Rational Chebyshev Approximations for the Exponential Integral $E_1(x)$

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Abstract. Rational Chebyshev approximations are presented for the exponential integral $E_1(x)$ in the intervals $(0, 1]$, $[1, 4]$, and $[4, \infty)$ with maximal relative errors ranging down to 10^{-21} . 25S coefficients are also given for a continued-fraction expansion for small x .

1. Introduction. Accurate function values for the exponential integral $E_1(x)$, defined [1] by

$$(1) \quad E_1(x) = -\text{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t} dt,$$

are frequently needed in a variety of physical applications. Approximations to this function previously given by Allen [2] and Hastings [3], [4] are of insufficient accuracy for many such applications and for the longer word-length of modern computers. While Clenshaw [5] has given 20D coefficients for expansions in Chebyshev polynomials, such expansions are generally less efficient than rational forms. In addition, the Clenshaw form loses significance through subtraction for $1 < x \leq 4$. In this paper we present a set of nearly-best rational approximations for $E_1(x)$ for $0 < x < \infty$ with accuracies up to 20S. The approximation forms used are efficient and are computationally stable.

2. Approximation Forms. The approximation forms and corresponding intervals used are

$$(2) \quad \begin{aligned} E_{lm}(x) &= -\ln(x) + R_{lm}(x), & 0 < x \leq 1, \\ &= e^{-x} R_{lm}(1/x), & 1 \leq x \leq 4, \\ &= \frac{e^{-x}}{x} \left[1 + \frac{1}{x} R_{lm}(1/x) \right], & 4 \leq x < \infty, \end{aligned}$$

where the $R_{lm}(z)$ are rational functions of degree l in the numerator and m in the denominator. All of the previously mentioned approximations have made use of the first and last of these forms, but not necessarily over the same intervals we use. The first form, based on the analytic expansion

$$(3) \quad E_1(x) + \ln(x) = -\gamma - \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!},$$

where γ is Euler's constant, involves a loss of significance through subtraction of

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TABLE II
 $E_1(x) \simeq -\ln(x) + \sum_{s=0}^n p_s x^s / \sum_{s=0}^n q_s x^s, \quad 0 < x \leq 1$

n	s	p_s		q_s	
1	0	-2.22420	7	3.85031	1
	1	3.28841	0	1.00000	0
2	0	-4.43668	255	7.68641	124
	1	4.42054	938	5.65655	216
	2	3.16274	620	1.00000	000
3	0	-1.64330	27495	2.84694	75956
	1	2.21255	49175	1.09905	66056
	2	2.91730	56556	1.65603	61842
	3	4.32318	05992	1.00000	00000
4	0	-4.34981	43832	7.53585	64359
	1	4.25696	82638	5.68052	52718
	2	2.92525	18866	1.50950	38744
	3	3.98941	53870	1.88992	88395
	4	4.29312	52343	1.00000	00000
5	0	-4.41785	47172	7.65373	32333
	1	5.77217	24713	3.25971	88129
	2	9.93831	38896	6.10610	79424
	3	1.84211	08866	6.35419	41837
	4	1.01093	80616	3.72298	35283
	5	5.03416	18409	1.00000	00000
6	0	-1.48151	02102	2.56664	93484
	1	1.50260	59476	1.84340	70063
	2	8.99049	72007	5.24405	29172
	3	1.59241	75980	8.12580	35174
	4	2.15006	72908	7.50431	63907
	5	1.16695	52669	4.02054	65640
	6	5.01967	85185	1.00000	00000

TABLE III
 $E_1(x) \simeq e^{-x} \sum_{s=0}^n p_s x^{-s} / \sum_{s=0}^n q_s x^{-s}, \quad 1 \leq x \leq 4$

