Natural Sorting over Permutation Spaces

By R. M. Baer and P. Brock

0. Introduction. In this paper we continue the study, begun in [1], of some combinatorial problems related to monotonicities that occur in certain spaces of finite sequences. These spaces are equipped with standard probability measures, so that one may study the distribution of monotonicities in such spaces and, in particular, the expected lengths of maximal monotonic subsequences. These, in turn, are upper bounds on the expected lengths of monotonic subsequences obtained by applying some selection process over the space of sequences. The problem which we have called *natural sorting* is concerned with the maximization of these expected lengths.

The scheme of the paper is as follows. In Section 1 we give definitions and review some background material. In Section 2 we describe the distribution of maximal sequences occurring in the space of permutation sequences.* These distributions have been computed exactly for spaces of permutations of length $n = 2(1)36$, and have been approximated by Monte Carlo computations for certain values of n ranging up to 10,000. In Section 3 we consider several selection strategies and the corresponding distribution of selected monotonic subsequences.

The computations were performed on the IBM 7094 at the Computer Center of the University of California, Berkeley. We are indebted to David M. Matula for some of the calculations in Section 3. We should like to thank Geri Stephen for her assistance in the preparation of the manuscript.

1. Definitions and Conventions.

1.1. Throughout this paper the term *sequence* is to be understood to mean *finite sequence*. Standard terms, if not here defined, are used according to the definitions given in [3].

1.2. *Definition.* Let $(X, <)$ be a (totally) ordered space and let n be a fixed natural number. A partial ordering is induced in each element of the cartesian product X^n in the following natural way. If $(x_1, \dots, x_n)$ is in X^n let S denote the space consisting of the set $\{x_1, \dots, x_n\}$ together with the partial ordering $x_i \leq x_j$ if and only if $x_i < x_j$ and $i < j$. In what follows, the space S varies over the set of permutation sequences obtained from $\langle 1, 2, \dots, n \rangle$.

1.3. *Definition.* A *chain* in S is a (totally) ordered subset of S . The length of a chain is the number of elements in it. A *maximal* chain in S is a chain that is not a proper subset of any other chain in S . A maximal chain in S which has length at least as great as that of any other chain in S is called a *spine* of S .

1.4. *Definition.* A maximal chain in S which has length at least as small as that of any other chain in S is called an *Erdős chain*. (For results on Erdős chains, see [1], [8], and [12].)

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* We shall treat cases of other sorting spaces—in particular sorting spaces of binary sequences—separately.

1.5. Definition. By an *n*th order selection algorithm (applied to sequences of length *n*) is meant an algorithm *A* which selects a monotonic subsequence from a sequence $(x_1, \dots, x_n)$ according to the following scheme. The first entry x_1 is selected or rejected on the basis of its size according to a rule of *A*, so we might write, according to $A(x_1)$. The second entry is selected or rejected in a manner determined by the ordered pair (x_1, x_2) so we might write, according to $A(x_1, x_2)$; and here we are to understand that *A* acts upon knowledge of whether x_1 was selected or not. Similarly x_3 is selected or rejected by *A* according to $A(x_1, x_2, x_3)$, and here we understand that *A* has the information as to which (if any) subset of $\{x_1, x_2\}$ was selected. And so, for each x_i , the selection or rejection of x_i is determined by *A* on the basis of the set $\{x_j : j \leq i\}$ and upon the selected subset of this set. So an *n*th order selection algorithm is essentially one in which all information (concerning selections already made and the preceding part of the original sequence) up to the current point of selection is available to the algorithm. The opposite notion is embodied in the definition of *0*th order selection algorithm, which, applied to a sequence $(x_1, \dots, x_n)$ selects or rejects each x_i ($i \leq n$) according to $A(x_i, s)$ where *s* is the value of the last selection (if there were such) preceding the decision to select x_i ; in other words, a 0th order algorithm simply selects or rejects each x_i according to a rule which depends only upon this x_i and the last previously selected entry and is oblivious to the past history of the sequence.

1.6. Definition. The *distinguished* (nondecreasing) subsequence of the sequence $(x_1, \dots, x_n)$ is obtained using the 0th order selection algorithm: Select x_1 . For each $i > 1$, select x_i if x_i is not less than the element selected last, prior to consideration of x_i .

2. The Distribution of Monotonocities in P_n . In [1] the authors considered the question of monotonocities in the space P_n consisting of the space of all permutations of $\langle 1, 2, \dots, n \rangle$. Our viewpoint was to consider simultaneously the maximal increasing and decreasing subsequences of the elements of this space. Some preliminary results concerning this distribution were obtained in [1], and an encompassing result was derived by Schensted [12], who showed the relationship between the distribution and representations of the symmetric group. However, the question of the actual distributions was left open both by [1] and [13]. We now proceed to fill in this gap by exhibiting the results of the requisite computations. First, however, we review the basis of the calculations.

2.1. Definition. A Young tableau of order *n* is an array of the integers $1, 2, \dots, n$ satisfying the following. The array consists of rows and columns. For each row, the entries in that row form an increasing sequence. For each column, the entries (moving down that column) form an increasing sequence. Each row contains at least as many entries as the row beneath it. Each column contains as least as many entries as the column to its right.

Each permutation of $\langle 1, 2, \dots, n \rangle$ uniquely determines a Young tableau. The determination proceeds as follows. Let the permutation be $\langle x_1, x_2, \dots, x_n \rangle$. For the moment, define the first entry in the first row of the tableau to be x_1 . Now, if at the *i*th step, the first *i* entries of the sequence have been used in the developing tableau then at the next step the element x_{i+1} is inserted into the first row of the tableau by displacing the smallest entry in the first row which is larger than x_{i+1}

or by appending x_{i+1} at the end of the first row if it is larger than all entries in the first row.

