -1, -2, \cdots$.

Proof. The determinant of the system is nonzero, so (44) has a unique solution. The lemma then results by substituting (45) in (44), interchanging the order of summation, and using Lemma 2 with $z = 0$.

Now, in (35) let $k = 0, 1, 2, \cdots, \sigma$. Then

$$f_k = \sum_{\nu=0}^{k} \frac{(-k)_\nu(-)^\nu \bar{A}_\nu}{(2n + \gamma + k + 1)_\nu} = \frac{(2n + \gamma + 1)_\sigma(n + a_{P+1} + k)\bar{A}_0}{(n + a_{P+1})(2n + \gamma + k + 1)_\sigma} \tag{46}$$

and this system is the form in Lemma 3 with $g_\nu = (-)^\nu \bar{A}_\nu$, $a = 2n + \gamma + 1$. Thus $\bar{A}_\nu$ and hence A_ν is easily found and the result is (13). $\bar{B}_\nu$ is similarly determined by applying Lemma 3 to (36).

The extension of the theorem to values λ such that $|\arg(1 - \lambda)| < \pi$ in Case A, $P = Q + 1$, or $|\arg(1 - 1/\lambda)| < \pi$ in Case B, $P = Q + 1$ is immediate by the permanence principle for functional equations [12].

The proof of Theorem 1 is complete.

Note that no restrictions on b_i enter in the proof of the theorem; the restriction that $b_i \neq 0, -1, -2, \cdots$, arises from the definition (6). In fact, by slightly modifying (12) (e.g., multiplying by $(n + a_{P+1})$) or the solutions of the difference equation (e.g., dividing $U_n(\lambda)$ by $\Gamma(b_Q)$), the theorem can be made valid for a_i, b_j negative integers. Also, Φ_n may be redefined so that the theorem will hold for all values of $\beta + 1$ and γ.

Now if no two of the quantities $[n, b_Q, -\gamma - n]$ differ by an integer or zero, all the solutions in Case A are distinct, and if no two of the quantities $[a_{P+1}]$ differ by an integer or zero, all the solutions in Case B are distinct. In fact, under these restrictions the functions in each group are linearly independent functions of λ, as is seen by comparing their behavior near $\lambda = 0$ or $\lambda = \infty$. This is not at all the same as asserting that the functions in either group are linearly independent as functions of n.

If $2n + \gamma$ is an integer, $\psi_n(\lambda)$ is proportional to $U_n(\lambda)$, while if two of the quantities $[b_Q]$ (or $[a_{P+1}]$) differ by an integer or zero, then two of the functions $[\phi_n^{[Q]}]$ (or $[\theta_n^{[P+1]}]$) are proportional. However, in any of these cases a distinct set of solutions can be constructed. For example, let $a_i = a_j + m$, $m = 0, 1, 2, 3, \cdots$. Then one forms an appropriate difference of the functions $\theta_n^{[i]}$, $\theta_n^{[j]}$ for $a_i = a_j + m + \epsilon$, divides by ϵ, and lets $\epsilon \to 0$. See [13] for the mechanics of this procedure.

We will subsequently need the following integral representations of (13) and (14).

LEMMA 4. *Let none of the quantities γ, a_i, $i = 1, 2, \cdots, P + 1$ be negative integers or zero. Then, for general σ, we have*

$$A_\nu = \frac{v_{n,\nu}}{2\pi i}\int_{\Gamma_\nu} \frac{\Gamma(2n+\gamma+\nu+z)\Gamma(-z)(n+a_{P+1}+z)\,dz}{\Gamma(2n+\gamma+\sigma+1+z)\Gamma(\nu+1-z)}, \tag{47}$$

$$B_\nu = \frac{v_{n,\nu}}{2\pi i}\int_{\Gamma_{\nu-1}} \frac{\Gamma(2n+\gamma+\nu+1+z)\Gamma(-z)(n+b_Q+z)\,dz}{\Gamma(2n+\gamma+\sigma+1+z)\Gamma(\nu-z)}, \tag{48}$$

$$v_{n,\nu} = \frac{(-)^{\nu+1}(2n+\gamma)_{\sigma+1}(n+\beta+1)_\nu}{(n+\gamma)_\nu(a_{P+1}+n)} \tag{49}$$

and Γ_m denotes a simple closed path enclosing the points $z = 0, 1, 2, \cdots, m$ but no other singularities of the integrand.