n	s	p_s		q_s	
1	0	1.4867	(-02)	1.0000	(00)
	1	8.6607	(-01)	4.7924	(-01)
2	0	1.21022 05	(-03)	1.00000 00	(00)
	1	9.81479 89	(-01)	1.61986 45	(00)
	2	7.53397 42	(-01)	2.91351 51	(-01)
3	0	1.33017 0319	(-04)	1.00000 0000	(00)
	1	9.97074 3175	(-01)	3.22255 1838	(00)
	2	2.25390 9654	(00)	2.08498 9311	(00)
	3	6.30676 6097	(-01)	2.01741 7317	(-01)
4	0	1.77505 46228 63	(-05)	1.00000 00000 00	(00)
	1	9.99485 00852 88	(-01)	5.19483 84424 40	(00)
	2	4.20224 42231 86	(00)	6.93251 45941 50	(00)
	3	3.66443 11027 66	(00)	2.49230 34830 53	(00)
	4	5.39370 23406 26	(-01)	1.52272 58442 11	(-01)
5	0	2.72428 14258 1277	(-06)	1.00000 00000 0000	(00)
	1	9.99901 07561 7661	(-01)	7.48056 46585 6657	(00)
	2	6.48236 27686 3502	(00)	1.64853 53756 8922	(01)
	3	1.09823 28074 9419	(01)	1.25009 56251 1548	(01)
	4	5.20167 43818 4078	(00)	2.89289 71153 3891	(00)
	5	4.75185 76444 3999	(-01)	1.22430 89892 0495	(-01)

6	0	4.65627	10797	50956	6	(-07)	1.00000	00000	00000	0	(00)		
	1	9.99979	57705	15949	7	(-01)	1.00411	64382	90544	8	(01)		
	2	9.04161	55694	63286	6	(00)	3.24264	21069	51380	5	(01)		
	3	2.43784	08879	13167	3	(01)	4.12807	84189	14243	4	(01)		
	4	2.30192	55939	13334	6	(01)	2.04494	78501	37941	7	(01)		
	5	6.90522	52278	44435	7	(00)	3.31909	21359	33016	0	(00)		
7	6	4.30967	83946	93887	8	(-01)	1.03400	13040	48739	8	(-01)		
	0	8.67745	95483	84437	437	(-08)	1.00000	00000	00000	000	(00)		
	1	9.99995	51930	13903	006	(-01)	1.28481	93537	91566	499	(01)		
	2	1.18483	10555	49458	443	(01)	5.64433	56956	18031	986	(01)		
	3	4.55930	64425	33898	233	(01)	1.06645	18376	99138	825	(02)		
	4	6.99279	45129	10030	229	(01)	8.97311	09712	52898	022	(01)		
8	5	4.25202	03476	88407	791	(01)	3.14971	84917	04407	502	(01)		
	6	8.83671	80880	38439	386	(00)	3.79559	00376	21222	428	(00)		
	7	4.01377	66494	06647	203	(-01)	9.08804	56918	88692	188	(-02)		
	0	1.73733	17607	20576	03093	2	(-08)	1.00000	00000	00000	0	(00)	
	1	9.99998	96423	47613	06843	7	(-01)	1.58796	45707	58947	92790	3	(01)
	2	1.48796	77028	40464	06661	3	(01)	9.02165	84505	29372	64231	4	(01)
	3	7.63362	88437	05946	89089	6	(01)	2.34257	35047	17625	15305	3	(02)
	4	1.69810	67637	64238	38270	5	(02)	2.95313	63356	77908	51742	3	(02)
	5	1.70063	29783	11516	12932	8	(02)	1.77572	81867	17289	79967	7	(02)
	6	7.24668	97828	58597	02119	9	(01)	4.66217	96103	56861	75681	2	(01)
	7	1.10732	66277	86831	74380	9	(01)	4.34483	63355	09282	08336	0	(00)
	8	3.82857	31210	22477	16910	8	(-01)	8.25816	00085	64488	03469	8	(-02)

TABLE IV

$$E_1(x) \simeq \frac{e^{-x}}{x} \left[1 + \frac{1}{x} \sum_{s=0}^n p_s x^{-s} / \sum_{s=0}^n q_s x^{-s} \right], \quad 4 \leq x$$