If an entry y is displaced from the first row by x_{i+1} then y is inserted into the second row by letting it displace the smallest entry in the second row which is larger than y or by simply appending y to the second row if there is no such element. The process is continued from row to row until either the original x_{i+1} or a displaced element is appended to the end of a row. Then the whole process is renewed for $x_{i+2}, \dots$ until all of the entries of the original permutation sequence have been entered into the tableau.

2.2. THEOREM (SCHENSTED). *If T is the Young tableau generated by the permutation sequence $\langle x_1, \dots, x_n \rangle$ then the greatest length of a maximal monotone increasing subsequence of $\langle x_1, \dots, x_n \rangle$ is equal to the number of columns of T , and the greatest length of a maximal monotone decreasing subsequence of $\langle x_1, \dots, x_n \rangle$ is equal to the number of rows of T .*

2.3. Definition. By a *partition* of the positive integer n we mean a monotonically nonincreasing sequence of positive integers which sum to n . By a *partition tableau* corresponding to a partition $(m_1, \dots, m_k)$ we mean an upper-left rectangular array of cells (for example, the figure below)

in which there are m_1 cells in the first row, m_2 cells in the second row, $\dots$, and m_k cells in the k th row. To each cell of a partition array is assigned a number h called its *hook number*. If b is the number of cells below a designated cell and if r is the number of cells to the right of this designated cell, then the value of h corresponding to the designated cell is $h = b + r + 1$.

We illustrate these matters in 2.4.

2.4. Example. Consider the four sequences:

- (i) 2, 1, 5, 4, 3, 6 (ii) 3, 2, 1, 6, 5, 4 (iii) 3, 1, 4, 2, 5, 6 (iv) 6, 5, 4, 3, 2, 1

According to our convention, these sequences correspond to the four partially ordered sets with Hasse diagrams [3]:

The corresponding Young tableaux are:

The Young tableaux with their associated hook numbers are:

$w = [6!/5 \cdot 3^2]^2$
 $= 16^2$

$w = [6!/4 \cdot 3^2 \cdot 2^2]^2$
 $= 5^2$

$w = [6!/5 \cdot 4 \cdot 2^2]^2$
 $= 9^2$

$w = 1$

If one considers the Young tableaux generated by permutations of $(1, 2, \dots, n)$ it is evident that different permutations can give rise to the same tableau shape (i.e., partition tableau). The number of tableaux with the same shape arising in this fashion is given by a powerful combinatorial theorem:

2.5. THEOREM (FRAME-ROBINSON-THRALL [10]). *The number of tableaux with given shape that contain the integers $1, 2, \dots, n$ is $n!/\prod_i h_i$, where the h_i are the hook numbers associated with the cells of the tableau.*

Finally we need [12, Theorem 3], which is obtained from 2.2 and 2.5.

2.6. THEOREM (SCHENSTED). *The number of sequences consisting of the distinct numbers $x_1, \dots, x_n$ and having a longest increasing subsequence of length j and a longest decreasing subsequence of length k is the sum of the squares of the numbers of identically shaped partition tableaux with j columns and k rows.*

Based upon the preceding results, the distribution of monotonicities has been computed exactly for the spaces p_n ($n \leq 36$). The procedure is to generate the distinct partitions of n , to generate the partition tableaux, load the cells with their hook numbers, evaluate the square of the value of the Frame-Robinson-Thrall function. This result is then added to one of several running sums, corresponding to the maximal partition element m_1 (for the increasing subsequences of greatest length where this length happens to be m_1) or to one of several running sums corresponding to k (for the case of the decreasing subsequences of greatest length where this length happens to be k , and $(m_1, \dots, m_k)$ is the current partition of n), or to one of several running sums corresponding to $j = \max [m, k]$ for the case of

monotone subsequences of greatest length when this length happens to be j .

These exact distributions were calculated on a 7094 using the multiple-precision fixed point routines described in [2].

The results are shown in Table 1, giving the distribution of permutations of $(1, 2, \dots, n)$ which contain increasing subsequences of greatest length as well as the distribution according to monotonic subsequences of greatest length. According to the statement of a celebrated theorem of Erdős [8], every sequence of length $k^2 + 1$ contains a maximal monotone subsequence of length at least $k + 1$. Hence the zeros as the first k entries in Table 1b.

The exact calculation for the distribution of monotonicities could be carried out on the 7094 only for sequences of length ≤ 36 . For sequences of greater length, we examined the expected length of the monotonically increasing subsequences of greatest length by a Monte Carlo procedure. That is, using a pseudo-random number generator we generated sequences of real numbers x ($0 < x < 1$) with uniform distribution, and verified that the distribution for sequences of length ≤ 36 matched the exact distributions. We then examined the distributions in Monte Carlo fashion for sequences of length up to 10,000. The results are summarized in Fig. 1 (and Table 2). It may be observed that the expected length of the spine, over the range $n \leq 10,000$, approaches the value $L_n = 2\sqrt{n}$. This gives us a standard for estimating efficiencies of sorting algorithms which may be applied over P_n .

FIGURE 1. Mean values of lengths of monotone increasing subsequences of greatest length over the space of permutations of length n , using Monte Carlo trials on sequences of pseudo-random x , $0 < x < 1$.