Proof. By the residue theorem. Note that Γ_m is a feasible path since, were any of the poles of $\Gamma(2n + \gamma + \nu + z)$ (or $\Gamma(2n + \gamma + \nu + z + 1)$) to coincide with any of the poles of $\Gamma(-z)$, then γ would be zero or a negative integer.

We now give alternate representations of A_ν, B_ν which are useful when ν is larger than $[\sigma/2]$.

THEOREM 2. *Let none of the quantities γ, $\beta + 1$, a_i, $i = 1, 2, \cdots, P$ be negative integers or zero. Then*

$$\begin{aligned} A_\nu = {}& \frac{(-)^{\nu+P+1}(2n+\gamma)_{\sigma+1}(n+\beta+1)_\nu(n+\gamma+\nu-a_{P+1})}{\Gamma(\sigma+1-\nu)(n+\gamma)_\nu(2n+\gamma+\nu)_{\nu+1}(n+a_{P+1})} \\ & \times {}_{P+3}F_{P+2}\left(\begin{matrix}\nu-\sigma, 2n+\gamma+\nu, n+\gamma+\nu+1-a_{P+1} \\ 2n+\gamma+2\nu+1, n+\gamma+\nu-a_{P+1}\end{matrix}\middle|\, 1\right), \end{aligned} \tag{50}$$

$$(51)\qquad \begin{aligned} B_\nu = {} & \frac{(-)^{\nu+Q}(2n+\gamma)_{\sigma+1}(n+\beta+1)_\nu(n+\gamma+\nu+1-b_Q)}{\Gamma(\sigma-\nu)(n+\gamma)_\nu(2n+\gamma+\nu+1)_\nu(n+a_{P+1})} \\ & \times {}_{Q+2}F_{Q+1}\left(\begin{matrix}\nu+1-\sigma, 2n+\gamma+\nu+1, n+\gamma+\nu+2-b_Q \\ 2n+\gamma+2\nu+1, n+\gamma+\nu+1-b_Q\end{matrix}\,\middle|\, 1\right), \end{aligned}$$

and in particular

$$(52)\qquad A_\sigma = \frac{(-)^{\sigma+P+1}(2n+\gamma)_\sigma(n+\beta+1)_\sigma(n+\gamma+\sigma-a_{P+1})}{(n+\gamma)_\sigma(2n+\gamma+\sigma+1)_\sigma(n+a_{P+1})}.$$

Proof. We prove (50) only, since (51) follows similarly. Denote the integrand of (47) by $L_n(z)$. It has poles at the points $\delta_m = -2n-\gamma-m$, $m = \nu, \nu+1, \cdots, \sigma$ and γ_m, $m = 0, 1, 2, \cdots, \nu$. The integral around any large circle containing both $\{\gamma_m\}$ and $\{\delta_m\}$ is zero, since $L_n(z) = O\{z^{P-\sigma-1}\}$, $|z| \to \infty$, and is a rational function of z. If Δ_ν is any simple closed curve containing the points $\{\gamma_m\}$ but none of the points $\{\delta_m\}$, then

$$(53)\qquad \int_{\Gamma_\nu} = -\int_{\Delta_\nu}$$

and (50), and hence (52), follow immediately by the residue theorem. (Note the hypotheses separate the points $\{\gamma_m\}$ from $\{\delta_m\}$.)

Because of the form of the functions $\theta_n^{[h]}(\lambda)$, Theorems 1 and 2 enable us to give explicit recurrence formulae for the classes of hypergeometric polynomials studied in [4].

COROLLARY 1. *Let R and T be integers $\geqq 0$, $\tau = \max\,[T+1, R+2]$. Let γ, c_i, d_j, $i = 1, 2, \cdots, R$, $j = 1, 2, \cdots, T+1$, ($d_j = 1$ for $j = T+1$) be complex constants such that none of the quantities γ, $\gamma+1-d_j$, $j = 1, 2, \cdots, T$ are negative integers or zero. Then the hypergeometric polynomials $P_n(z)$, see (7), satisfy the recursion relationship*

$$(54)\qquad \sum_{\nu=0}^{\tau}[C_\nu + zD_\nu]P_{n-\nu}(z) = 0\,, \qquad n = \tau, \tau+1, \tau+2, \cdots,$$