<i>n</i>	<i>s</i>	<i>p_s</i>		<i>q_s</i>	
0	0	-7.370	(-01)	1.000	(00)
	1	-9.96969 8	(-01)	1.00000 0	(00)
	1	-4.25784 9	(-01)	2.31826 1	(00)
2	0	-9.99964 218	(-01)	1.00000 000	(00)
	1	-3.73313 946	(00)	5.72998 011	(00)
	2	-4.46320 868	(-01)	6.00442 247	(00)
3	0	-9.99999 54096 2	(-01)	1.00000 00000 0	(00)
	1	-8.13255 72741 8	(00)	1.01324 79271 3	(01)
	2	-1.24514 81283 9	(01)	2.67210 71019 7	(01)
	3	-6.42505 77883 8	(-01)	1.71463 13845 1	(01)
4	0	-9.99999 99359 628	(-01)	1.00000 00000 000	(00)
	1	-1.34511 14977 055	(01)	1.54511 13160 377	(01)
	2	-4.79714 42385 415	(01)	7.28738 48194 513	(01)
	3	-4.20689 28555 297	(01)	1.19100 78997 159	(02)
	4	-1.14468 73348 374	(00)	5.32225 12691 417	(01)
5	0	-9.99999 99990 36009	(-01)	1.00000 00000 00000	(00)
	1	-1.96304 08535 93865	(01)	2.16304 08494 23774	(01)
	2	-1.19557 61038 37175	(02)	1.56818 43364 53856	(02)
	3	-2.54376 33976 88951	(02)	4.62230 27156 14783	(02)
	4	-1.47982 19500 50448	(02)	5.30685 09610 81167	(02)
	5	-2.39099 64453 13573	(00)	1.77600 70940 35063	(02)

6	0	-9.99999	99999	84469	10	(-01)	1.00000	00000	00000	00	(00)	
	1	-2.66271	06043	18114	52	(01)	2.86271	06042	21918	96	(01)	
	2	-2.41055	82709	70148	47	(02)	2.92310	03938	85332	47	(02)	
	3	-8.95927	95777	29368	14	(02)	1.33278	53774	82572	31	(03)	
	4	-1.29885	68874	64841	01	(03)	2.77761	94950	91629	62	(03)	
	5	-5.45374	15888	31328	73	(02)	2.40401	71322	59089	47	(03)	
	6	-5.66575	20653	38687	35	(00)	6.31657	48328	08002	29	(02)	
7	0	-9.99999	99999	99734	1443	(-01)	1.00000	00000	00000	0000	(00)	
	1	-3.44061	99500	66848	9491	(01)	3.64061	99500	64598	0400	(01)	
	2	-4.27532	67120	19885	3941	(02)	4.94345	07020	99036	4527	(02)	
	3	-2.39601	94324	74905	4028	(03)	3.19027	23748	95433	0383	(03)	
	4	-6.16885	21005	54763	5088	(03)	1.03370	75308	58409	7698	(04)	
	5	-6.57609	69874	80211	7925	(03)	1.63241	45355	77835	0289	(04)	
	6	-2.10607	73714	26332	8896	(03)	1.11497	75287	10966	2000	(04)	
8	7	-1.48990	84997	29481	6902	(01)	2.37813	89910	21602	2120	(03)	
	0	-9.99999	99999	99995	19501	(-01)	1.00000	00000	00000	00000	(00)	
	1	-4.29386	24114	77102	57520	(01)	4.49386	24114	77048	90829	(01)	
	2	-6.95170	14067	13531	07041	(02)	7.79047	38890	11024	39770	(02)	
	3	-5.40015	52494	29216	58167	(03)	6.71261	82825	01531	33866	(03)	
	4	-2.12023	11271	04813	65124	(04)	3.09117	90486	38225	40045	(04)	
	5	-4.06389	28671	55509	35195	(04)	7.62113	01980	23102	09672	(04)	
9	6	-3.34895	49619	12054	50827	(04)	9.53743	96753	45534	18034	(04)	
	7	-8.50640	61297	99729	23790	(03)	5.30984	66398	43201	92097	(04)	
	8	-4.28026	86155	59885	31046	(01)	9.42559	71270	68271	36132	(03)	
	0	-9.99999	99999	99999	90878	(-01)	1.00000	00000	00000	00000	(00)	
	1	-5.21996	32588	52257	24810	(01)	5.41996	32588	52255	94149	24	(01)
	2	-1.06117	77263	55033	17668	(03)	1.16357	69915	32084	80354	59	(03)
	3	-1.08168	52399	09591	56224	(04)	1.28428	08586	62729	73659	98	(04)
	4	-5.93468	41538	83711	91723	(04)	7.92317	87945	27904	36987	18	(04)
	5	-1.75032	73087	49708	13147	(05)	2.78581	34710	52084	21393	57	(05)
	6	-2.61814	54937	20563	96473	(05)	5.46168	42050	69115	57357	58	(05)
	7	-1.72833	75773	77759	39268	(05)	5.59037	56210	02286	40033	80	(05)
	8	-3.58461	98743	99690	43086	(04)	2.59897	62083	60848	97774	11	(05)
	9	-1.32768	81505	63744	46229	(02)	3.91478	56245	55634	56270	78	(04)