N	15	16	17	18	19
1	1	1	1	1	1
2	9694844	35357669	129644789	477638699	1767263189
3	8017098273	56928364553	409558170361	2981386305018	21935294881644
4	164161815768	1744049683213	18865209953045	207591285198178	2321616416280982
5	485334447114	6835409506841	9796603326993	1429401763567226	21226755241285022
6	434119587475	7583461369373	134533482045389	2426299018270338	44506885647682026
7	172912977525	3615907795025	76340522760097	1631788075873114	35378058306185002
8	37558353900	927716186325	22904111472825	568209449266202	14216730315766814
9	4927007100	143938455225	4142847526101	118504614869214	3389618010035458
10	410474625	14353045401	484748595081	16029615164446	523952747921310
11	22128576	947236425	38094121561	1470147102730	55235843005474
12	766221	41662441	2043822961	93574631242	4087226730670
13	16381	1203441	74797417	4166173834	215285274766
14	196	21841	1830561	128922442	8088065845
15	1	225	28561	2708305	214496074
16		1	256	36721	3910885
17			1	289	46513
18				1	324
19					1

<i>N</i>	<i>20</i>	<i>21</i>	<i>22</i>
1	1	1	1
2	6564120419	24466267019	91482563639
3	162951791097669	1221201051018189	9225637750090023
4	26362085777156567	303635722412859447	3544040394934246209
5	320692032888290224	4926576077469905280	76913478420068425515
6	830512607486659272	15764082963927084216	304295666452406076997
7	778860477345867008	17423197016288134608	3961690708392366609236
8	359666061054003144	9216708503647774264	239524408949706575548
9	97376389179852540	2818543211543628620	82388635477750176388
10	17044414451764396	554568496974014588	18113988555378974988
11	2041043061768988	74723654734001996	2725298085020712539
12	172898075436668	7159734192739823	292523675918642499
13	10556881208783	498970549878348	22987539301199111
14	468217205543	25597176520323	1342940739033279
15	15050633363	969739466151	58746150699644
16	345989051	26985380213	1924101911964
17	5527861	543028571	46819431044
18	58141	7666121	831771172
19	361	71821	10451981
20	1	400	87781
21		1	441
22			1

<i>N</i>	<i>23</i>	<i>24</i>	<i>25</i>
1	1	1	1
2	343059613649	1289904147323	4861946401451
3	70209505971502533	537934326588404973	4147337689049888701
4	41881891423602685193	500690223797206725847	6050434838705784406551
5	1219520974164038038455	19625674731276275749737	320348783206253047567401
6	5971518739677370493811	119087070548532813807947	2412379484535302037916891
7	9157097111888617643722	215143361542096212159897	5136996996820257019680841
8	6317740398995612513164	16920749997274346326579	4602911809939402715164066
9	2436180769576352799396	72958306889459609898731	2214789502139053994814716
10	595604303387826752023	19755504320385394380715	662039152774864970449891
11	9942808099387084396	3640046755032713093843	134046618461374364168411
12	11861492339537464775	479491040778079234419	19390512879546189292111
13	1040829060468117119	46606795474062898831	2073904870905091148211
14	68403703794493420	3411981675691129279	167759928098294047111
15	3401595419412851	190527252376282087	10415612075920450911
16	128468428322048	8168500125865719	500718113865946741
17	3675343389664	269170287475509	18709292281579191
18	78885640857	6786436811013	542824153010541
19	1246532772	129464551053	12155099539941
20	14033405	1831706109	207490079241
21	106261	18582345	2643997501
22	484	127513	24297201
23	1	529	151801
24		1	576
25			1

N	26	27	28
1	1	1	1
2	18367353072151	695333550916003	263747951750359
3	32159889269079495426	250717468966793970202	1964346822230479484023
4	73852382382545858737126	909943177632220199263042	11310232116090782070465841
5	5300292212652610734928566	88832931884770251957700042	1507202979025523814325637146
6	49616581304399446635438226	1035620403406365159860967667	21925826354844065315916897256
7	124619442258128879178524666	3070563111316053793393797817	76815934189294635140961483586
8	127185386263877438987727916	3569565523674458604017507932	101744071361900204239163084176
9	68192223459336813448498166	2130316698942051016198015702	675387776302361298233506320226
10	22441579408623486032950936	770150147621700978419561077	26774766892473521664006347716
11	4974684788366042119649866	186321569087821491256958077	7050722553537961699341091246
12	786573753963611623878186	32074009903173294316954102	1316919181912414288817275336
13	92050364310704637832186	4087354609490748901647222	181995073612039972330379626
14	8172068412490464108186	395979688058468617286752	19146538134094625864605786
15	559546105197085784116	29697830210588581628302	1563800962990859091545746
16	29864196368967346036	1745384459871184579552	100513204250216012020066
17	1249784830144144516	80996819431842935752	5130101633704123358126
18	4107307871752516	2978887120344535882	209018576537102684926
19	1057716176138766	86826165506931592	6811596755141818486
20	21179941102126	19979553535068042	177313615826634406
21	325456288366	35994624514642	3668913368754571
22	3755016526	500568628717	59792609043091
23	31405401	5254253902	756189714295
24	179401	40166101	7252484527
25	625	210601	50873005
26	1	676	245701
27	1	1	729
28			1