where

$$(55)\qquad \begin{aligned} C_\nu = {} & \frac{(-)^\nu(n+1-\nu)_\nu(1-\gamma-2n)_{2\nu}(n-\nu-1+d_{T+1})}{\nu!(n+\gamma-\nu)_\nu(\tau+1-\gamma-2n)_\nu(n+d_{T+1}-1)} \\ & \times {}_{T+3}F_{T+2}\left(\begin{matrix}-\nu, 2n+\gamma-\tau-\nu, n-\nu+d_{T+1} \\ 2n+\gamma+1-2\nu, n-\nu-1+d_{T+1}\end{matrix}\,\middle|\, 1\right) \end{aligned}$$

and

$$(56)\qquad \begin{aligned} D_\nu = {} & \frac{(-)^{\nu+1}(n+1-\nu)_\nu(1-\gamma-2n)_{2\nu}(n-\nu+c_R)}{\Gamma(\nu)(n+\gamma-\nu)_\nu(1+\tau-\gamma-2n)_{\nu-1}(n+d_{T+1}-1)} \\ & \times {}_{R+2}F_{R+1}\left(\begin{matrix}1-\nu, 2n+\gamma+1-\tau-\nu, n+1-\nu+c_R \\ 2n+\gamma+1-2\nu, n-\nu+c_R\end{matrix}\,\middle|\, 1\right) \end{aligned}$$

and $D_0 = D_\tau = 0$.

Proof. In $\theta_n^{[P+1]}(\lambda)$ let $Q = R$, $P = T$, $a_j = \gamma+1-d_j$ ($d_{T+1} = 1$), $b_j = \gamma+1-c_j$, $\beta+1 = \gamma$, $z = (-)^{Q+P+1}/\lambda$, $\sigma = \tau$. Then (55) and (56) follow from Theorem 2 when the sums are turned around and n is replaced by $n - \tau$; since the polynomials are computed in the forward direction, this is the more useful form of the recursion relationship. Note that it is not necessary to assume $P > Q + 1$ in using Theorem 2. Since $\theta_n^{[P+1]}(\lambda)$ terminates, the recursion formula is valid for all P, Q. Also, alternate forms for C_ν, D_ν which are useful when $\nu > [\sigma/2]$ can be determined from Theorem 1.

COROLLARY 2. *Let R and T be integers $\geqq 0$, $\tau = \max[T+1, R+2]$, and let c_i, d_j, $i = 1, 2, \cdots, R$, $j = 1, 2, \cdots, T+1$ be complex constants, ($d_j = 1$ for $j = T + 1$). Then the hypergeometric polynomials $Q_n(z)$, see* (8), *satisfy the recursion relationship*

$$\sum_{\nu=0}^{l_1} E_\nu Q_{n-\nu}(z) + z\sum_{\nu=1}^{l_2} F_\nu Q_{n-\nu}(z) = 0\,, \tag{57}$$

$l_1 = \min[\tau, T+1]$, $l_2 = \min[\tau - 1, R+1]$, $n = \tau + \delta, \tau + \delta + 1, \tau + \delta + 2, \cdots$, $\delta = 0$ or -1, *where*

$$E_\nu = \frac{(n+1-\nu)_\nu(n-\nu-1+d_{T+1})}{\nu!(n+d_{T+1}-1)}\;{}_{T+2}F_{T+1}\left(\begin{matrix}-\nu, n-\nu+d_{T+1}\\ n-\nu-1+d_{T+1}\end{matrix}\middle|\,1\right), \tag{58}$$

$$F_\nu = \frac{(n+1-\nu)_\nu(n-\nu+c_R)}{\Gamma(\nu)(n+d_{T+1}-1)}\;{}_{R+1}F_R\left(\begin{matrix}1-\nu, n+1-\nu+c_R\\ n-\nu+c_R\end{matrix}\middle|\,1\right). \tag{59}$$

Proof. Let

$$Q_n^{(\gamma)}(z) = P_n(z/\gamma)\,. \tag{60}$$

Then

$$\lim_{\gamma\to\infty} Q_n^{(\gamma)}(z) = Q_n(z)\,. \tag{61}$$

If we form the difference equation for $Q_n^{(\gamma)}(z)$ we see we must have

$$\lim_{\gamma\to\infty} C_\nu = E_\nu\,, \qquad \lim_{\gamma\to\infty}\gamma^{-1}D_\nu = E_\nu\,. \tag{62}$$

Using (55), (56) to take the limits term by term gives (58) and (59).

Note that E_ν vanishes for $\nu > T + 1$ and F_ν for $\nu > R + 1$ since they may be expressed as the νth difference of $(n + d_{T+1} - 1 - \nu + x)$ or the $(\nu - 1)$th difference of $(n + c_R - \nu + x)$ respectively evaluated at $x = 0$.

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