On the Relations between Eigenvalues and Eigenvectors for Matrices Resulting from Pre and Post Multiplication by the Transpose

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Abstract. Relationships between the eigenproblem associated with the matrices $A^T A$ and $A A^T$ are derived. If the larger problem is solved first, then the eigenvalues and eigenvectors associated with the smaller problem may easily be computed from the derived relationships.

Consider a nonsquare matrix A of rank m and of dimension $c \times d$. The product $A^T A$ is symmetric and positive semidefinite, and hence possesses an eigenvector-eigenvalue relationship of the form

$$(1) \quad S^T (A^T A) S = D,$$

where S is a normal orthogonal matrix containing eigenvectors as columns, and D is a diagonal matrix containing eigenvalues as the diagonal elements. S and D , both of dimension $d \times d$, can be ordered so that any zero eigenvalues, resulting from $A^T A$ having excess dimension over the rank m , are at the bottom:

$$(2) \quad D = \left[\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array} \right]$$

where E is an $m \times m$ submatrix with nonzero diagonal elements. Correspondingly, the ordered S can be partitioned as

$$(3) \quad S = [S_m \mid S_p],$$

so that the left-hand part S_m contains those eigenvectors associated with nonzero eigenvalues. Here $p = d - m$.

The positive semidefinite property of $A^T A$ assures that the diagonal elements of E are positive, hence E can be "square-rooted":

$$(4) \quad E = E^{1/2} E^{1/2}$$

with $E^{1/2}$ being a diagonal matrix containing the square roots of the nonzero eigenvalues of $A^T A$.

The basic relationship (1) can be modified by premultiplication by S to give

$$(5) \quad (A^T A) S = S D.$$

If (5) is written in partitioned form, the following result becomes obviously valid:

$$(6) \quad A^T A S_m = S_m E.$$

Received December 11, 1967.

A normal orthogonal vector set can be constructed as follows: First substitute (4) into (6) and premultiply by S_m^T , then premultiply and postmultiply by $E^{-1/2}$ to give

$$(7) \quad (AS_mE^{-1/2})^T(AS_mE^{-1/2}) = I.$$

Equation (6) is a good starting point for investigating the eigenproblem of AA^T . Premultiplying (6) by A and postmultiplying that result by $E^{-1/2}$ gives

$$(8) \quad (AA^T)(AS_mE^{-1/2}) = AS_mE^{1/2} = (AS_mE^{-1/2})E.$$

Now, from (7) and (8) it follows that

$$(9) \quad (AS_mE^{-1/2})^T(AA^T)(AS_mE^{-1/2}) = E.$$

Now denote the eigenvector-eigenvalue relationship for AA^T similarly to (1) as

$$(10) \quad R^T(AA^T)R = D^*,$$

where R and D^* are of dimension $c \times c$. Reorder R and D^* to obtain

$$R = [R_m; R_q] \quad \text{and} \quad D^* = \begin{bmatrix} E^* & \vdots & 0 \\ \hline & & \\ 0 & \vdots & 0 \end{bmatrix}.$$

Here $q = c - m$ and E^* is an $m \times m$ diagonal submatrix.

The following properties are well known:

(1) $E^* = E$.

(2) R_q is arbitrary, subject to the restrictions $R_q^T R_m = 0$, $R_q^T R_q = I$.

The important result of the paper is that

$$(11) \quad R_m = AS_mE^{-1/2}$$

which follows from an examination of (9).

In conclusion, if the larger problem (i.e., the eigenproblem associated with $A^T A$) is solved first, then the complete eigenproblem associated with AA^T may be computed by using the above-mentioned well-known properties together with the result (11).

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