1	1	1	1
2	1002242216651367	3814986502092303	145444636039226908
3	15462158343386281664951	122238896672891001069665	970325450637821942142140
4	141740653186926189324638372	1790036582998939530743648877	22770455209205915603060025597
5	25871893550358473180365541399	449044243619862872721423598179	78761105037412253622359167480199
6	470631631288386313363280613284	10236819433951393776243660748875	225527222131530886314493531870530
7	195039004021483258890097740134	50241067877038219983230124657600	1312475888601307252438342452786525
8	2944633130025648248522014500254	86511371455863277882723853476200	2579359193231328192634102461101400
9	2173205974659557190038205835604	70971582765623356071324810857700	2352171204427805439293130821881500
10	943409440670609713571889909584	33700117351593715495661064101700	1220678270222172955204768869925305
11	269805363161880496284204974114	10447178628714722178634866396630	409537040100177324029704062492225
12	54517248993798658282931531009	2277900847905046253535807880680	96144246401064295441641257152680
13	814130867656818373762688529	366440157064983378222220318530	16615605064690814678338772535330
14	926170396512403010651743634	44912755712412555783652789980	2187028591109076615105275286855
15	81973005417740643915053114	4289203871330156652985437480	2245243802471403955475550730525
16	5728775881849435165638734	324301002215082697285357800	18286697815711493218117882080
17	319376306355915037772354	19633107355949074371195000	1196174724725300729578276800
18	14297487439649796716054	9590642295461783875326600	63388140573056826813361425
19	515761997928294967934	37982369568044622191625	2737060291816929872883300
20	15001117556281650059	1222055891584247185425	96614587298003699934909
21	350943858097699574	31925927141978856309	2790511127801226016125
22	6505817250924743	675007128155925069	65853587607447398904
23	97270960754675	11475430101232224	1264668868266067329
24	1123637013101	155228816648544	19624301369534804
25	9885634319	1644397829384	243364536846276
26	63857305	13319151176	2372835665009
27	285013	79490741	17752922056
28	784	328861	98188781
29	1	841	377581
30			900
31		1	1

N	32	33	34
1	1	1	1
2	5534064877048197	212336130412243109	812944042149730763
3	7731901699077394587687705	61832653155697977964046841	496160667633390522403119618
4	291634367197743500729356426685	3759180275300445999022196163965	48750519654702845837670090343690
5	139529391281472957898846922554177	249535731853436745675666735320729	45030549929498371344718980159549618
6	503010841377922887807240206908229	113526878157525401424493275730864805	2591590094916509302194350734175723642
7	34757316644086125833855818210629081	932721236417411178945416488871094393	25353527824408968292697801668489332618
8	78021383414618228991895594624384125	2393529094511913346236680508422151237	7446327525109871020936280154425059442
9	79103234852498901475016639807359305	2698866785171713218689926232232182145	93397918235878046419269724247743498698
10	44838509270283683161801441967761005	167028453206945517676243567807176205	63096075267224041401539583781897117650
11	1625886539223347142486988285081905	653884729493472367356718097456293825	26643987678124309733086378500759601170
12	4101889128787944119368835836805205	176985923134010840234658522454420185	772607169062109957701553827419159450
13	759734193449732487433651283328585	35057851502535900625566612868833945	1633674263137736224481910975930836130
14	107087715263351031368724617545005	5278522355384573352283039598435085	262163587785580774313722100829067050
15	11779377563652142010462547688305	620305092258993791288024881652625	32828599452830145581256610566628170
16	102955858601572310417090968405	57988267420258090595629883010105	3272729269090490786274074268040650
17	7244932799546083407848773305	4373124531832276616938310101845	263612845059752600845112121561774
18	414392763537970664647378425	268801257453378459965223500889	17346036203890648128109387808046
19	193944375806296670833026009	13567132517272171232545253241	940024606611956915670586054722
20	7458944916310495249735449	565160788944400127086416969	42199282317113259559913583810
21	236220803698044253404489	19489532317209393713131545	1575292943311758744546072386
22	6160162790565939425289	557008975918218063593513	48999881305502827905367746
23	131990124545762873513	13182832535508092797577	12704134804739796772908698
24	2312819723545285865	257635374171351028337	27414607341140712516858
25	32891644638784625	4136163375619184361	490721103325697233626
26	37325561053745	54110750735698449	7245202714152021114
27	3379471497105	570076699965873	87490246706863122
28	23426779857	4754933730273	853666255814802
29	120413921	306206645457	6614681604114
30	431521	146679105	39691351954
31	961	491041	177551265
32	1	1024	556513
33		1	1089
34			1

1 3116285494907301261
 2 399408621135593946349383548
 3 635848553797809170295520418710782
 4 819588244871579975094742632650930358
 5 59812066183117095755447194860322648806
 6 697312273978076490697755421027696727478
 7 2346831372517025163776957078701792456870
 8 3277611718424868517776769808161439844462
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 13 13129491403244799666927129567113776890
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 15 185330636189134919022484650048294414
 16 158968039654996938651780896898834
 17 1116003041710932482624313323727894
 18 648627893924224479276254311634
 19 3115625942782130518017444226214
 20 125263010041612539140025735398
 21 4215653028099462112368819038
 22 118905644693483891716868358
 23 2809856400891651324384238
 24 55518761604460803409598
 25 913685147570122124342
 26 12448894244607115902
 27 139195797135863382
 28 1261485212494102
 29 9104618690917
 30 51020199826
 31 213654981
 32 628321
 33 1156
 34 1
 35 1
 36 1

1 11959798385860453491
 2 32249846192152237908327459893
 3 838432923207513905107942840897207
 4 15083912583528984220392081381933578208
 5 1395020405364298522836760442796104706656
 6 19439642570069266048240891724790133009888
 7 74956422241816331218926228837298911186720
 8 116608638100360116710442324796876756408352
 9 93859283446101269401691563039073753552064
 10 46025756664927326758904180662495739416960
 11 15264266027718982566224773594719844868640
 12 3660517378342559966873860780713566493240
 13 6634449927112151517704628177014442337840
 14 93703493478435896513152031028151961064
 15 10542497140124782901363704728467537040
 16 960400377652127813719842557392971204
 17 71716805772727233527588176423578804
 18 4430941771361782282147424398422404
 19 228095611162787120114225791488548
 20 9833255433124590718722650114864
 21 356237120025927944601115628528
 22 10866695854770117187576414928
 23 279258361123714921066189328
 24 6040431285173037746028992
 25 109700112356207611240448
 26 1665622494029241660512
 27 21008924568821050912
 28 218144396451607687
 29 1841133016223551
 30 12407725834267
 31 65081293411
 32 255676261
 33 706861
 34 1225
 35 1
 36 1