TABLE V

$$E_1(x) = -\ln(x) - \gamma + \frac{a_1x}{1-} \frac{a_2x}{1-} \dots$$

<i>i</i>	<i>a_i</i>					
1	1.00000	00000	00000	00000	0000	(00)
2	-2.50000	00000	00000	00000	0000	(-01)
3	2.77777	77777	77777	77777	7778	(-02)
4	-2.77777	77777	77777	77777	7778	(-01)
5	1.64000	00000	00000	00000	0000	(-01)
6	-4.81463	41463	41463	41463	4146	(-02)
7	1.86603	29429	42170	56679	5923	(-02)
8	-1.60205	68757	07501	54525	5703	(-01)
9	1.23861	39210	41923	32991	7815	(-01)
10	-2.31794	05112	45808	29915	6950	(-02)
11	1.29681	11650	66612	32574	9017	(-02)
12	-1.12427	07106	81348	55498	0946	(-01)
13	9.48210	68248	17627	18042	2609	(-02)
14	-1.50705	56940	70993	41586	7282	(-02)
15	9.91703	79965	35417	38979	0346	(-03)
16	-8.65217	39735	62029	78930	2325	(-02)
17	7.61891	65113	71489	38461	1831	(-02)
18	-1.11260	95393	81208	17324	8394	(-02)
19	8.02137	38225	66099	94103	6526	(-03)
20	-7.03079	19046	68019	59276	8798	(-02)
21	6.35254	92288	39041	18840	0415	(-02)
22	-8.80574	51173	87599	83602	9885	(-03)
23	6.73156	43159	35130	57355	3415	(-03)
24	-5.92093	68551	88450	02360	8312	(-02)
25	5.44193	72539	14569	37403	6161	(-02)
26	-7.28137	18497	92553	59228	5206	(-03)
27	5.79799	42107	94125	58486	8766	(-03)
28	-5.11362	97085	60959	55012	6918	(-02)
29	4.75746	03552	29856	08437	2345	(-02)
30	-6.20470	20824	22709	71921	3328	(-03)
31	5.09130	38779	10156	55165	2889	(-03)
32	-4.50002	85546	37539	73594	7098	(-02)
33	4.22487	22218	42761	47612	1945	(-02)
34	-5.40431	89612	94834	49318	0440	(-03)
35	4.53788	95713	23179	68530	3408	(-03)
36	-4.01789	51290	74937	23947	5927	(-02)
37	3.79896	24606	33396	75543	1572	(-02)
38	-4.78623	11965	44072	42723	6975	(-03)
39	4.09283	20621	35569	26450	8815	(-03)
40	-3.62907	03638	13851	03882	9943	(-02)

the QD algorithm [8], using 40S arithmetic, into a continued fraction

$$(4) \quad E_1(x) + \ln(x) = -\gamma + \frac{a_1x}{1-} \frac{a_2x}{1-} \frac{a_3x}{1-} \dots$$

which converges rapidly with little inherent error except near the zero at $x \simeq .675$. Using this continued fraction for $0 < x \leq 4$, and the standard continued fraction [1, Eq. 5.1-22] for $x > 4$, at least 23S master function values were obtained for E_1 for all $x > 0$.

4. Results. Table I lists the values of

$$\varepsilon_{lm} = -100 \log_{10} \max \left| \frac{E_1(x) - E_{lm}(x)}{E_1(x)} \right|,$$

where the maximum is taken over the appropriate interval, for the initial segments of the various L_∞ Walsh arrays. Tables II-IV present the coefficients for the main diagonals of these arrays. All coefficients are given to accuracies greater than that justified by the maximal errors, but reasonable additional rounding should not greatly affect the overall accuracies. Each approximation listed, with the coefficients rounded as they appear in the table, was tested against the master function routines for 5000 pseudo-random arguments. In all cases maximal errors agreed, within roundoff, in magnitude and location with those for the unrounded approximations.

Table V lists the coefficients of the continued fraction (4) to 25S. While these coefficients are not all correct to 25S, the evaluation of (4) using these coefficients appears correct to nearly 25S for $x \leq 8$, (25D for $x \simeq .675$), slowly deteriorating in accuracy for larger x .

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