TABLE 1b
Distributions of maximal monotone subsequences over the space of permutations of $(1, 2, \dots, N)$

[illegible]

<i>N</i>	<i>15</i>	<i>16</i>	<i>17</i>	<i>18</i>	<i>19</i>
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	577152576	577152576	0	0	0
5	217908143024	1882013490500	15137332742532	111495773096136	736543205438040
6	663703605292	9933107320916	144868857114644	2058529251860328	28477732544688960
7	339659751732	6934946649572	140700564230692	2834419897105696	56633617912329112
8	75104929176	1853582024436	45645218200032	1125758970041192	27855001516505912
9	9854014200	287876910450	8285529415302	236974544144628	6775208774870652
10	820949250	28706090802	969497190162	32059230328892	1047903131938220
11	44257152	1894472850	76188243122	2940294205460	110471686010948
12	1532442	83324882	4087645922	187149262484	8174453461340
13	32762	2406882	149594834	8332347668	430570549532
14	392	43682	3661122	257844884	16176131690
15	2	450	57122	5416610	428992148
16		2	512	73442	7821770
17			2	578	93026
18				2	648
19					2

<i>N</i>	<i>20</i>	<i>21</i>	<i>22</i>
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	4254132236622840	20794481008587000	82244764498782600
6	382902517231190424	4989011841687509064	62711531490290646392
7	1121088586380202928	21963654061311148288	425442532345718472872
8	691709550866489296	17230668107410810776	430164924248070169032
9	194409066733408704	5612953233636388664	163297299176471000376
10	34088191453068288	1109042572932268208	36217781367812519128
11	4082086123537976	149447275333224456	5450584636999631766
12	345796150873336	14319468385479646	585047351837284998
13	21113762417566	997941099756696	45975078602398222
14	936434411086	51194353040646	2685881478066558
15	30101266726	1939478932302	117492301399288
16	691978102	53970760426	3848203823928
17	11055722	1086057142	93638862088
18	116282	15332242	1663542344
19	722	143642	20903962
20	2	800	175562
21		2	882
22			2

N	23	24	25
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	245804974123480200	491609948246960400	491609948246960400
6	75971089727178127088	8676145262210193012960	94003717658533097844000
7	8140032932924164032080	153675090362576773465488	2859133318231049063998800
8	10750511568454997569192	268637736880089119920928	6704129404938223686722000
9	4790143278215489101336	141677498323618085096158	4223257115141515056146350
10	1190309508929724579654	39442211487452719258622	1319337197720643808194350
11	198854038310659029760	7279810433848890021742	268062617677311763906750
12	23722984181440622926	958981875514820763726	38780979573723991401750
13	2081658120936234238	93213590948125797662	4147809734497722624086
14	136807407588986840	6823963351382258558	335519856196588094222
15	6803190838825702	381054504752564174	20831224151840901822
16	256936856644096	16337000251731438	1001436227731893482
17	7350686779328	538340574951018	37418584563158382
18	157771281714	13572873622026	1085648306021082
19	2493065544	258929102106	24310199079882
20	28066810	3663412218	414980158482
21	212522	37164690	5287995002
22	968	255026	48594402
23	2	1058	303602
24		2	1152
25			2

N	26	27	28
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	952382027853194853314400	8925592101113301299829000	76425573055466553008547600
7	52341940024485713702458800	941037757041119409946427400	16575282453100233870595785600
8	166907338705800981751802000	4141095209547848247104514000	102290188725079565161442454000
9	126787030401135363262237600	3829965323205241372302462600	116300812211682276193750329300
10	44581491460640052185514200	1522264573503362764833831000	52523195747848467041038765500
11	9946512181305155408896900	372404222345803710911969700	14083087501046107918253600100
12	1573140059093914882150500	64147050297343993386275100	263373004327306538195696300
13	184100724978419751092436	8174708442035329661568972	363989959709917641189074676
14	16344136824980928216372	791959376008764754213504	38293076204333270491391572
15	1119092210394171568232	59395660421177163256604	3127601925981718183091492
16	59728392737934692072	3490768919742369159104	201026408500432024040132
17	2499569660288289032	161993638863685871504	10260203271408246716252
18	82156615743505032	5957774240689071764	418037153074205369852
19	2115432352277532	173652331013863184	13623193510283636972
20	42359882204252	3995910670137284	354627231653268812
21	650912576732	71989249029284	7337826737509142
22	7510033052	1001137257434	119585218086182
23	62810802	10508507804	1512379428590
24	358802	80332202	14504969054
25	1250	421202	101746010
26	2	1352	491402
27		2	1458
28			2

N	29	30	31
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	589207733618083635076678800	4017834614129468263414826400	23692354719303941547720386400
7	285202153278958513785707520000	477672028403661379299805820600	77623700024019150137247624840600
8	2513137284768128335678924779600	61354192232985708594062524631400	1486899898340427286368998222579400
9	3546523725021576401940276722100	108498874067996587999017464032200	3326862986586311241555650840434200
10	1830683985233926033276259240700	64426925473775882732381174069400	2287930539402346328092953064629900
11	538290932005594148436878681700	20804390394983947746227177299800	813200022570515550808858717299300
12	109023704504845696062407712600	4554817493732644750790195030400	192204847001779358208731530304400
13	16282588447793719958338703304	732876506258667437085324970680	33230764580423680716981408825480
14	1852340772834412694565585668	89825506876602139897193098560	4374056362868704012595427126060
15	163946010833871946234546228	8578407741549180952620369360	449048760085161503360103812100
16	11457551763698870331277468	648602004430165394570715600	36573395631398924991224813760
17	638732612711830075544708	39266214711898148742390000	2392349449450601459156553600
18	28594974879299593432108	1918128459092356775065200	126776281146113653626722850
19	1031523995856589935868	75964739136089244383250	5474120583638859745766600
20	30002235112563300118	2444111783168494370850	193229174596007399869818
21	701887716195399148	36851854283957712618	5581022255602452032250
22	13131634501849486	1350014256311850138	131707175214894797808
23	194541921509350	22950860202464448	2529337736532134658
24	2247274026202	310457633297088	39248602739009608
25	19771268638	3288795658768	486729073692552
26	127714610	26638302352	4745671330018
27	570026	158981482	35505844112
28	1568	657722	196377562
29	2	1682	755162
30		2	1800
31			2

N	S2	S3	S4
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	117239808798376941090171700800	466537502959586314018384364800	1397195216117310308303763100800
7	1218028738549680917003439208163200	18374567256228810031700251202812800	265170170591976087538367224240243200
8	35731933969901471743292605066276800	850441482101675510240319406760601600	20019612952745687325073221493720761600
9	102149478451453503725655595992317700	3137998326590381466122532791507132100	963664047077382117938559553103329800
10	81927367586821122917342698830532500	2955913746239917536402680534032391700	107370883921209174050195411816886297000
11	32147481661001074987395360190510500	1285098171830687847235480636082667300	51932576136493461969700853994704901000
12	8197061613892990320149108104590900	353456937224263313066214631017011700	1541414891711286888099328686581957800
13	1519420800841425827079577453336260	70110994517157770278807222474601220	32669092180261672118660801160561355080
14	214175304882927153555062226802260	10557027660684041574234887535896020	52432507348094276390095522492791400
15	23558755020416536108098219522660	1240610162269720189226878940201700	65657194977601776429750506532740640
16	205911171718393093052820598260	115976534832365504921572288232160	654545535736399512361315896644200
17	144898655990921668156975466610	8746249063664191936241377651590	527225690119174755622602852617748
18	8287855270759413329294756850	537602514906756919930447001778	34692072407781296256218775616092
19	387888751612593341666052018	27134265034544342465090506482	1880049213223913831341172109444
20	14917889832620990499470898	1130321577888800254172833938	83498564634226519119827167620
21	472441607396088506808978	38979064634418787426263090	3150585886623517489092144772
22	12320325581131878850378	1114017951836436127187026	97999762611005655810735492
23	263980249091525747026	26365665071136185595154	2540826960947959354417396
24	4625639447090371730	515270748342702056674	54829214682281425033716
25	65783289277569250	8272326751238368722	981442206651394507252
26	750651122107490	108221501471396898	14490405428304042228
27	6758942994210	1140153399931746	174980493413726244
28	46853559714	9509867460546	1707332511629604
29	240827842	61253290914	13229363208228
30	863024	293358210	79382703908
31	1922	982082	355102530
32	2	2048	1113026
33		2	2178
34			2

N	35	36
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	2794390432234620616607526201600	2794390432234620616607526201600
7	3641303371464170140381289444094019200	47302385876751262940791680364789651200
8	465398762365262092373752560180569107200	10666155857406751349088621857909306035200
9	2956033630424242850168930272742455579800	90503674086599235538008607944288376981800
10	3923448021404298815368599763876489462200	144110704271685827074239405831683400108000
11	2120729276021540978123490370436277859800	87470915270995458872125418341705078404000
12	679941588966377982290083835615230414200	3033942620565271155315161503380509159200
13	153796050449988087049755243386090817240	7317709103493134296844739138087484074840
14	26258742863974898853251857892785234080	132687413768104388998211439348353055480
15	3495758975783663056219222404208369080	18740690067748451831329589758203838928
16	37066127179609469009583907209585528	21084994163019612030162518457673914480
17	31793607930839377511589787895627368	1920800755249650541615397967562798608
18	2232006083421859519530636624767388	143433611545448810668577419836608208
19	12937255787848448958552508623268	8861883542723564564294848796844808
20	623125188556426103603488452428	456191222325574240228451582977096
21	250326020083225078280051470796	19666510866249181437445300229728
22	8431306056198924224737638076	712474240051855889202231257056
23	237811289386967783433736716	21733391709540234375152829856
24	5619712801783302648668476	558516722247429842132378656
25	111037523208921606819196	12080862570346075492057984
26	1827370295140244248684	219400224712415222480896
27	24897788489214231804	3331244988058483321024
28	278391594271726764	42017849137642101824
29	2522970424988204	436288792903215374
30	18209237381834	3682266032447102
31	102040399652	24815451668534
32	427309962	130162586822
33	1256642	511352522
34	2312	1413722
35	2	2450
36		2

3. Two Sorting Strategies over P_n . We consider first the 0th order sorting strategy over P_n . This consists of selecting the initial element (in a permutation sequence), and, having selected an element, selecting the first following element in the sequence which is greater. The resulting selected subsequence is called the *distinguished subsequence*.

3.1. THEOREM. *In the space P_n the expected length of the distinguished subsequence is*

$$L_n = \sum_{k=1}^n k^{-1}.$$

Proof. The statement of the theorem is clearly true for $n = 1$. We proceed by induction. Suppose that the statement of the theorem holds for $k \leq n$. Given any particular sequence in P_n , there are $n + 1$ ways in which sequences of P_{n+1} may be formed through the adjunction of the number $n + 1$ to this particular sequence, and with each of these new sequences is associated the probability $(n + 1)^{-1}$. For each new sequence so constructed, the corresponding distinguished subsequence ends with the adjoined number. Hence the expected length of the corresponding distinguished subsequence is greater by unity than the expected length of a distinguished subsequence in that part of the original sequence preceding the adjoined integer, averaged over the positions that the adjoined integer may occupy. This is just

$$L_{n+1} = (n + 1)^{-1} \sum_{k=0}^n L_k + 1$$

where L_0 is taken to be zero. It follows that

$$L_{n+1} = L_n + (n + 1)^{-1}.$$

This concludes the proof.

We now consider the n th order strategy for natural sorting in P_n . According to the definition, the sorting algorithm of an n th order strategy has available, at the point where it decides whether to select the i th sequence element, the identity of the previously considered sequence elements x_k , $k < i$. The selection algorithm for P_n accordingly has the following form. A choice level C_1 (where C_1 is a natural number, $0 < C_1 \leq n$) determines the selection of the first sequence element x_1 (i.e. if $x_1 \leq C_1$ then x_1 is selected). A selection level $C_2 = f_1(x_1)$ is defined, depending upon x_1 . The second sequence element x_2 is selected accordingly (if $x_1 \leq x_2 \leq C_2$ in the case that x_1 was selected, or else simply $x_2 \leq C_2$ in the alternate case). What is required is the sequence of functions $f_0(S) = C_1$, $f_1(S) = C_2$, $\dots$ (where S is an arbitrary sequence in P_n) which maximizes the expected length of the selected subsequence.

To show how the choice levels are made, we will consider the example of P_4 . We suppose that the expected lengths from the optimal choice functions are known for P_1 , P_2 , and P_3 . (These expected lengths are: $\langle L_1 \rangle = 1$, $\langle L_2 \rangle = 3/2$, $\langle L_3 \rangle = 2$.) There are four possibilities for the value of C_1 (i.e. $C_1 = 1, 2, 3$, or 4). Consider the expected lengths of the selected subsequence which results from each possible value of C_1 . Thus, if $C_1 = 1$ the first element of a permutation sequence will be selected if and only if it has the value unity. In this case the remaining sequence consists of a permutation of the sequence $(2, 3, 4)$ and by, assumption, the strategy

and expected length of P_3 are known and may be applied to the permutation of (2, 3, 4)—with certain obvious adjustments. The expected length in this case, which we shall denote $L_4(1)$, is just the value of $L_3 + 1$ weighted by the probability (which is $1/4$) that unity occurs as the initial entry of the sequence from P_4 added to the value $\langle L_3 \rangle$ which in turn is weighted by the probability that unity does not occur as the first sequence entry. So, $L_4(1) = 4^{-1}[(1 + \langle L_3 \rangle) + 3\langle L_3 \rangle]$.

TABLE 2

Expected lengths of (1) monotonically increasing subsequences of greatest length, (2) of monotone subsequences of greatest length, and (3) observed (Monte Carlo) means of monotonically increasing subsequences of greatest length, (4) of monotone increasing subsequences selected according to the n th order strategy; (5) is the initial selection level used in (4).

N	(1)	(2)	(3)	(4)	(5)
3	2.000	2.333			
4	2.416	2.916			
5	2.791	3.300		2.73	3
6	3.140	3.650	3.155	3.04	3
7	3.465	4.021	3.446	3.33	3
8	3.770	4.350	3.760	3.60	4
9	4.059	4.647	4.049	3.86	4
10	4.334	4.938	4.333	4.10	4
11	4.598	5.222	4.577	4.32	4
12	4.852	5.490	4.849	4.54	5
13	5.096	5.745	5.115	4.75	5
14	5.332	5.991	5.323	4.95	5
15	5.561	6.232	5.560	5.15	5
16	5.783	6.465	5.786	5.33	5
17	5.999	6.691	6.013	5.51	5
18	6.209	6.910	6.213	5.69	6
19	6.414	7.123	6.417	5.86	6
20	6.614	7.332	6.618	6.03	6
21	6.810	7.536	6.832	6.19	6
22	7.002	7.736	7.002	6.35	6
23	7.189	7.931	7.165	6.50	6
24	7.373	8.122	7.354	6.65	7
25	7.554	8.309	7.556	6.80	7
26	7.731	8.493	7.741	6.94	7
27	7.905	8.673	7.926	7.09	7
28	8.076	8.851	8.074	7.23	7
29	8.244	9.025	8.258	7.36	7
30	8.410	9.196	8.420	7.50	7
31	8.573	9.365	8.553	7.63	8
32	8.734	9.531	8.717	7.76	8
33	8.892	9.695	8.871	7.87	8
34	9.049	9.857	9.026	8.01	8
35	9.203	10.016	9.206	8.14	8
36	9.355	10.173	9.368	8.26	8
100			16.723	14.05	14
200			24.508	20.02	20
1000			58.154	44.99	44
10000			192.2	142.07	141

If we take $C_1 = 2$ then the argument proceeds in a similar way, except that now if either 1 or 2 occurs as the initial sequence element it is selected, and if 2 occurs and is selected then since the selection algorithm applies now only to sequence entries with values greater than 2, the strategy and expected length of the selected subsequence from P_2 comes into play, thus

$$L_4(2) = 4^{-1}[(1 + \langle L_3 \rangle) + (1 + \langle L_2 \rangle) + 2\langle L_3 \rangle] .$$

And similarly $L_4(3)$ and $L_4(4)$ may be evaluated. Then C_1 is defined as that value of k which maximizes $L_4(k)$. In the general case

$$\langle L_{n+1} \rangle = \max_{0 < k \leq n} (n + 1)^{-1} \left[k + (n + 1 - k) \langle L_n \rangle + \sum_{i=2}^n \langle L_{n-i+1} \rangle \right]$$

and it follows that the expected lengths of the selected subsequences and the choice levels are determined at the same time.

The required functions may be computed, using a course-of-values recursion. For this we are indebted to Mr. David Matula, who provided us with columns 4 and 5 of Table 2, computed in double-precision directly from the above equation. It will be observed that the expected length of the selected monotone subsequences, in sequences of length n , is approximately $(2n)^{1/2}$.

Conclusions. Over the range which has been examined ($n \leq 10,000$), let r_1 be the ratio of the expected value of a monotone subsequence, selected according to the 0th order strategy, to the expected length of the monotone subsequence of greatest length. Then

$$r_1 \sim \frac{\sum_1^n k^{-1}}{2\sqrt{n}} \sim \frac{\log n}{2\sqrt{n}} \rightarrow 0 .$$

Let r_2 be the ratio of the expected length of the monotone subsequence, selected according to the n th order strategy, to the expected length of the monotone subsequence of greatest length. Then

$$r_2 \sim (2n)^{1/2}/2\sqrt{n} = 1/\sqrt{2} .$$

TABLE 3
Tableaux with maximal weights for group of order n ; $10 \leq n \leq 36$

<i>order 10</i>	<i>order 11</i>	<i>order 12</i>	<i>order 13</i>	<i>order 14</i>	<i>order 15</i>
7 5 3 1	8 6 4 2 1	9 6 4 2 1	9 6 4 3 1	10 7 5 4 2 1	9 7 5 3 1
5 3 1	5 3 1	6 3 1	7 4 2 1	7 4 2 1	7 5 3 1
3 1	3 1	4 1	4 1	4 1	5 3 1
1	1	2	2	2	3 1
		1	1	1	1
<i>order 16</i>	<i>order 17</i>	<i>order 18</i>	<i>order 19</i>	<i>order 20</i>	
10 8 6 4 2 1	11 8 6 4 2 1	12 9 7 5 3 2 1	12 9 7 5 4 2 1	12 10 7 5 4 2 1	
7 5 3 1	8 5 3 1	8 5 3 1	9 6 4 2 1	9 7 4 2 1	
5 3 1	6 3 1	6 3 1	6 3 1	6 4 1	
3 1	4 1	4 1	4 1	4 2	
1	2	2	2	3 1	
	1	1	1	1	

TABLE 3—*Continued*

<i>order 21</i>	<i>order 22</i>	<i>order 23</i>	<i>order 24</i>
13 10 7 5 4 2 1	12 10 8 6 4 2 1	13 10 8 6 4 2 1	14 11 9 7 5 3 2 1
10 7 4 2 1	9 7 5 3 1	10 7 5 3 1	10 7 5 3 1
7 4 1	7 5 3 1	8 5 3 1	8 5 3 1
5 2	5 3 1	6 3 1	6 3 1
4 1	3 1	4 1	4 1
2	1	2	2
1		1	1
<i>order 25</i>	<i>order 26</i>	<i>order 27</i>	<i>order 28</i>
14 11 9 7 5 4 2 1	15 11 9 7 5 4 2 1	15 12 9 7 5 4 2 1	15 12 9 7 6 4 2 1
11 8 6 4 2 1	12 8 6 4 2 1	12 9 6 4 2 1	12 9 6 4 3 1
8 5 3 1	9 5 3 1	9 6 3 1	10 7 4 2 1
6 3 1	7 3 1	7 4 1	7 4 1
4 1	5 1	5 2	5 2
2	3	4 1	4 1
1	2	2	2
	1	1	1
<i>order 29</i>	<i>order 30</i>	<i>order 31</i>	
14 12 10 8 6 4 2 1	15 12 10 8 6 4 2 1	16 13 11 9 7 5 3 2 1	
11 9 7 5 3 1	12 9 7 5 3 1	12 9 7 5 3 1	
9 7 5 3 1	10 7 5 3 1	10 7 5 3 1	
7 5 3 1	8 5 3 1	8 5 3 1	
5 3 1	6 3 1	6 3 1	
3 1	4 1	4 1	
1	2	2	
	1	1	
<i>order 32</i>	<i>order 33</i>	<i>order 34</i>	
13 16 11 9 7 5 4 2 1	17 13 11 9 7 5 4 2 1	17 14 11 9 7 5 4 2 1	
13 10 8 6 4 2 1	14 10 8 6 4 2 1	14 11 8 6 4 2 1	
10 7 5 3 1	11 7 5 3 1	11 8 5 3 1	
8 5 3 1	9 5 3 1	9 6 3 1	
6 3 1	7 3 1	7 4 1	
4 1	5 1	5 2	
2	3	4 1	
1	2	2	
	1	1	
<i>order 35</i>	<i>order 36</i>		
17 14 11 9 7 6 4 2 1	17 14 12 9 7 6 4 2 1		
14 11 8 6 4 3 1	14 11 9 6 4 3 1		
12 9 6 4 2 1	12 9 7 4 2 1		
9 6 3 1	9 6 4 1		
7 4 1	7 4 2		
5 2	6 3 1		
4 1	4 1		
2	2		
1	1		

Open questions. This study has raised some questions of which the following, in the area of asymptotic combinatorics, are particularly interesting:

(1) What is the asymptotic form of the distribution of spines in the case of permutation spaces?

Presumably an analytical derivation of the answer to (1) would require an answer to

(2) For given n , which Young tableau maximizes the value of the Frame-Robinson-Thrall function? If the maximizing tableau is denoted T_n then

(3) Is there an algorithm which permits the immediate construction of T_{n+1} from T_n ?

In computing the exact distributions of the spines (for $n \leq 36$), the corresponding T_n were obtained automatically. These are exhibited (for $10 \leq n \leq 36$) in Table 3, although it is not believed that they suggest the form of T_n for large n .

University of California
Computer Center
Berkeley, California

General Electric Company
Santa Barbara, California

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